

General Relativity - Problem Sheet 2

Please return solutions

1. 2d de Sitter Space

Consider the two-dimensional de Sitter metric

$$ds^2 = -du^2 + \cosh^2 u d\varphi^2$$

where $-\infty < u < \infty$ and $0 \leq \varphi \leq 2\pi$.

1. What is the proper length of the space-like curve defined by $u = u_c$?
2. Write down the lagrangian for (affinely parametrized) geodesics and, using Lagrange's equations, compute the non-vanishing Christoffel symbols:

$$\Gamma_{\varphi\varphi}^u \quad \Gamma_{u\varphi}^\varphi \quad \Gamma_{\varphi u}^\varphi .$$

3. Show that

$$J = \cosh^2 u \dot{\varphi} \quad E = \dot{u}^2 - \cosh^2 u \dot{\varphi}^2$$

are both conserved along geodesics. *Hint: for an elegant derivation of the second conserved quantity, you may want to compute the hamiltonian and explain why it is conserved.*

4. Consider a geodesic with initial condition $\dot{\varphi}(0) = 0$. Show that $J = 0$ and $E = \dot{u}^2$.
5. Now consider the case $J \neq 0$. Introduce the variable $v = \tanh u$ and show that

$$\left(\frac{dv}{d\varphi}\right)^2 = \left(\frac{E}{J^2} + 1\right) - v^2 .$$

Write down the most general solution $v(\varphi)$ and discuss its behaviour as a function of the ratio E/J^2 .

2. Infinitesimal Symmetries

The Lie derivative of a type $(2, 0)$ tensor T_{ab} with respect to a vector field X^a is defined by

$$\mathcal{L}_X T_{ab} = X^c \partial_c T_{ab} + (\partial_a X^c) T_{cb} + (\partial_b X^c) T_{ac}.$$

1. Show that you can replace ∂_a with any covariant derivative ∇_a in this expression and so argue that the Lie derivative transforms as a tensor.
2. Consider the infinitesimal coordinate transformation $\delta x^a = \epsilon K^a$ generated by a vector field K^a . Show that the action of an (affinely parametrized) geodesic

$$S = \int_{s_1}^{s_2} g_{ab}(x(s)) \frac{dx^a}{ds} \frac{dx^b}{ds}$$

is invariant under this transformation if

$$\mathcal{L}_K g_{ab} = 0.$$

Show that this can be equivalently expressed as

$$\nabla_{(a} K_{b)} = 0$$

where ∇_a is the covariant derivative associated to the metric g_{ab} . A vector field with this property is known as a ‘Killing vector’ and generates an infinitesimal symmetry of the geometry defined by the metric g_{ab} .

3. Show that the inner product $g_{ab} K^a \dot{x}^b$ is conserved along the geodesic.
4. Consider the two-dimensional de Sitter metric from problem 1. Show that the vector field K^a with components $K^u = 0$ and $K^\varphi = 1$ is a Killing vector and the corresponding conserved quantity is J .
- 5* Show that the sum $X^a + Y^a$ and commutator $[X, Y]^a = X^b \nabla_b Y^a - Y^b \nabla_b X^a$ of two Killing vectors are also Killing vectors. As the commutator obeys the Jacobi identity, Killing vectors generate a Lie algebra.

3. Independent Components of The Riemann Tensor

We have shown in the lecture that the Riemann tensor $R_{abc}{}^d$ associated to any covariant derivative ∇_a obeys the following algebraic identities:

1. $R_{abc}{}^d = -R_{bac}{}^d$
2. $R_{abc}{}^d + R_{bca}{}^d + R_{cab}{}^d = 0$

What is the additional algebraic identity obeyed when ∇_a is the Levi-Civita covariant derivative associated to a metric g_{ab} ?

In this case, show that the Riemann tensor in four dimensions has, in general, 20 independent components.

Optional: How many independent components does it have in D dimensions?

4. Riemann tensor in two dimensions

1. Prove that in two dimensions, the Riemann tensor has the form

$$R_{abcd} = \frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

where R is the Ricci scalar. *Hint: express the Ricci scalar R in terms of the one independent component of the Riemann tensor, say R_{1212} .*

2. Show that Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ vanishes automatically in two dimensions.

5. 2d de Sitter Revisited

Consider again the two-dimensional de Sitter metric

$$ds^2 = -du^2 + \cosh^2 u d\varphi^2$$

where $-\infty < u < \infty$ and $0 \leq \varphi \leq 2\pi$.

1. Using your results from question 1, compute the one independent component of the Riemann tensor, say $R_{u\varphi u\varphi}$.
2. Compute the Ricci scalar R and show that $R_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$.