Physical laws in curved spacetime

Minimal coupling, equivalence principle

Physical laws in curved spacetime should exhibit *general covariance*: they should be independent of any choice of basis or coordinate chart. In special relativity, we restrict attention to coordinate systems corresponding to inertial frames. The laws of physics should exhibit *special covariance*, i.e, take the same form in any inertial frame (this is the principle of relativity). The following procedure can be used to convert such laws of physics into generally covariant laws:

1. Replace the Minkowski metric by a curved spacetime metric.

2. Replace partial derivatives with covariant derivatives (associated to the Levi-Civita connection). This rule is called *minimal coupling* in analogy with a similar rule for charged fields in electrodynamics.

3. Replace coordinate basis indices μ, ν etc (referring to an inertial frame) with abstract indices a, b etc.

Examples. Let x^μ denote the coordinates of an inertial frame, and $\eta^{\mu\nu}$ the inverse Minkowski metric (which has the same components as $\eta_{\mu\nu}$).

1. The simplest Lorentz invariant field equation is the wave equation for a scalar field Φ

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = 0. \tag{5.1}$$

Follow the rules above to obtain the wave equation in a general spacetime:

$$g^{ab}\nabla_a\nabla_b\Phi = 0, \qquad \text{or} \qquad \nabla^a\nabla_a\Phi = 0 \qquad \text{or} \qquad \Phi_{;a}{}^{a} = 0. \tag{5.2}$$

A simple generalization of this equation is the *Klein-Gordon equation* describing a scalar field of mass m:

$$\nabla^a \nabla_a \Phi - m^2 \Phi = 0. \tag{5.3}$$

2. In special relativity, the electric and magnetic fields are combined into an antisymmetric tensor $F_{\mu\nu}$. The electric and magnetic fields in an inertial frame are obtained by the rule (i, j, k take values from 1 to 3) $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk} B_k$. The (source-free) Maxwell equations take the covariant form

$$\eta^{\mu\nu} \partial_\mu F_{\nu\rho} = 0, \qquad \partial_{[\mu} F_{\nu\rho]} = 0. \tag{5.4}$$

Hence in a curved spacetime, the electromagnetic field is described by an antisymmetric tensor F_{ab} satisfying

$$g^{ab} \nabla_a F_{bc} = 0, \qquad \nabla_{[a} F_{bc]} = 0. \tag{5.5}$$

The Lorentz force law for a particle of charge q and mass m in Minkowski spacetime is

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{q}{m} \eta^{\mu\nu} F_{\nu\rho} \frac{dx^\rho}{d\tau} \tag{5.6}$$

where τ is proper time. We saw previously that the LHS can be rewritten as $u^\nu \partial_\nu u^\mu$ where $u^\mu = dx^\mu/d\tau$ is the 4-velocity. Now following the rules above gives the generally covariant equation

$$u^b \nabla_b u^a = \frac{q}{m} g^{ab} F_{bc} u^c = \frac{q}{m} F^a{}_b u^b. \tag{5.7}$$

Note that this reduces to the geodesic equation when $q = 0$.

Remark. The rules above ensure that we obtain generally covariant equations. But how do we know they are the *right* equations? The Einstein equivalence principle states that, in a local inertial frame, the laws of physics should take the same form as in an inertial frame in Minkowski spacetime. But we saw above, that in a local inertial frame at p, $\Gamma^\mu_{\nu\rho}(p) = 0$ and hence (first) covariant derivatives reduce to partial derivatives at p. For example, $\nabla^\mu \nabla_\mu \Phi = g^{\mu\nu} \nabla_\mu \partial_\nu \Phi$ (in any chart) and, at p, this reduces to $\eta^{\mu\nu} \partial_\mu \partial_\nu \Phi$ in a local inertial frame at p (since the metric at p is $\eta_{\mu\nu}$). Hence all of our generally covariant equations reduce to the equations of special relativity in a local inertial frame at any given point. The Einstein equivalence principle is satisfied automatically if we use the above rules. Nevertheless, there is still some scope for ambiguity, which arises from the possibility of including terms in an equation involving the curvature of spacetime (see later). These vanish identically in Minkowski spacetime. Sometimes, such terms are fixed by mathematical consistency. However, this is not always possible: there is no reason why it should be possible to derive laws of physics in curved spacetime from those in flat spacetime. The ultimate test is comparison with observations.

Energy-momentum tensor

in GR, the curvature of spacetime is related to the energy and momentum of matter. So we need to discuss how the latter concepts are defined in GR. We shall start by discussing the energy and momentum of particles.

In special relativity, associated to any particle is a scalar called its *rest mass* (or simply its mass) m. If the particle has 4-velocity u^μ (again x^μ denote inertial frame coordinates) then its 4-momentum is

$$P^\mu = m u^\mu \tag{5.8}$$

The time component of P^μ is the particle's energy and the spatial components are its 3-momentum with respect to the inertial frame.

If an observer at some point p has 4-velocity $v^\mu(p)$ then he measures the particle's energy, when the particle is at q, to be

$$E = -\eta_{\mu\nu} v^\mu(p) P^\nu(q). \tag{5.9}$$

The way to see this is to choose an inertial frame in which, at p, the observer is at rest at the origin, so $v^\mu(p) = (1, 0, 0, 0)$ so E is just the time component of $P^\nu(q)$ in this inertial frame.

By the equivalence principle, GR should reduce to SR in a local inertial frame. Hence in GR we also associate a rest mass m to any particle and define the 4-momentum of a particle with 4-velocity u^a as

$$P^a = m u^a \tag{5.10}$$

Note that

$$g_{ab} P^a P^b = -m^2 \tag{5.11}$$

The EP implies that when the observer and particle *both* are at p then (5.9) should be valid so the observer measures the particle's energy to be

$$E = -g_{ab}(p) v^a(p) P^b(p) \tag{5.12}$$

However, an important difference between GR and SR is that there is no analogue of equation (5.9) for $p \neq q$. This is because $v^a(p)$ and $P^a(q)$ are vectors defined at different points, so they live in different tangent spaces. There is no way they can be combined to give a scalar quantity. An observer at p cannot measure the energy of a particle at q.

Now let's consider the energy and momentum of continuous distributions of matter.

Example. Consider Maxwell theory (without sources) in Minkowski spacetime. Pick an inertial frame and work in pre-relativity notation using Cartesian tensors. The electromagnetic field has energy density

$$\mathcal{E} = \frac{1}{8\pi}\left(E_i E_i + B_i B_i\right) \tag{5.13}$$

and the momentum density (or energy flux density) is given by the Poynting vector:

$$S_i = \frac{1}{4\pi}\epsilon_{ijk} E_j B_k. \tag{5.14}$$

The Maxwell equations imply that these satisfy the conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \partial_i S_i = 0. \tag{5.15}$$

The momentum flux density is described by the stress tensor:

$$t_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}\left(E_k E_k + B_k B_k\right)\delta_{ij} - E_i E_j - B_i B_j\right], \tag{5.16}$$

with the conservation law

$$\frac{\partial S_i}{\partial t} + \partial_j t_{ij} = 0. \tag{5.17}$$

If a surface element has area dA and normal n_i then the force exerted on this surface by the electromagnetic field is $t_{ij} n_j dA$.

In special relativity, these three objects are combined into a single tensor, called variously the "energy-momentum tensor", the "stress tensor", the "stress-energy-momentum tensor" etc. In an inertial frame it is

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma}\, \eta_{\mu\nu}\right) \tag{5.18}$$

where we've raised indices with $\eta^{\mu\nu}$. Note that this is a symmetric tensor. It has components $T_{00} = \mathcal{E}$, $T_{0i} = -S_i$, $T_{ij} = t_{ij}$. The conservation laws above are equivalent to the single equation

$$\partial^\mu T_{\mu\nu} = 0. \tag{5.19}$$

The definition of the energy-momentum tensor extends naturally to GR:

Definition. The energy-momentum tensor of a Maxwell field in a general spacetime is

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac} F_b{}^c - \frac{1}{4} F^{cd} F_{cd}\, g_{ab}\right) \tag{5.20}$$

Exercise (examples sheet 2). Show that Maxwell's equations imply that

$$\nabla^a T_{ab} = 0. \tag{5.21}$$

In GR (and SR) we assume that continuous matter always is described by a conserved energy-momentum tensor:

Postulate. The energy, momentum, and stresses, of matter are described by an *energy-momentum tensor*, a $(0,2)$ symmetric tensor T_{ab} that is *conserved*: $\nabla^a T_{ab} = 0$.

Remark. Let u^a be the 4-velocity of an observer $\mathcal{O}$ at p. Consider a local inertial frame (LIF) at p in which $\mathcal{O}$ is at rest. Choose an orthonormal basis at p $\{e_\mu\}$ aligned with the coordinate axes of this LIF. In such a basis, $e_0^a = u^a$. Denote the spatial basis vectors as e_i^a, $i = 1,2,3$. From the Einstein equivalence principle, $\mathcal{E} \equiv T_{00} = T_{ab}e_0^a e_0^b = T_{ab}u^a u^b$ is the energy density of matter at p measured by $\mathcal{O}$. Similarly, $S_i \equiv -T_{0i}$ is the momentum density and $t_{ij} \equiv T_{ij}$ the stress tensor measured by $\mathcal{O}$. The *energy-momentum current* measured by $\mathcal{O}$ is the 4-vector $j^a = -T^a{}_b u^b$, which has components $(\mathcal{E}, S_i)$ in this basis.

Remark. In an inertial frame x^μ in Minkowski spacetime, local conservation of T_{ab} is equivalent to equations of the form (5.15) and (5.17). If one integrates these over a fixed volume V in surfaces of constant $t = x^0$ then one obtains global conservation equations. For example, integrating (5.15) over V gives

$$\frac{d}{dt}\int_V \mathcal{E} = -\int_S \mathbf{S}\cdot\mathbf{n}dA \tag{5.22}$$

where the surface S (with outward unit normal $\mathbf{n}$) bounds V. In words: the rate of increase of the energy of matter in V is equal to minus the energy flux across S. In a general curved spacetime, such an interpretation is not possible. This is because the gravitational field can do work on the matter in the spacetime. One might think that one could obtain global conservation laws in curved spacetime by introducing a definition of energy density etc for the gravitational field. This is a subtle issue. The gravitational field is described by the metric g_{ab}. In Newtonian theory, the energy density of the gravitational field is $-(1/8\pi)(\nabla\Phi)^2$ so one might expect that in GR the energy density of the gravitational field should be some expression quadratic in first derivatives of g_{ab}. But we have seen that we can choose normal coordinates to make the first partial derivatives of g_{ab} vanish at any given point. Gravitational energy certainly exists but not in a local sense. For example one can define the *total* energy (i.e. the energy of matter and the gravitational field) for certain spacetimes (this will be discussed in the black holes course).

Example. A *perfect fluid* is described by a 4-velocity vector field u^a, and two scalar fields ρ and p. The energy-momentum tensor is

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab} \tag{5.23}$$

ρ and p are the energy density and pressure measured by an observer co-moving with the fluid, i.e., one with 4-velocity u^a (check: $T_{ab}u^a u^b = \rho + p - p = \rho$). The equations of motion of the fluid can be derived by conservation of T_{ab}:

Exercise (examples sheet 2). Show that, for a perfect fluid, $\nabla^a T_{ab} = 0$ is equivalent to

$$u^a \nabla_a \rho + (\rho + p)\nabla_a u^a = 0, \qquad (\rho + p)u^b \nabla_b u_a = -(g_{ab} + u_a u_b)\nabla^b p \tag{5.24}$$

These are relativistic generalizations of the mass conservation equation and Euler equation of non-relativistic fluid dynamics. Note that a pressureless fluid moves on timelike geodesics. This makes sense physically: zero pressure implies that the fluid particles are non-interacting and hence behave like free particles.

Curvature

Parallel transport

On a general manifold there is no way of comparing tensors at different points. For example, we can't say whether a vector at p is the same as a vector at q. However, with a connection we can define a notion of "a tensor that doesn't change along a curve":

Definition. Let X^a be the tangent to a curve. A tensor field T is *parallelly transported along the curve* if $\nabla_X T = 0$.

Remarks.

1. Sometimes we say "parallelly propagated" instead of "parallelly transported".

2. A geodesic is a curve whose tangent vector is parallelly transported along the curve.

3. Let p be a point on a curve. If we specify T at p then the above equation determines T uniquely everywhere along the curve. For example, consider a $(1,1)$ tensor. Introduce a chart in a neighbourhood of p. Let t be the parameter along the curve. In a coordinate chart, $X^\mu = dx^\mu/dt$ so $\nabla_X T = 0$ gives

$$\begin{aligned} 0 &= X^\sigma T^\mu{}_{\nu;\sigma} = X^\sigma T^\mu{}_{\nu,\sigma} + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho X^\sigma \\ &= \frac{dT^\mu{}_\nu}{dt} + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho X^\sigma \end{aligned} \tag{6.1}$$

Standard ODE theory guarantees a unique solution given initial values for the components $T^\mu{}_\nu$.

4. If q is some other point on the curve then parallel transport along a curve from p to q determines an isomorphism between tensors at p and tensors at q.

Consider Euclidean space or Minkowski spacetime with the Levi-Civita connection, and use Cartesian/inertial frame coordinates so the Christoffel symbols vanish everywhere. Then a tensor is parallelly transported along a curve iff its components are constant along the curve. Hence if we have two different curves from p to q then the result of parallelly transporting T from p to q is independent of which curve we choose. However, in a general spacetime this is no longer true: parallel transport is path-dependent. The path-dependence of parallel transport is measured by the *Riemann curvature tensor*. For Euclidean space or Minkowski spacetime, the Riemann tensor (of the Levi-Civita connection) vanishes and we say that the spacetime is *flat*.

The Riemann tensor

We shall return to the path-dependence of parallel transport below. First we define the Riemann tensor is as follows:

Definition. The *Riemann curvature tensor* $R^a{}_{bcd}$ of a connection ∇ is defined by $R^a{}_{bcd} Z^b X^c Y^d = (R(X,Y)Z)^a$, where X, Y, Z are vector fields and $R(X,Y)Z$ is the vector field

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{6.2}$$

To demonstrate that this defines a tensor, we need to show that it is linear in X, Y, Z. The symmetry $R(X,Y)Z = -R(Y,X)Z$ implies that we need only check linearity in X and Z. The non-trivial part is to check what happens if we multiply X or Z by a function f:

$$\begin{aligned}
R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\
&= f\nabla_X \nabla_Y Z - \nabla_Y (f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X} Z \\
&= f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - Y(f)\nabla_X Z - \nabla_{f[X,Y]} Z + \nabla_{Y(f)X} Z \\
&= f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]} Z + Y(f)\nabla_X Z \\
&= fR(X,Y)Z
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
R(X,Y)(fZ) &= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) \\
&= \nabla_X (f\nabla_Y Z + Y(f)Z) - \nabla_Y (f\nabla_X Z + X(f)Z) \\
&\quad - f\nabla_{[X,Y]} Z - [X,Y](f)Z \\
&= f\nabla_X \nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z \\
&\quad - f\nabla_Y \nabla_X Z - Y(f)\nabla_X Z - X(f)\nabla_Y Z - Y(X(f))Z \\
&\quad - f\nabla_{[X,Y]} Z - [X,Y](f)Z \\
&= fR(X,Y)Z
\end{aligned} \tag{6.4}$$

It follows that our definition does indeed define a tensor. Let's calculate its components in a *coordinate* basis $\{e_\mu = \partial/\partial x^\mu\}$ (so $[e_\mu, e_\nu] = 0$). Use the notation $\nabla_\mu \equiv \nabla_{e_\mu}$,

$$\begin{aligned}
R(e_\rho, e_\sigma)e_\nu &= \nabla_\rho \nabla_\sigma e_\nu - \nabla_\sigma \nabla_\rho e_\nu \\
&= \nabla_\rho (\Gamma^\tau_{\nu\sigma} e_\tau) - \nabla_\sigma (\Gamma^\tau_{\nu\rho} e_\tau) \\
&= \partial_\rho \Gamma^\mu_{\nu\sigma} e_\mu + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} e_\mu - \partial_\sigma \Gamma^\mu_{\nu\rho} e_\mu - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma} e_\mu
\end{aligned} \tag{6.5}$$

and hence, in a coordinate basis,

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma} \tag{6.6}$$

Remark. It follows that the Riemann tensor vanishes for the Levi-Civita connection in Euclidean space or Minkowski spacetime (since one can choose coordinates for which the Christoffel symbols vanish everywhere).

The following contraction of the Riemann tensor plays an important role in GR:

Definition. The *Ricci curvature tensor* is the $(0,2)$ tensor defined by

$$R_{ab} = R^c{}_{acb} \tag{6.7}$$

We saw earlier that, with vanishing torsion, the second covariant derivatives of a function commute. The same is not true of covariant derivatives of tensor fields. The failure to commute arises from the Riemann tensor:

Exercise. Let ∇ be a torsion-free connection. Prove the *Ricci identity*:

$$\nabla_c \nabla_d Z^a - \nabla_d \nabla_c Z^a = R^a{}_{bcd} Z^b \tag{6.8}$$

Hint. Show that the equation is true when multiplied by arbitrary vector fields X^c and Y^d.

Parallel transport again

Now we return to the relation between the Riemann tensor and the path-dependence of parallel transport. Let X and Y be vector fields that are linearly independent everywhere, with $[X,Y]=0$. Earlier we saw that we can choose a coordinate chart

$(s,t,\ldots)$ such that $X = \partial/\partial s$ and $Y = \partial/\partial t$. Let $p \in M$ and choose the coordinate chart such that p has coordinates $(0,\ldots,0)$. Let q, r, u be the point with coordinates $(\delta s, 0, 0, \ldots)$, $(\delta s, \delta t, 0, \ldots)$, $(0, \delta t, 0, \ldots)$ respectively, where δs and δt are small. We can connect p and q with a curve along which only s varies, with tangent X. Similarly, q and r can be connected by a curve with tangent Y. p and u can be connected by a curve with tangent Y, and u and r can be connected by a curve with tangent X. The result is a small quadrilateral (Fig. 6.1).

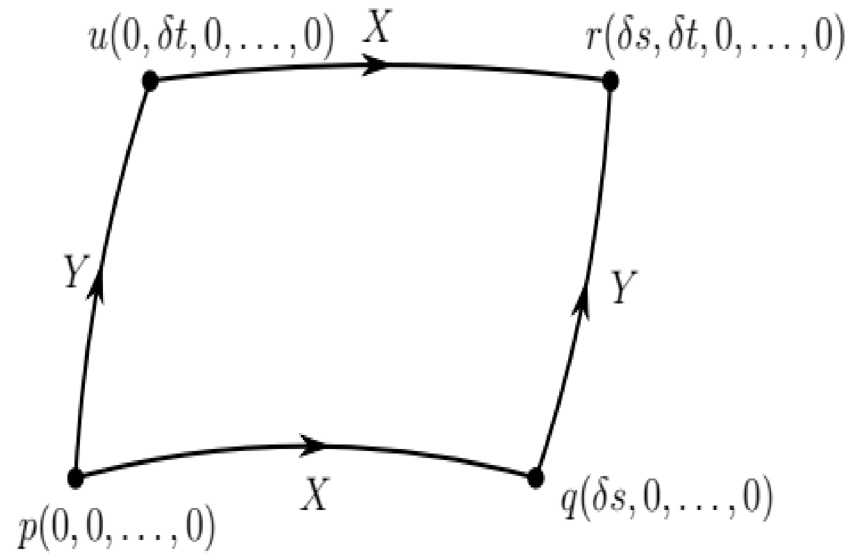

Figure 6.1: Parallel transport

Now let $Z_p \in T_p(M)$. Parallel transport Z_p along pqr to obtain a vector $Z_r \in T_r(M)$. Parallel transport Z_p along pur to obtain a vector $Z'_r \in T_r(M)$. We shall calculate the difference $Z'_r - Z_r$ for a torsion-free connection.

It is convenient to introduce a new coordinate chart: normal coordinates at p. Henceforth, indices $\mu, \nu, \ldots$ will refer to this chart. s and t will now be used as *parameters* along the curves with tangent X and Y respectively.

pq is a curve with tangent vector X and parameter s. Along pq, Z is parallely transported: $\nabla_X Z = 0$ so $dZ^\mu/ds = -\Gamma^\mu_{\nu\rho} Z^\nu X^\rho$ and hence $d^2 Z^\mu/ds^2 = -(\Gamma^\mu_{\nu\rho} Z^\nu X^\rho)_{,\sigma} X^\sigma$. Now Taylor's theorem gives

$$\begin{aligned} Z^\mu_q &= Z^\mu_p + \left(\frac{dZ^\mu}{ds}\right)_p \delta s + \frac{1}{2}\left(\frac{d^2 Z^\mu}{ds^2}\right)_p \delta s^2 + \mathcal{O}(\delta s^3) \\ &= Z^\mu_p - \frac{1}{2}\left(\Gamma^\mu_{\nu\rho,\sigma} Z^\nu X^\rho X^\sigma\right)_p \delta s^2 + \mathcal{O}(\delta s^3) \end{aligned} \quad (6.9)$$

where we have used $\Gamma^\mu_{\nu\rho}(p) = 0$ in normal coordinates at p (assuming a torsion-free connection). Now consider parallel transport along qr to obtain

$$Z^\mu_r = Z^\mu_q + \left(\frac{dZ^\mu}{dt}\right)_q \delta t + \frac{1}{2}\left(\frac{d^2 Z^\mu}{dt^2}\right)_q \delta t^2 + \mathcal{O}(\delta t^3)$$

$$
\begin{aligned}
&= Z_q^\mu - \left(\Gamma^\mu_{\nu\rho} Z^\nu Y^\rho\right)_q \delta t - \frac{1}{2}\left(\left(\Gamma^\mu_{\nu\rho} Z^\nu Y^\rho\right)_{,\sigma} Y^\sigma\right)_q \delta t^2 + \mathcal{O}(\delta t^3) \\
&= Z_q^\mu - \left[\left(\Gamma^\mu_{\nu\rho,\sigma} Z^\nu Y^\rho X^\sigma\right)_p \delta s + \mathcal{O}(\delta s^2)\right] \delta t \\
&\quad -\frac{1}{2}\left[\left(\left(\Gamma^\mu_{\nu\rho,\sigma} Z^\nu Y^\rho Y^\sigma\right)_p + \mathcal{O}(\delta s)\right] \delta t^2 + \mathcal{O}(\delta t^3) \\
&= Z_p^\mu - \frac{1}{2}\left(\Gamma^\mu_{\nu\rho,\sigma}\right)_p \left[Z^\nu \left(X^\rho X^\sigma \delta s^2 + Y^\rho Y^\sigma \delta t^2 + 2 Y^\rho X^\sigma \delta s \delta t\right)\right]_p + \mathcal{O}(\delta^3)
\end{aligned}
\tag{6.10}
$$

Here we assume that δs and δt both are $\mathcal{O}(\delta)$ (i.e. $\delta s = a\delta$ for some non-zero constant a and similarly for δt). Now consider parallel transport along pur. The result can be obtained from the above expression simply by interchanging X with Y and s with t. Hence we have

$$
\begin{aligned}
\Delta Z_r^\mu \equiv Z_r'^\mu - Z_r^\mu &= \left[\Gamma^\mu_{\nu\rho,\sigma} Z^\nu \left(Y^\rho X^\sigma - X^\rho Y^\sigma\right)\right]_p \delta s \delta t + \mathcal{O}(\delta^3) \\
&= \left[\left(\Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma}\right) Z^\nu X^\rho Y^\sigma\right]_p \delta s \delta t + \mathcal{O}(\delta^3) \\
&= \left(R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma\right)_p \delta s \delta t + \mathcal{O}(\delta^3) \\
&= \left(R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma\right)_r \delta s \delta t + \mathcal{O}(\delta^3)
\end{aligned}
\tag{6.11}
$$

where we used the expression (6.6) for the Riemann tensor components (remember that $\Gamma^\mu_{\nu\rho}(p) = 0$). In the final equality we used that quantities at p and r differ by $\mathcal{O}(\delta)$. We have derived this result in a coordinate basis defined using normal coordinates at p. But now both sides involve tensors at r. Hence our equation is basis-independent so we can write

$$
\left(R^a{}_{bcd} Z^b X^c Y^d\right)_r = \lim_{\delta\to 0} \frac{\Delta Z_r^a}{\delta s \delta t} \tag{6.12}
$$

The Riemann tensor measures the path-dependence of parallel transport.

Remark. We considered parallel transport along two different curves from p to r. However, we can reinterpret the result as describing the effect of parallel transport of a vector Z_r^a around the closed curve $rqpur$ to give the vector $Z_r'^a$. Hence ΔZ_r^a measures the change in Z_r^a when parallel transported around a closed curve.

Symmetries of the Riemann tensor

From its definition, we have the symmetry $R^a{}_{bcd} = -R^a{}_{bdc}$, equivalently:

$$
R^a{}_{b(cd)} = 0. \tag{6.13}
$$

Proposition. If ∇ is torsion-free then

$$R^a{}_{[bcd]} = 0. \tag{6.14}$$

Proof. Let $p \in M$ and choose normal coordinates at p. Vanishing torsion implies $\Gamma^\mu_{\nu\rho}(p) = 0$ and $\Gamma^\mu_{[\nu\rho]} = 0$ everywhere. We have $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho}$ at p. Antisymmetrizing on $\nu\rho\sigma$ now gives $R^\mu{}_{[\nu\rho\sigma]} = 0$ at p in the coordinate basis defined using normal coordinates at p. But if the components of a tensor vanish in one basis then they vanish in any basis. This proves the result at p. However, p is arbitrary so the result holds everywhere.

Proposition. (Bianchi identity). If ∇ is torsion-free then

$$R^a{}_{b[cd;e]} = 0 \tag{6.15}$$

Proof. Use normal coordinate at p again. At p,

$$R^\mu{}_{\nu\rho\sigma;\tau} = \partial_\tau R^\mu{}_{\nu\rho\sigma} \tag{6.16}$$

In normal coordinates at p, $\partial R = \partial\partial\Gamma - \Gamma\partial\Gamma$ and the latter terms vanish at p, we only need to worry about the former:

$$R^\mu{}_{\nu\rho\sigma;\tau} = \partial_\tau \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\tau \partial_\sigma \Gamma^\mu_{\nu\rho} \qquad \text{at } p \tag{6.17}$$

Antisymmetrizing gives $R^\mu{}_{\nu[\rho\sigma;\tau]} = 0$ at p in this basis. But again, if this is true in one basis then it is true in any basis. Furthermore, p is arbitrary. The result follows.

Geodesic deviation

Remark. In Euclidean space, or in Minkowski spacetime, initially parallel geodesics remain parallel forever. On a general manifold we have no notion of "parallel". However, we can study whether nearby geodesics move together or apart. In particular, we can quantify their "relative acceleration".

Definition. Let M be a manifold with a connection ∇. A *1-parameter family of geodesics* is a map $\gamma : I \times I' \to M$ where I and I' both are open intervals in $\mathbb{R}$, such that (i) for fixed s, $\gamma(s,t)$ is a geodesic with affine parameter t (so s is the parameter that labels the geodesic); (ii) the map $(s,t) \mapsto \gamma(s,t)$ is smooth and one-to-one with a smooth inverse. This implies that the family of geodesics forms a 2d surface $\Sigma \subset M$.

Let T be the tangent vector to the geodesics and S to be the vector tangent to the curves of constant t, which are parameterized by s (see Fig. 6.2). In a chart x^μ, the geodesics are specified by $x^\mu(s,t)$ with $S^\mu = \partial x^\mu / \partial s$. Hence

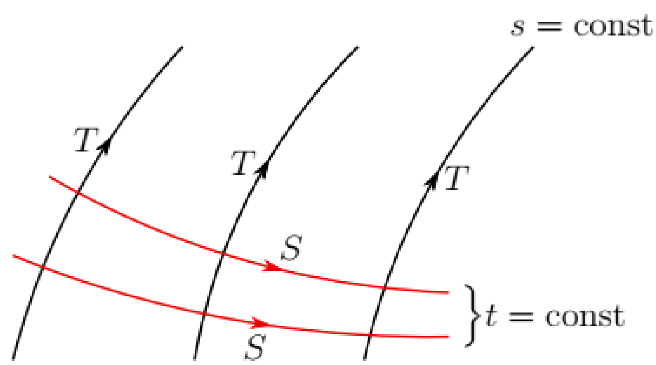

Figure 6.2: 1-parameter family of geodesics

$x^\mu(s+\delta s,t) = x^\mu(s,t) + \delta s S^\mu(s,t) + \mathcal{O}(\delta s^2)$. Therefore $(\delta s)S^a$ points from one geodesic to an infinitesimally nearby one in the family. We call S^a a *deviation vector*.

On the surface Σ we can use s and t as coordinates. We can extend these to coordinates $(s,t,\ldots)$ defined in a neighbourhood of Σ. This gives a coordinate chart in which $S = \partial/\partial s$ and $T = \partial/\partial t$ on Σ. We can now use these equations to extend S and T to a neighbourhood of the surface. S and T are now vector fields satisfying

$$[S,T] = 0 \tag{6.18}$$

Remark. If we fix attention on a particular geodesic then $\nabla_T(\delta s S) = \delta s \nabla_T S$ can be regarded as the rate of change of the relative position of an infinitesimally nearby geodesic in the family i.e., as the "relative velocity" of an infinitesimally nearby geodesic. We can define the "relative acceleration" of an infinitesimally nearby geodesic in the family as $\delta s \nabla_T \nabla_T S$. The word "relative" is important: the acceleration of a curve with tangent T is $\nabla_T T$, which vanishes here (as the curves are geodesics).

Proposition. If ∇ has vanishing torsion then

$$\nabla_T \nabla_T S = R(T,S)T \tag{6.19}$$

Proof. Vanishing torsion gives $\nabla_T S - \nabla_S T = [T,S] = 0$. Hence

$$\nabla_T \nabla_T S = \nabla_T \nabla_S T = \nabla_S \nabla_T T + R(T,S)T, \tag{6.20}$$

where we used the definition of the Riemann tensor. But $\nabla_T T = 0$ because T is tangent to (affinely parameterized) geodesics.

Remark. This result is known as the *geodesic deviation equation*. In abstract index notation it is:

$$T^c\nabla_c(T^b\nabla_b S^a) = R^a{}_{bcd}T^bT^cS^d \tag{6.21}$$

This equation shows that curvature results in relative acceleration of geodesics. It also provides another method of measuring $R^a{}_{bcd}$: at any point p we can pick our 1-parameter family of geodesics such that T and S are arbitrary. Hence by measuring the LHS above we can determine $R^a{}_{(bc)d}$. From this we can determine $R^a{}_{bcd}$:

Exercise. Show that, for a torsion-free connection,

$$R^a{}_{bcd} = \frac{2}{3}\left(R^a{}_{(bc)d} - R^a{}_{(bd)c}\right) \tag{6.22}$$

Remarks.

1. Note that the relative acceleration vanishes for *all* families of geodesics if, and only if, $R^a{}_{bcd} = 0$.

2. In GR, free particles follow geodesics of the Levi-Civita connection. Geodesic deviation is the tendency of freely falling particles to move together or apart. We have already met this phenomenon: it arises from *tidal forces*. Hence the Riemann tensor is the quantity that measures tidal forces.

Curvature of the Levi-Civita connection

Remark. From now on, we shall restrict attention to a manifold with metric, and use the Levi-Civita connection. The Riemann tensor then enjoys additional symmetries. Note that we can lower an index with the metric to define R_{abcd}.

Proposition. The Riemann tensor satisfies

$$R_{abcd} = R_{cdab}, \qquad R_{(ab)cd} = 0. \tag{6.23}$$

Proof. The second identity follows from the first and the antisymmetry of the Riemann tensor. To prove the first, introduce normal coordinates at p, so $\partial_\mu g_{\nu\rho} = 0$ at p. Then, at p,

$$0 = \partial_\mu \delta^\nu_\rho = \partial_\mu\left(g^{\nu\sigma}g_{\sigma\rho}\right) = g_{\sigma\rho}\partial_\mu g^{\nu\sigma}. \tag{6.24}$$

Multiplying by the inverse metric gives $\partial_\mu g^{\nu\rho} = 0$ at p. Using this, we have

$$\partial_\rho \Gamma^\tau_{\nu\sigma} = \frac{1}{2}g^{\tau\mu}\left(g_{\mu\nu,\sigma\rho} + g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho}\right) \qquad \text{at } p \tag{6.25}$$

And hence (as $\Gamma^\mu_{\nu\rho} = 0$ at p)

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma}\right) \qquad \text{at } p \tag{6.26}$$

This satisfies $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ at p using the symmetry of the metric and the fact that partial derivatives commute. This establishes the identity in normal coordinates, but this is a tensor equation and hence valid in any basis. Furthermore p is arbitrary so the identity holds everywhere.

Proposition. The Ricci tensor is symmetric:

$$R_{ab} = R_{ba} \tag{6.27}$$

Proof. $R_{ab} = g^{cd}R_{dacb} = g^{cd}R_{cbda} = R^c{}_{bca} = R_{ba}$ where we used the first identity above in the second equality.

Definition. The *Ricci scalar* is

$$R = g^{ab}R_{ab} \tag{6.28}$$

Definition. The *Einstein tensor* is the symmetric $(0,2)$ tensor defined by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{6.29}$$

Proposition. The Einstein tensor satisfies the *contracted Bianchi identity*:

$$\nabla^a G_{ab} = 0 \tag{6.30}$$

which can also be written as

$$\nabla^a R_{ab} - \frac{1}{2}\nabla_b R = 0 \tag{6.31}$$

Proof. Examples sheet 2.

Einstein's equation

Postulates of General Relativity.

1. Spacetime is a 4d Lorentzian manifold equipped with the Levi-Civita connection.

2. Free particles follow timelike or null geodesics.

3. The energy, momentum, and stresses of matter are described by a symmetric tensor T_{ab} that is conserved: $\nabla^a T_{ab} = 0$.

4. The curvature of spacetime is related to the energy-momentum tensor of matter by the *Einstein equation* (1915)

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G\, T_{ab} \tag{6.32}$$

where G is Newton's constant.

We have discussed points 1-3 above. It remains to discuss the Einstein equation. We can motivate this as follows. In GR, the gravitational field is described by the curvature of spacetime. Since the energy of matter should be responsible for gravitation, we expect *some* relationship between curvature and the energy-momentum tensor. The simplest possibility is a linear relationship, i.e., a curvature tensor is proportional to T_{ab}. Since T_{ab} is symmetric, it is natural to expect the Ricci tensor to be the relevant curvature tensor.

Einstein's first guess for the field equation of GR was $R_{ab} = CT_{ab}$ for some constant C. This does not work for the following reason. The RHS is conserved hence this equation implies $\nabla^a R_{ab} = 0$. But then from the contracted Bianchi identity we get $\nabla_a R = 0$. Taking the trace of the equation gives $R = CT$ (where $T = T^a{}_a$) and hence we must have $\nabla_a T = 0$, i.e., T is constant. But, T vanishes in empty space and is usually non-zero inside matter. Hence this is unsatisfactory.

The solution to this problem is obvious once one knows of the contracted Bianchi identity. Take G_{ab}, rather than R_{ab}, to be proportional to T_{ab}. The coefficient of proportionality on the RHS of Einstein's equation is fixed by demanding that the equation reduces to Newton's law of gravitation when the gravitational field is weak and the matter is moving non-relativistically. We will show this later.

Remarks.

1. In vacuum, $T_{ab} = 0$ so Einstein's equation gives $G_{ab} = 0$. Contracting indices gives $R = 0$. Hence the *vacuum Einstein equation* can be written as

$$R_{ab} = 0 \tag{6.33}$$

2. The "geodesic postulate" of GR is redundant. Using conservation of the energy-momentum tensor it can be shown that a distribution of matter that is small (compared to the scale on which the spacetime metric varies), and sufficiently weak (so that its gravitational effect is small), must follow a geodesic. (See examples sheet 4 for the case of a point particle.)

3. The Einstein equation is a set of non-linear, second order, coupled, partial differential equations for the components of the metric $g_{\mu\nu}$. Very few physically interesting explicit solutions are known so one has to develop other methods to solve the equation, e.g., numerical integration.

4. How unique is the Einstein equation? Is there any tensor, other than G_{ab} that we could have put on the LHS? The answer is supplied by:

Theorem (Lovelock 1972). Let H_{ab} be a symmetric tensor such that (i) in any coordinate chart, at any point, $H_{\mu\nu}$ is a function of $g_{\mu\nu}$, $g_{\mu\nu,\rho}$ and $g_{\mu\nu,\rho\sigma}$ at that point; (ii) $\nabla^a H_{ab} = 0$; (iii) either spacetime is four-dimensional or $H_{\mu\nu}$ depends linearly on $g_{\mu\nu,\rho\sigma}$. Then there exist constants α and β such that

$$H_{ab} = \alpha G_{ab} + \beta g_{ab} \tag{6.34}$$

Hence (as Einstein realized) there is the freedom to add a constant multiple of g_{ab} to the LHS of Einstein's equation, giving

$$G_{ab} + \Lambda g_{ab} = 8\pi G\, T_{ab} \tag{6.35}$$

Λ is called the *cosmological constant*. This no longer reduces to Newtonian theory for slow motion in a weak field but the deviation from Newtonian theory is unobservable if Λ is sufficiently small. Note that $|\Lambda|^{-1/2}$ has the dimensions of length. The effects of Λ are negligible on lengths or times small compared to this quantity. Astronomical observations suggest that there is indeed a very small positive cosmological constant: $\Lambda^{-1/2} \sim 10^9$ light years, the same order of magnitude as the size of the observable Universe. Hence the effects of the cosmological constant are negligible except on cosmological length scales. Therefore we can set $\Lambda = 0$ unless we discuss cosmology.

Note that we can move the cosmological constant term to the RHS of the Einstein equation, and regard it as the energy-momentum tensor of a perfect fluid with $\rho = -p = \Lambda/(8\pi G)$. Hence the cosmological constant is sometimes referred to as *dark energy* or *vacuum energy*. It is a great mystery why it is so small because arguments from quantum field theory suggest that it should be 10^{120} times larger. This is the *cosmological constant problem*. One (controversial) proposed solution of this problem invokes the *anthropic principle*, which posits the existence of many possible universes with different values for constants such as Λ. If Λ was very different from its observed value then galaxies never would have formed and hence we would not be here.

Remark. We have explicitly written Newton's constant G throughout this section. Henceforth we shall choose units so that $G = c = 1$.