General Relativity - Problem Sheet 4

Please return solutions

1. Capture by a Black Hole

For geodesics in the Schwarzschild solution

$$\frac{E^2 - \kappa}{2} = \frac{1}{2}\dot{r}^2 + V(r)$$

where

$$V(r) = -\frac{\kappa m}{r} + \frac{J^2}{2r^2} - \frac{mJ^2}{r^3}$$

with $\kappa = 0$ for null geodesics and $\kappa = 1$ for time-like geodesics

In this question we are interested in when incoming geodesics will be captured by the black hole. For such problems it is convenient to define the 'impact parameter' b by

$$b \equiv \frac{J}{\sqrt{E^2 - \kappa}} \, .$$

a) Massless Particle

First consider an incoming null geodesic. Show that a massless particle is captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the capture cross-section $\sigma \equiv \pi b_c^2$ is

$$\sigma = 27\pi M^2.$$

a) Non-relativistic Massive Particle

Now consider an incoming time-like geodesic. We will assume that the massive particle starts at $r \to \infty$ with non-relativistic velocity $v \ll 1$ measured by a stationary observer. Explain why

$$b = \frac{J}{v} + \mathcal{O}(v)$$

and draw a diagram explaining the physical significance of the impact parameter in this case.

Show that the massive particle will be captured by the black hole if the impact parameter b is smaller than a critical value b_c. Show that the capture cross-section $\sigma \equiv \pi b_c^2$ is approximately

$$\sigma = 16\pi M^2/v^2.$$

2) Spatial Geometry

The constraints of homogeneity and isotropy determine the metric of the universe to have one of the three forms

$$ds^2 = -dt^2 + a(t)^2 \begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2) \\ d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2) \end{cases}$$

where the function $a(t)$ is known as the 'scale factor'.

In each case, find a coordinate transformation that brings the metric into the form

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \right)$$

with $k = +1, 0, -1$.

3) Einstein's Equations in Cosmology

You do not have to hand in this question, but you should perform the computations carefully in your own time.

Using the cosmological metric from question 2) in coordinates (τ, r, θ, ϕ) show that Einstein's equation in the presence of a perfect fluid imply

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P)\,.$$

Multiply the first equation by a^2, differentiate with respect to τ and eliminate a'' using the second equation to show that

$$\rho' + 3\frac{a'}{a}(\rho + P) = 0\,.$$

Derive the same equation directly from the local conservation of energy and momentum $\nabla^a T_{ab} = 0$.

4) Dust-filled Universe

Consider the implicit solutions

$$\begin{aligned} &i) \qquad a(\eta) = C(1-\cos\eta) \qquad && \tau(\eta) = C(\eta - \sin\eta) \\ &ii) \qquad a(\eta) = C(\cosh\eta - 1) \qquad && \tau(\eta) = C(\sinh\eta - \eta) \end{aligned}$$

where C is a constant. Plot the scale factor $a(\tau)$ as a function of the coordinate τ in each case, indicating the appropriate range of the parameter η. Show that $i)$ and $ii)$ are cosmological solutions of Einstein's equations with $P = 0$ and $k = +1$ and -1 respectively.

5) Conformal Time

In a spatially flat universe ($k = 0$) the 'conformal time' between two events at coordinate times τ_1 and τ_2 is defined by the integral

$$\Delta\eta = \int_{\tau_1}^{\tau_2} \frac{d\tau}{a(\tau)} .$$

Write down the line element ds^2 using conformal time and explain how the conformal time is used to detect the presence of 'particle horizons'.

Consider a cosmological model with contributions from both pressureless matter and radiation. Explain why the density $\rho(a)$ has the form

$$\rho(a) = \frac{\rho_m}{a^3} + \frac{\rho_r}{a^4}$$

(Here we have normalized so that $a = 1$ at the present time $\tau = \tau_0$ and ρ_m and ρ_r are the present day densities of matter and radiation.)

Show that that this cosmological model has a past horizon but no future horizon. Show that the conformal time at present is given by

$$\int_0^{\tau_0} \frac{d\tau}{a(\tau)} = C\left(\sqrt{1+a_{\rm eq}} - \sqrt{a_{\rm eq}}\right)$$

where $a_{\rm eq} = \rho_r/\rho_m$ and C is a constant that you should find. What is the physical significance of $a_{\rm eq}$?

6) Cosmological Constant

A cosmological constant Λ modifies Einstein's equations as follows

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 8\pi T_{ab} .$$

Show that the cosmological constant is mathematically equivalent to a perfect fluid with density $\rho_\Lambda = \Lambda/8\pi$ and pressure $P_\Lambda = -\Lambda/8\pi$. Hence show that for cosmological solutions we have

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad \frac{a''}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \,.$$

In an expanding universe with contributions from pressureless matter, radiation and a cosmological constant, which contribution will dominate the energy density at a) early times and b) late times? Consider a universe with a positive cosmological constant: how does the scalar factor $a(\tau)$ behave at late times? Does it have a future horizon?