General Relativity - Problem Sheet 3

Please return solutions

0. Schwarzschild Solution

Everyone should check that the Schwarzschild solution satisfies the vacuum field equations $R_{ab} = 0$. You should do this carefully, in your own time, and it is not neccessary to hand this calculation in.

1. Newtonian Limit of Geodesic Deviation

The relative acceleration of a one-parameter family of geodesics $x^a_s(\tau) = x^a(\tau, s)$ is determined by the geodesic deviation equation

$$\frac{D^2 X^d}{D\tau^2} = -R_{abc}{}^d V^a X^b V^d$$

where

$$V^a = \frac{dx^a}{d\tau} \qquad X^a = \frac{dx^b}{ds}$$

and $D/D\tau = V^a \nabla_a$ is the covariant derivative in the direction of V^a.

In the Newtonian limit, show that this reduces to

$$\frac{\partial^2 X^i}{\partial t^2} = -\sum_j \frac{\partial^2 \phi}{\partial x^i \partial x^j} X^j$$

where ϕ is the gravitational potential.

2. Circular Orbits

The lagrangian for affinely parametrized geodesics in the Schwarzschild solution is

$$L = -(1 - 2M/r)\dot{t}^2 + (1 - 2M/r)^{-1}\dot{r}^2 + r^2 (\, \dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \,)$$

Show that

$$E = (1 - 2M/r)\dot{t} \qquad J = r^2 \sin^2\theta \, \dot{\phi}$$

are conserved. What symmetries do they correspond to? Explain why the lagrangian itself is conserved and why $L = -1$ for time-like geodesics parametrized by the proper time. Show that without loss of generality, you can always restrict attention to time-like geodesics lying in the equatorial plane, $\theta = \pi/2$. Now consider a circular orbit in the equatorial plane at coordinate distance R. Show that this is a time-like geodesic if $R > 3M$ and find the conserved quantities (E, J).

3. Kepler's 3rd Law

What is the physical meaning of the coordinate time t?

Show that the proper-time τ of an observer on a circular orbit at radius R and the coordinate time t are related by

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{1 - 3M/R}\,.$$

Using this result, show that

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{M}{R^3}$$

for a circular orbit at radius R.

4. Stability of Circular Orbits

Consider time-like geodesic that is a small perturbation from a circular orbit at radius R in the equatorial plane

$$r(\tau) = R + \varepsilon(\tau)\,.$$

Show that the perturbation must solve an equation of the form

$$\ddot{\varepsilon} + f(R)\varepsilon = 0$$

and find the function $f(R)$. Plot this function, showing clearly its asymptote and intercept. Hence, re-derive the fact that circular orbits only exist if $R > 3M$ and show that these orbits are stable if $R > 6M$.

5. Meaning of E

Consider a stationary observer at radius R in the Schwarzschild geometry and a massive test particle moving on a time-like geodesic $x^a(\tau)$ that intersect at some point P. Show that the stationary observer measures the energy per-unit-rest mass of the test particle to be

$$\sqrt{1 - \frac{2M}{R}}\;\dot{t}\,.$$

Let the test particle and the stationary observer have relative velocity v at the point P. Explain why

$$\gamma(v) = \sqrt{1 - \frac{2M}{R}}\,\dot{t}\,.$$

Now derive an expression for the conserved quantity E. Expanding this expression for large distances ($R \gg 2M$) and small velocities ($v \ll 1$), show that E is approximately the sum of the rest mass, kinetic energy and potential energy.

This is an approximate characterization of the conserved quantity E. To find the precise meaning, suppose the stationary observer converts the energy measured into a photon and sends it out to another stationary observer at $r \to \infty$. What is the energy of the photon measured by a stationary observer at infinity?

6. Acceleration

Observers not freely falling experience acceleration forces. This is encoded in the acceleration 4-vector $A^a = V^b \nabla_b V^a$ where V^a is the 4-velocity. This measures the failure of the corresponding curve to be a geodesic.

Consider a stationary observer at radius R in the Schwarzschild geometry and show that his acceleration four-vector is

$$A^a = (0, -m/R^2, 0, 0)\,.$$

You will need to compute the Christoffel symbol Γ^r_{tt} for the Schwarzschild metric from the r-equation of motion. Now compute the proper acceleration $a = (g_{ab}A^a A^b)^{1/2}$. Show that this agrees with the Newtonian expectation for $r \gg 2M$ and that stationary observers can only exists for $R > 2M$.