FINAL EXAM

TIME: 3 HOURS
INSTRUCTION: ATTEMPT ALL QUESTIONS.

Question 1: (50 marks a=10 marks, b - f= 20 marks each)

(a) Give at least least 3 (a list is fine) experimental or observational confirmations of general relativity.

Sol.

- Precession of Mercury
- Gravitational red-shifting (observable on earth or in WDs)
- Gravitational Lensing
- LIGO gravitational wave detection
- This isn't quite a GR test, but there's strong observational evidence for black holes.

(b) One of the great triumphs of modern cosmology is the determination that the universe is comprised of approximately 30% ordinary and dark matter (with $w \simeq 0$), and 70% dark energy (with $w = -1$). Assume for simplicity that the density of matter and dark energy are currently equal (they're not, see above).

What happens to the ratio of densities of DE/Matter once the universe increases in scale (a) by a factor of 2?

Sol.

Matter scales as a^{-3}, while DE is a constant. Thus, if the universe doubles, the ratio $\boxed{DE/Mat = 8}$.

(c) You are dangled at radial coordinate, r above a black hole of mass, M. How far above the black hole (in units of M) must you be held so that your clock runs exactly half as fast as a starship some distance away?

Sol.

The ratio is:

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2M}{r}} = \frac{1}{2}$$

or

$$1 - \frac{2M}{r} = \frac{1}{4}$$

or

$$\frac{2M}{r} = \frac{3}{4}$$

yielding:

$$\boxed{r = \frac{8}{3}M}$$

(d) In a sentence or two, describe the Equivalence Principle (any of them). For a point of extra credit, explain the distinction between the weak equivalence principle, and what your book describes as the Einstein Equivalence Principle.

Sol.

- Weak: Freely falling particles move on timelike geodesics of the spacetime.
- Einstein: Any local physical experiment not involving gravity will have the same result if performed in a freely falling inertial frame as if it were performed in the flat spacetime of special relativity.

(e) Using only a pencil, straight-edge and a protractor (and not even necessarily any equations), describe how you might determine whether you are currently on a curved surface, and if so, how you might figure out the radius of curvature.

Sol.

Any version of this will generally involve going around a simple closed polygon (triangle or square) and tracing the path. Once you're done, compute the interior angles. If they deviate from expected (180 degrees for a triangle, 360 for a square, etc.) then you're on a curved surface. If the deviation is of order unity, then you've traversed a distance comparable to the radius of curvature.

(f) Consider a planet in a circular orbit in the equatorial plane of a central star (the Schwarzschild metric). Compute a relationship between the reduced energy, $\tilde{E} = -U_0$ as a function of the reduced angular momentum, $\tilde{L} = -p_\phi$, as a function of r.

What is the limit for $r \gg 2M$, including only the most significant term that include $\tilde{L}$?

Sol.

We have:

$$g^{00}(U_0)^2 + g^{22}(U_1)^2 = -1$$

so

$$\begin{aligned}\tilde{E}^2 &= \frac{(-1 - g^{22}\tilde{L}^2)}{g^{00}} \\ &= \boxed{\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)}\end{aligned}$$

BTW, if you looked at the equation sheet, this is simply the solution for $dr/d\tau = 0$. That's fine, too.

Expanding out, and dropping the fasted dropping term in r^{-3}, we get:

$$\boxed{\tilde{E}^2 = 1 - \frac{2M}{r} + \frac{\tilde{L}^2}{r^3}}$$

Question 2: (40 Marks)

Be aware that though this appears to be a single *very* long problem, it's actually 2 separate parts in disguise. In other words, if you have a tough time on parts a,b,c, and h, the others can be done independently. In other words, don't lose heart!

Consider a metric of the form:

$$g_{\mu\nu} = \begin{pmatrix} -f(y) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where "y" in this context is specifically the y-coordinate, and $f(y)$ is some as-yet unknown function. At the origin:

$$f(0) = 1$$

I'd also like you (for clarity) to use $f' \equiv df/dy$, $f'' \equiv d^2f/dy^2$ in your answers below.

(a) What are the conserved dynamic quantities for the metric?

Sol.

The metric isn't dependent upon t, x or z, and thus:

$$\boxed{p_0, p_1, p_3}$$

are all conserved.

(b) A particle moves in the metric starting *at the origin* (where the metric is locally Minkowskian), but with an initial 4-momentum in the **x direction** only (also, in time, if you need that reminder).

Some later time, it's at the coordinates $\mathcal{P} = (x, y)$. In terms of the final position and the initial 4-momentum (above) what is the p^2 component of the 4-momentum at $\mathcal{P}$?

First, note that initially:

$$-E^2 + (p^x)^2 = -m^2$$

and thus:

$$p_0 = -E$$

We immediately get:

$$p^0 = g^{00} p_0 = \frac{E}{f(y)}$$

Because the conservation of p_1 and mass, the $p \cdot p$ term simplifies significantly.

Thus:

$$\boxed{(p^2)^2 = \frac{E^2}{f(y)} - E^2}$$

(c) Using the result from the previous part, compute $\frac{dp^2}{d\tau}$:

Sol.

First, we get:

$$2p^2\frac{dp^2}{d\tau} = -\frac{E^2}{f(y)^2}f'\frac{dy}{d\tau}$$

But, $dy/d\tau = p^2/m$, so:

$$\boxed{\frac{dp^2}{d\tau} = -\frac{E^2}{f(y)^2}\frac{f'}{2m}}$$

(d) Compute all non-zero components of the Christoffel symbols of the metric.

Sol.

This is relatively straightforward since the only non-zero derivative is:

$$g_{00,2} = -f'$$

so

$$\boxed{\Gamma^2_{00} = \frac{1}{2}f' \quad ; \quad \Gamma^0_{20} = \Gamma^0_{02} = \frac{1}{2}\frac{f'}{f}}$$

where again we've used $g^{00} = 1/f$.

(e) Compute all non-zero components of the Riemann tensor.

Sol.

The only non-zero terms are going to involve the coupling between the g_{00} and yy derivatives. Thus:

$$\begin{aligned} R^0{}_{202} &= \Gamma^0_{22,0} - \Gamma^0_{20,2} + \Gamma^0_{\sigma 0}\Gamma^\sigma_{22} - \Gamma^0_{\sigma 2}\Gamma^\sigma_{02} \\ &= 0 - \frac{1}{2}\frac{f''}{f} + \frac{1}{2}\frac{(f')^2}{f^2} + 0 - \frac{1}{4}\frac{(f')^2}{f^2} \end{aligned}$$

or

$$\boxed{R_{0202} = \frac{2f(y)f'' - (f')^2}{4f}}$$

with the relevant perturbations.

(f) Compute all non-zero components of the Einstein tensor (you may need to do a bit of work with the Ricci tensor and scalar first).

Sol.

We quickly get:

$$R_{00} = R^2{}_{020} = \frac{2f(y)f'' - (f')^2}{4f}$$

and

$$R_{22} = R^0{}_{202} = \frac{(f')^2 - 2f(y)f'}{4f^2}$$

So:

$$R = R^0{}_0 + R^2{}_2 = \frac{(f')^2 - 2f(y)f'}{2f^2}$$

You'll note that: $R_{22} = \frac{1}{2}R$ and $R_{00} = -\frac{f}{2}R$. Combining everything, we get find that the 00 and 22 components of the einstein tensor disappear. This yields.

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}R \end{pmatrix}}$$

(g) As it happens, the actual functional form of f is:

$$f(y) = (1 + ky)^2$$

where k is some constant. This solution clearly satisfies the boundary condition($f(0) = 1$) of the metric.

What is the stress-energy tensor which gives rise to this solution?

Sol.

Note that:

$$f' = 2k(1 + ky)$$

and

$$f'' = 2k^2$$

so

$$R = \frac{2(2k^2)(1 + ky)^2 - (2k(1 + ky))^2}{(1 + ky)^2} = 0$$

so

$$\boxed{G_{\mu\nu} = 0}$$

Empty space!

(h) Using this form of the metric, describe the initial (non-relativistic) behavior of a particle *placed* near the origin.

Sol.

Near the origin:

$$\boxed{\Gamma^2_{00} = \frac{1}{2}(2k)(1 + ky) \simeq k}$$

There is a gravitational acceleration of strength, k in the $-y$ direction, which is totally consistent with the result in part c.