Convolution

Fourier Convolution

Outline

- Review linear imaging model
- Instrument response function vs Point spread function
- Convolution integrals
- Fourier Convolution
- Reciprocal space and the Modulation transfer function
- Optical transfer function
- Examples of convolutions
- Fourier filtering
- Deconvolution
- Example from imaging lab
- Optimal inverse filters and noise

Instrument Response Function

The Instrument Response Function is a conditional mapping, the form of the map depends on the point that is being mapped.

$$IRF(x,y\,|\,x_0,y_0) = S\{\delta(x-x_0)\delta(y-y_0)\}$$

This is often given the symbol h(r|r').

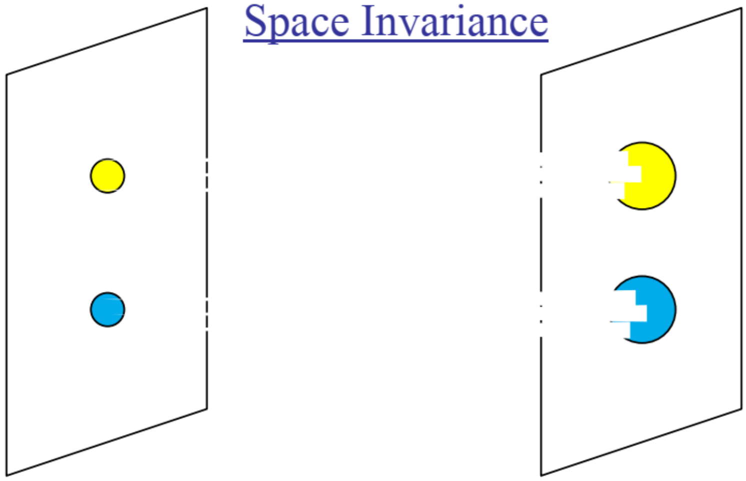
Of course we want the entire output from the whole object function,

$$E(x,y) = \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} I(x,y) S\{\delta(x-x_0)\delta(y-y_0)\} dx dy dx_0 dy_0$$

$$E(x,y) = \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} I(x,y) IRF(x,y\,|\,x_0,y_0) dx dy dx_0 dy_0$$

and so we need to know the IRF at all points.

Space Invariance



Now in addition to every point being mapped independently onto the detector, imaging that the form of the mapping does not vary over space (is independent of r_0). Such a mapping is called isoplantic. For this case the instrument response function is not conditional.

$$IRF(x,y\,|\,x_0,y_0)=PSF(x-x_0,y-y_0)$$

The Point Spread Function (PSF) is a spatially invariant approximation of the IRF.

Space Invariance

Since the Point Spread Function describes the same blurring over the entire sample,

$$IRF(x,y\,|\,x_0,y_0)\Rightarrow PSF(x-x_0,y-y_0)$$

The image may be described as a convolution,

$$E(x,y)=\iint_{-\infty}^{\infty} I(x_0,y_0)PSF(x-x_0,y-y_0)dx_0dy_0$$

or,

$$Image(x,y)=Object(x,y)\otimes PSF(x,y)+noise$$

Convolution Integrals

Let's look at some examples of convolution integrals,

$$f(x) = g(x) \otimes h(x) = \int_{-\infty}^{\infty} g(x')h(x-x')dx'$$

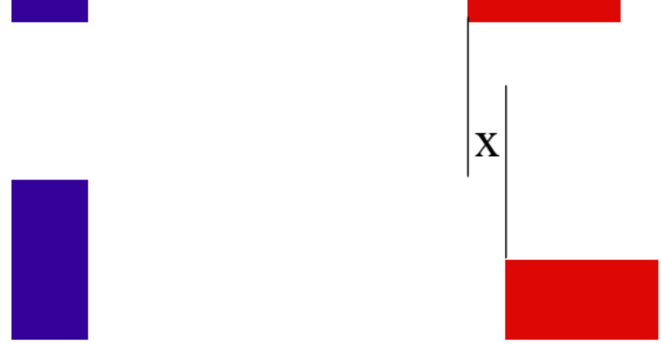
So there are four steps in calculating a convolution integral:

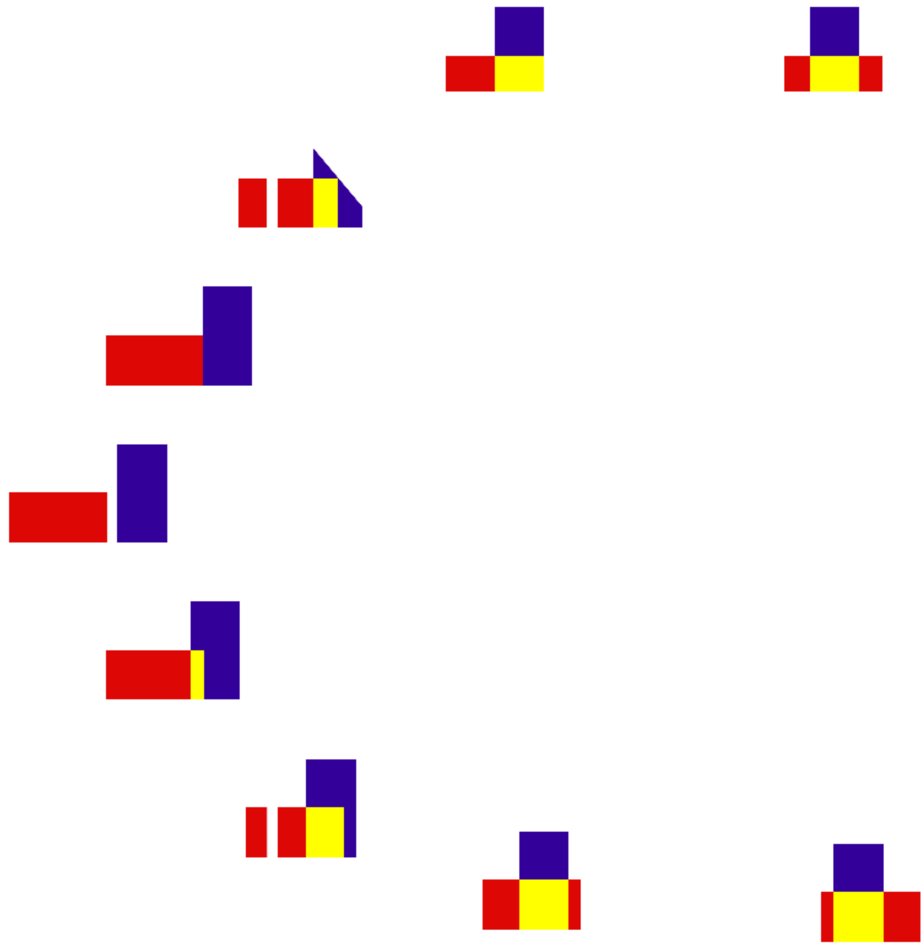
#1. Fold h(x') about the line x'=0
#2. Displace h(x') by x
#3. Multiply h(x-x') * g(x')
#4. Integrate

Consider the following two functions:

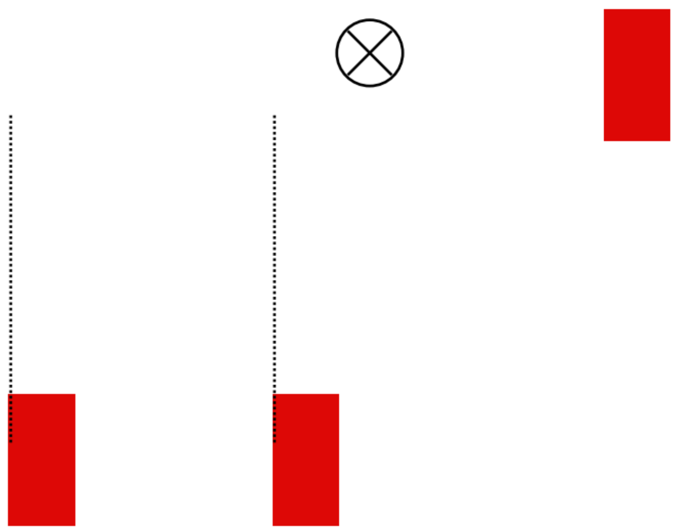
#1. Fold h(x') about the line x'=0

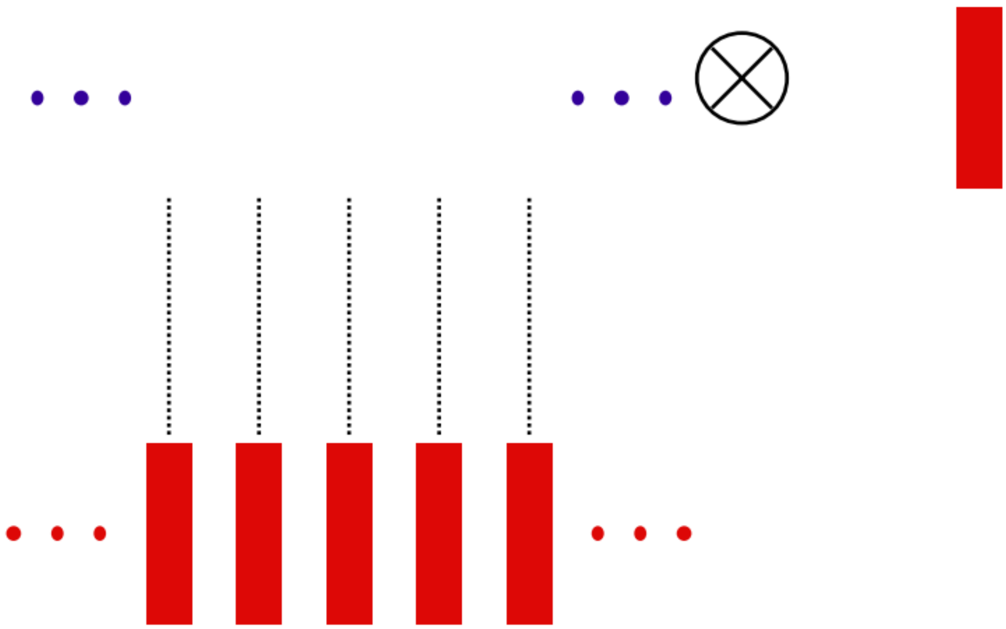
#2. Displace h(x') by x





Consider the following two functions:





Some Properties of the Convolution

commutative:

$$f \otimes g = g \otimes f$$

associative:

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h$$

multiple convolutions can be carried out in any order.

distributive:

$$f \otimes (g + h) = f \otimes g + f \otimes h$$

Convolution Integral

Recall that we defined the convolution integral as,

$$f \otimes g = \int_{-\infty}^{\infty} f(x) g(x' - x) dx$$

One of the most central results of Fourier Theory is the convolution theorem (also called the Wiener-Khitchine theorem.

$$\Im\{f \otimes g\} = F(k) \cdot G(k)$$

where,

$$f(x) \Leftrightarrow F(k)$$
$$g(x) \Leftrightarrow G(k)$$

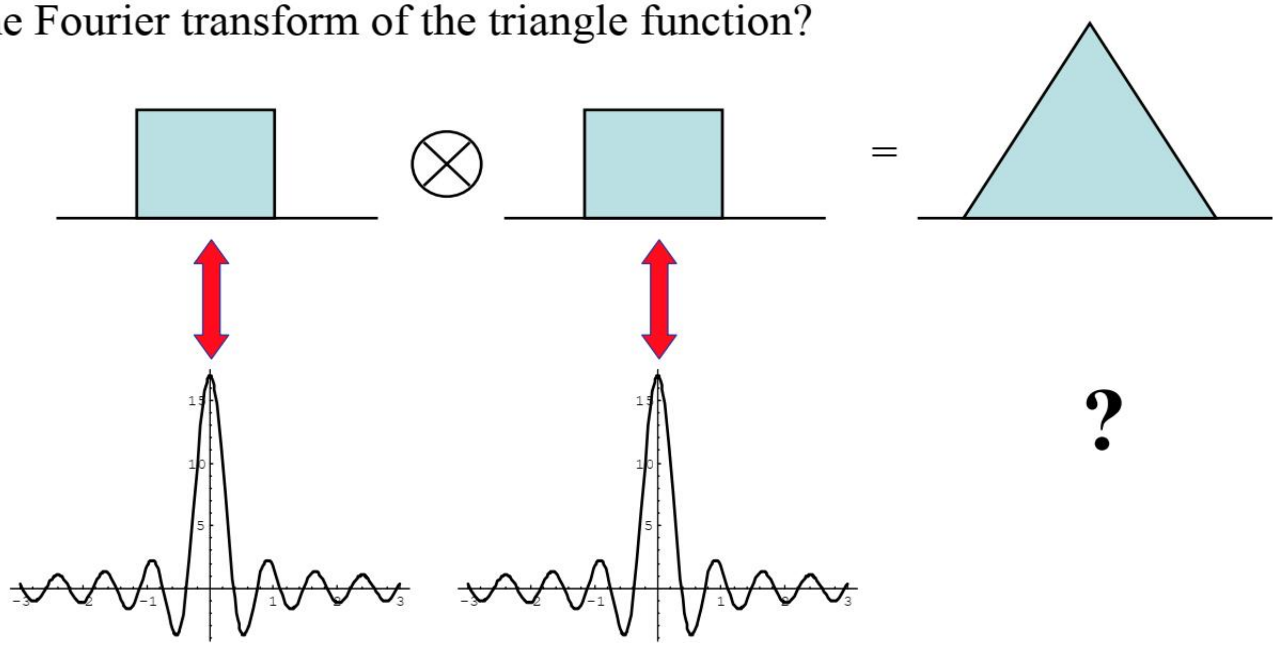
Convolution Theorem

$$\Im\{f \otimes g\} = F(k) \cdot G(k)$$

In other words, convolution in real space is equivalent to multiplication in reciprocal space.

Convolution Integral Example

We saw previously that the convolution of two top-hat functions (with the same widths) is a triangle function. Given this, what is the Fourier transform of the triangle function?



Proof of the Convolution Theorem

$$f \otimes g = \int_{-\infty}^{\infty} f(x) g(x'-x) dx$$

The inverse FT of f(x) is,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

and the FT of the shifted g(x), that is g(x'-x)

$$g(x'-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k') e^{ik'(x'-x)} dk'$$

So we can rewrite the convolution integral,

$$f \otimes g = \int_{-\infty}^{\infty} f(x) g(x' - x) dx$$

as,

$$f \otimes g = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} F(k) e^{ikx} dk \int_{-\infty}^{\infty} G(k') e^{ik'(x'-x)} dk'$$

change the order of integration and extract a delta function,

$$f \otimes g = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk F(k) \int_{-\infty}^{\infty} dk' G(k') e^{ik'x'} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix(k-k')} dx}_{\delta(k-k')}$$

$$f \otimes g = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk F(k) \int_{-\infty}^{\infty} dk' G(k') e^{ik'x'} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix(k-k')} dx}_{\delta(k-k')}$$

or,

$$f \otimes g = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk F(k) \int_{-\infty}^{\infty} dk' G(k') e^{ik'x'} \delta(k - k')$$

Integration over the delta function selects out the k'=k value.

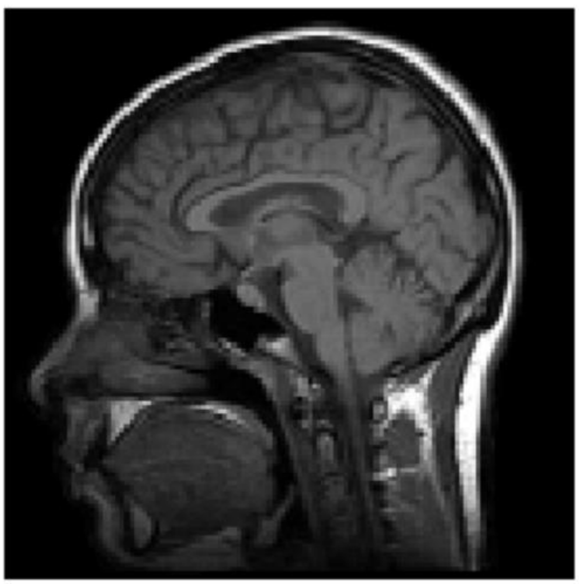
$$f \otimes g = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk F(k) G(k) e^{ikx'}$$

$$f \otimes g = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk F(k) G(k) e^{ikx'}$$

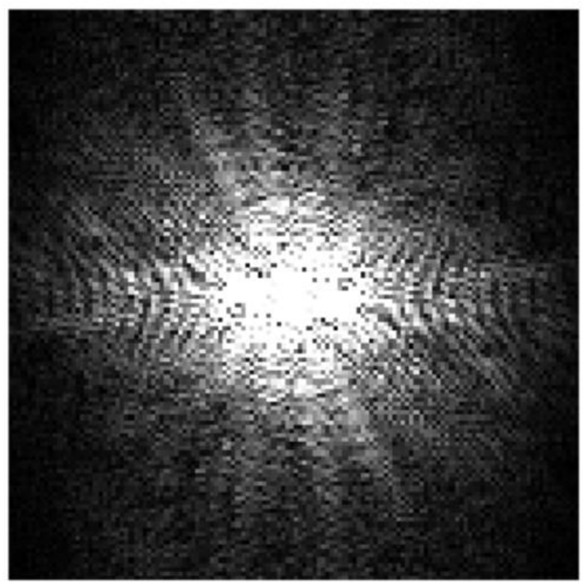
This is written as an inverse Fourier transformation. A Fourier transform of both sides yields the desired result.

$$\Im\{f \otimes g\} = F(k) \cdot G(k)$$

Reciprocal Space



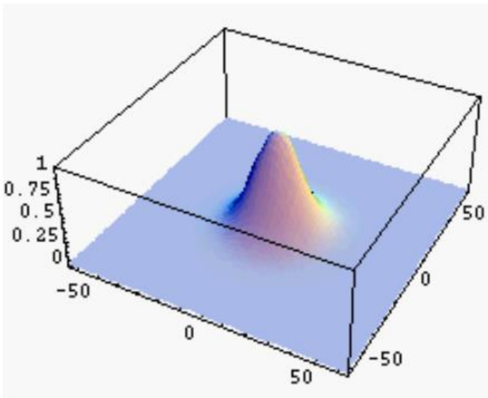
real space



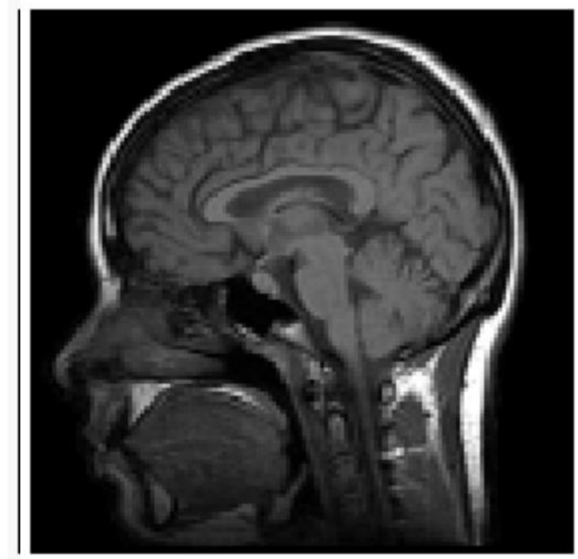
reciprocal space

Filtering

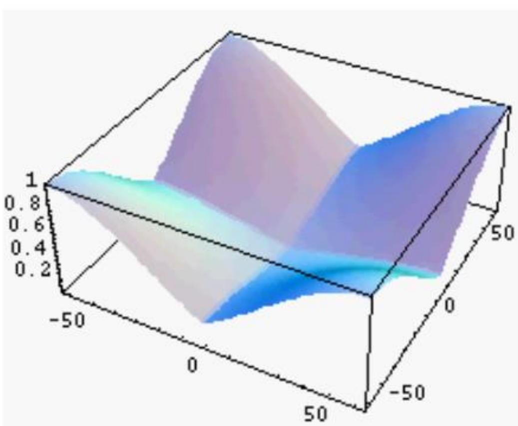
We can change the information content in the image by manipulating the information in reciprocal space.



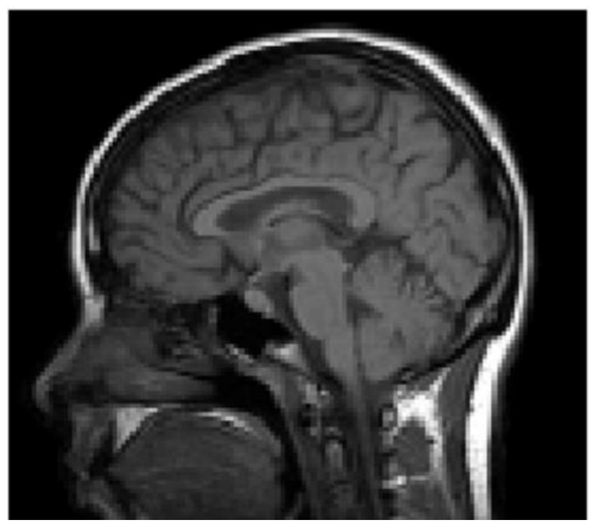
Weighting function in k-space.



We can also emphasis the high frequency components.



Weighting function in k-space.



Modulation transfer function

$$
\begin{array}{ccccccc}
i(x,y) & = & o(x,y)\otimes & PSF(x,y) & + & noise \\
\Updownarrow & & \Updownarrow & \Updownarrow & & \Updownarrow \\
\end{array}
$$

$$
I(k_x,k_y)=O(k_x,k_y)\cdot MTF(k_x,k_y)+\Im\{noise\}
$$

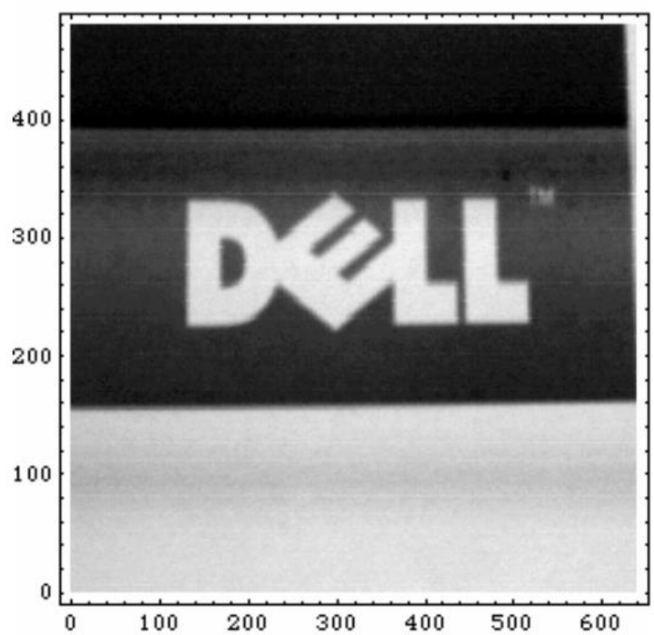
$$E(x,y)=\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} I(x,y)S\{\delta(x-x_0)\delta(y-y_0)\}dxdydx_0dy_0$$

$$E(x,y)=\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} I(x,y)IRF(x,y\,|\,x_0,y_0)dxdydx_0dy_0$$

Optics with lens

‡ Input bit mapped image

```
sharp = Import @"sharp . bmp"D;

Shallow @InputForm @sharp DD

Graphics@Raster@<< 4>>D, Rule@<< 2>>DD

s = sharp @@1 , 1 DD;

Dimensions @s D

8480, 640<
```

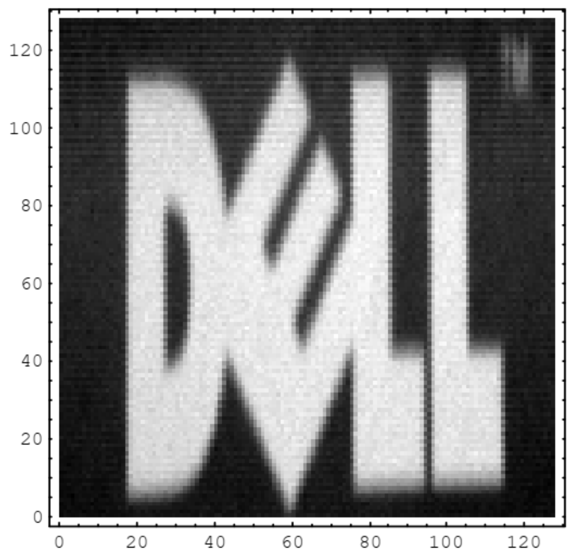



```
ListDensityPlot @s , 8PlotRange ÆAll , Mesh ÆFalse <D
```

Optics with lens

crop = Take @s , 8220 , 347 <, 864 , 572 , 4<D;

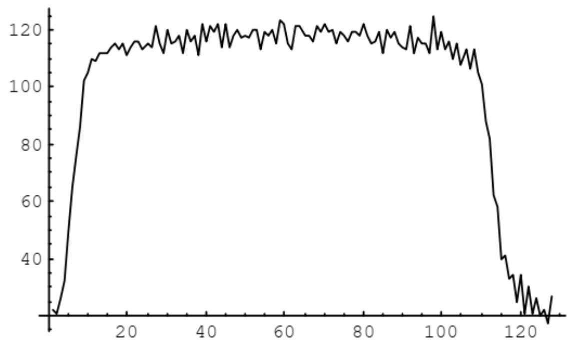
‡ look at artifact in vertical dimension



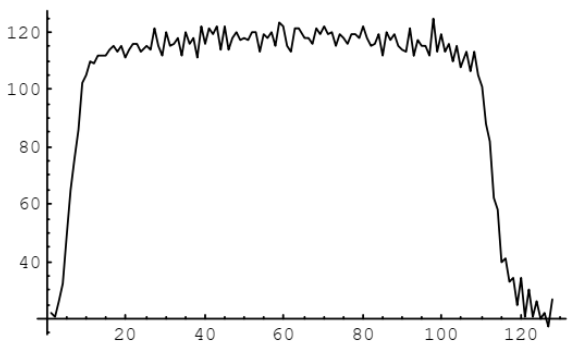
rot = Transpose @crop D;

line = rot @@20 DD;

ListPlot @line , 8PlotRange ÆAll , PlotJoined ÆTrue <D



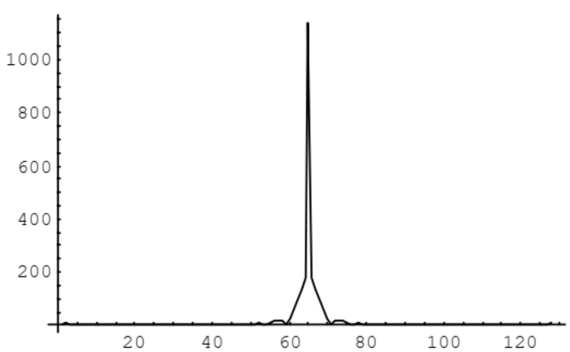
ÖGraphicsÖ



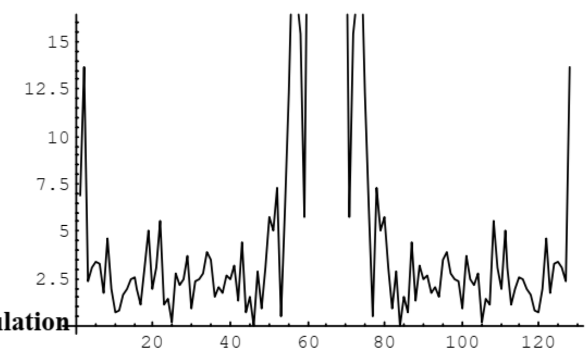
ü Fourier transform of vertical line to show modulation

ftline = Fourier @line D;

ListPlot @RotateLeft @Abs @ftline D, 64 D, 8PlotRange ÆAll , PlotJoined ÆTrue <D



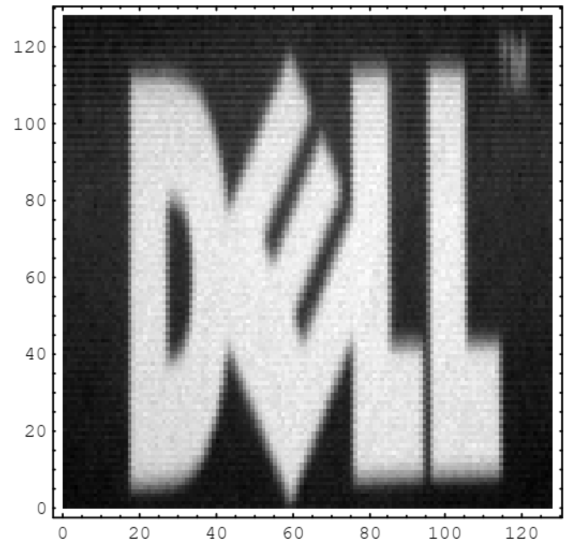
ListPlot @RotateLeft @Abs @ftline D, 64 D, 8PlotJoined Æ

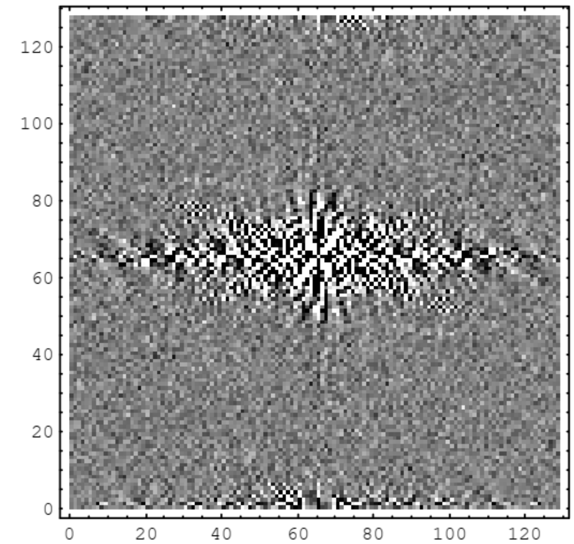


ÖGraphicsÖ

2D FT

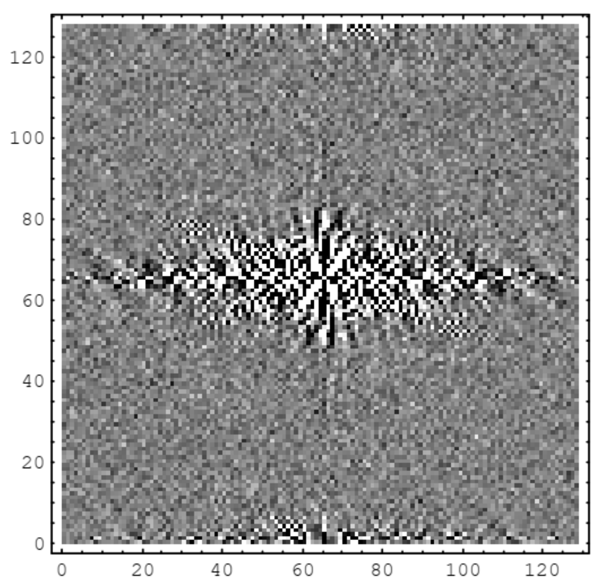
```
ftcrop = Fourier @crop D;
```

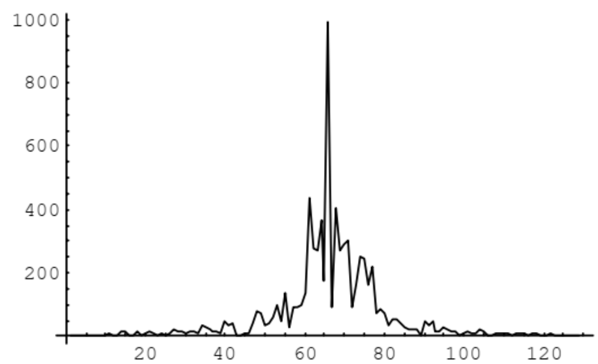


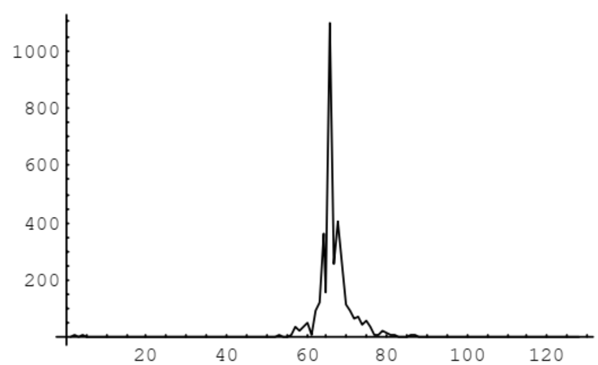


Optics with lens

Projections

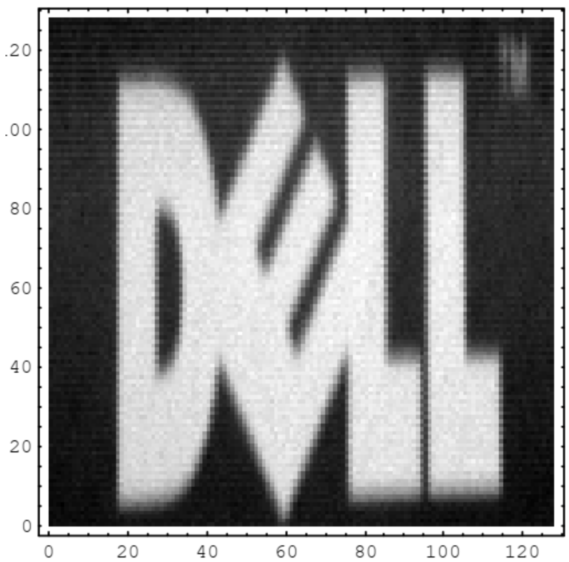


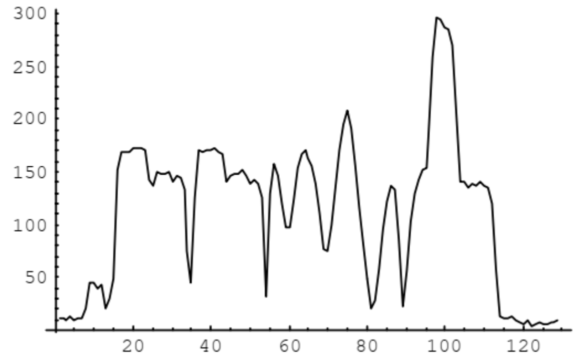


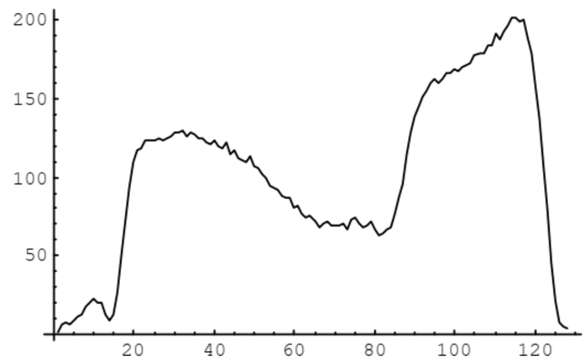


Projections

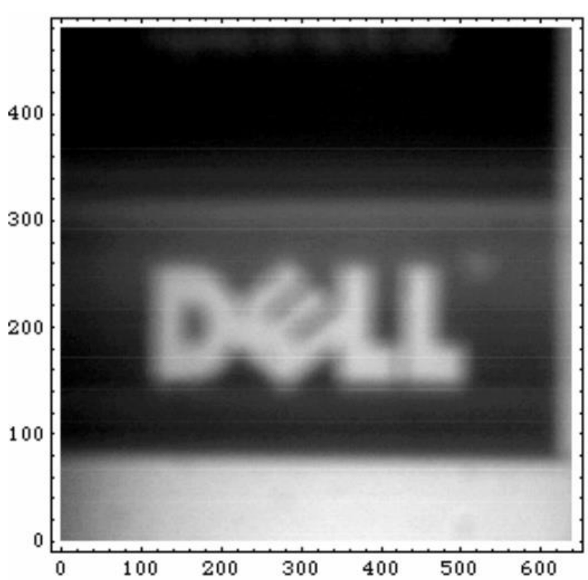
xprojection = Fourier @RotateLeft @rotft @@64 DD, 64 DD;

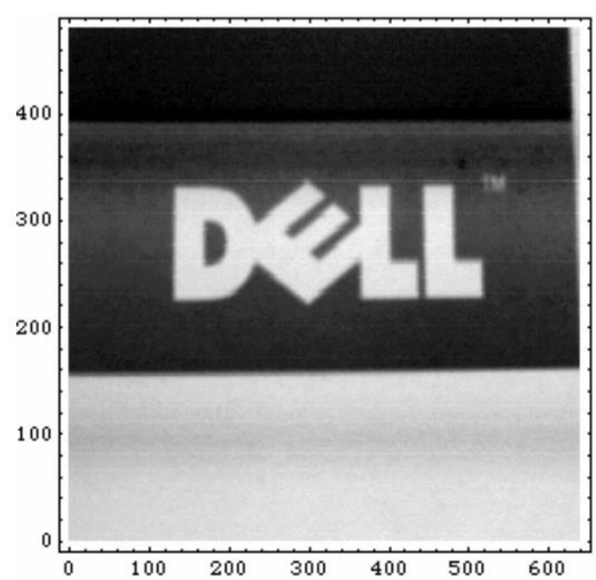


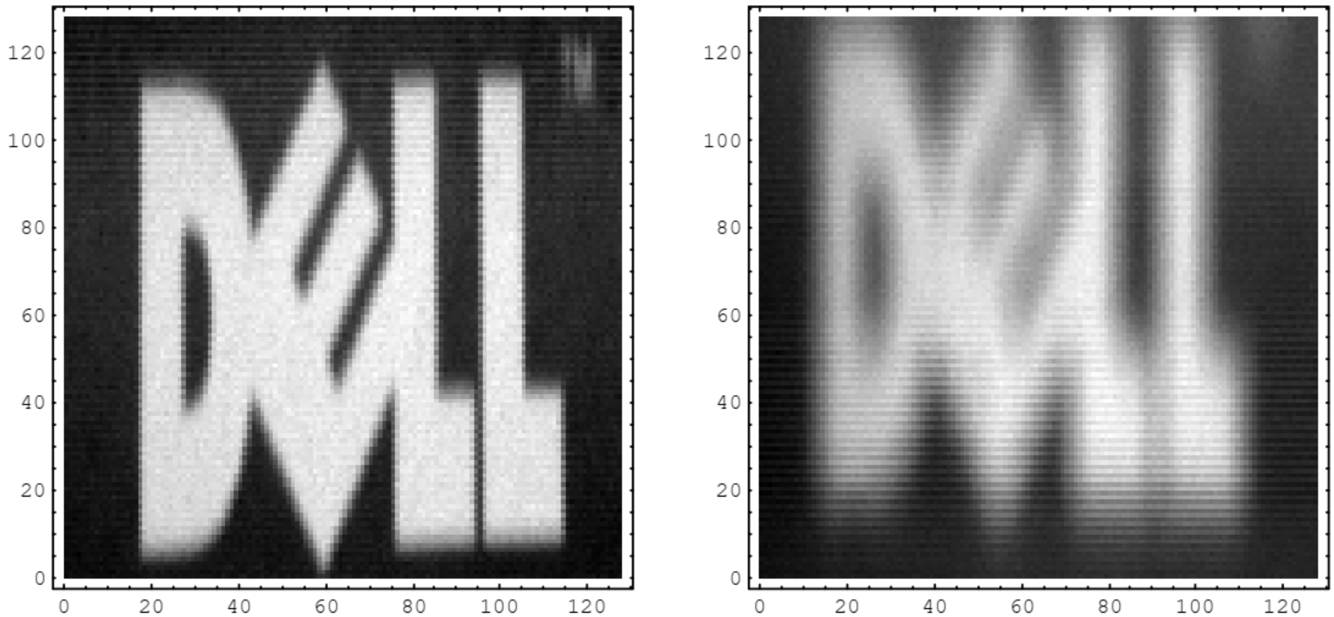




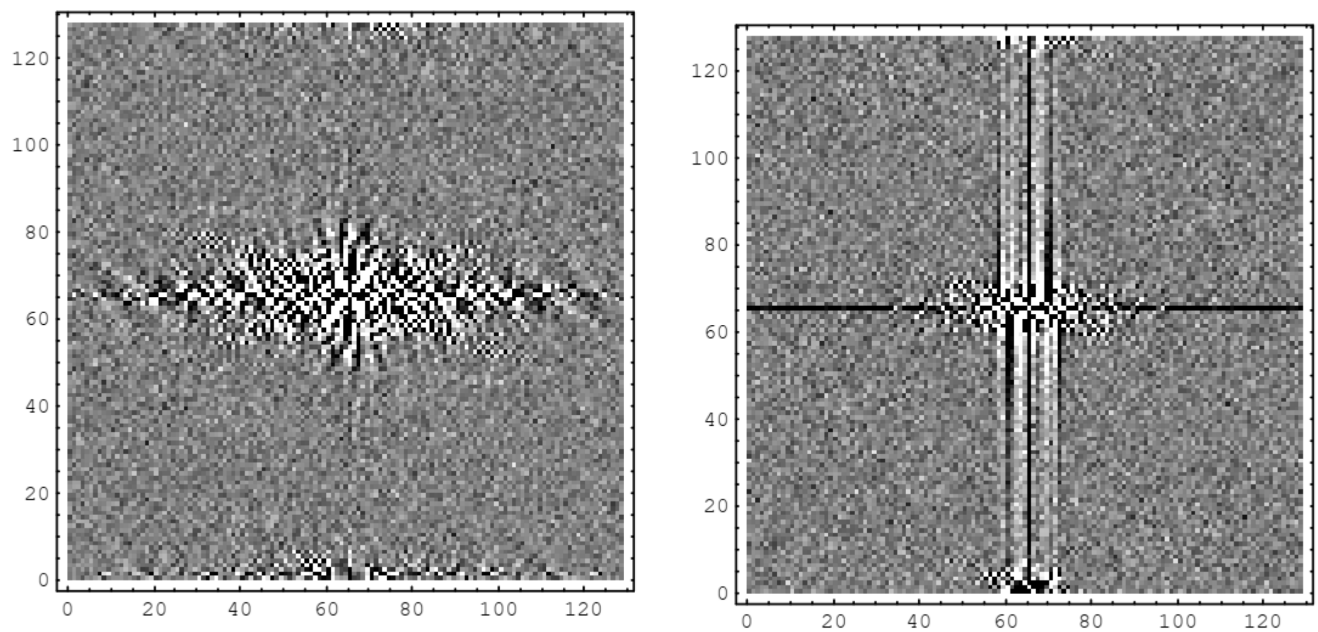
Optics with pinhole





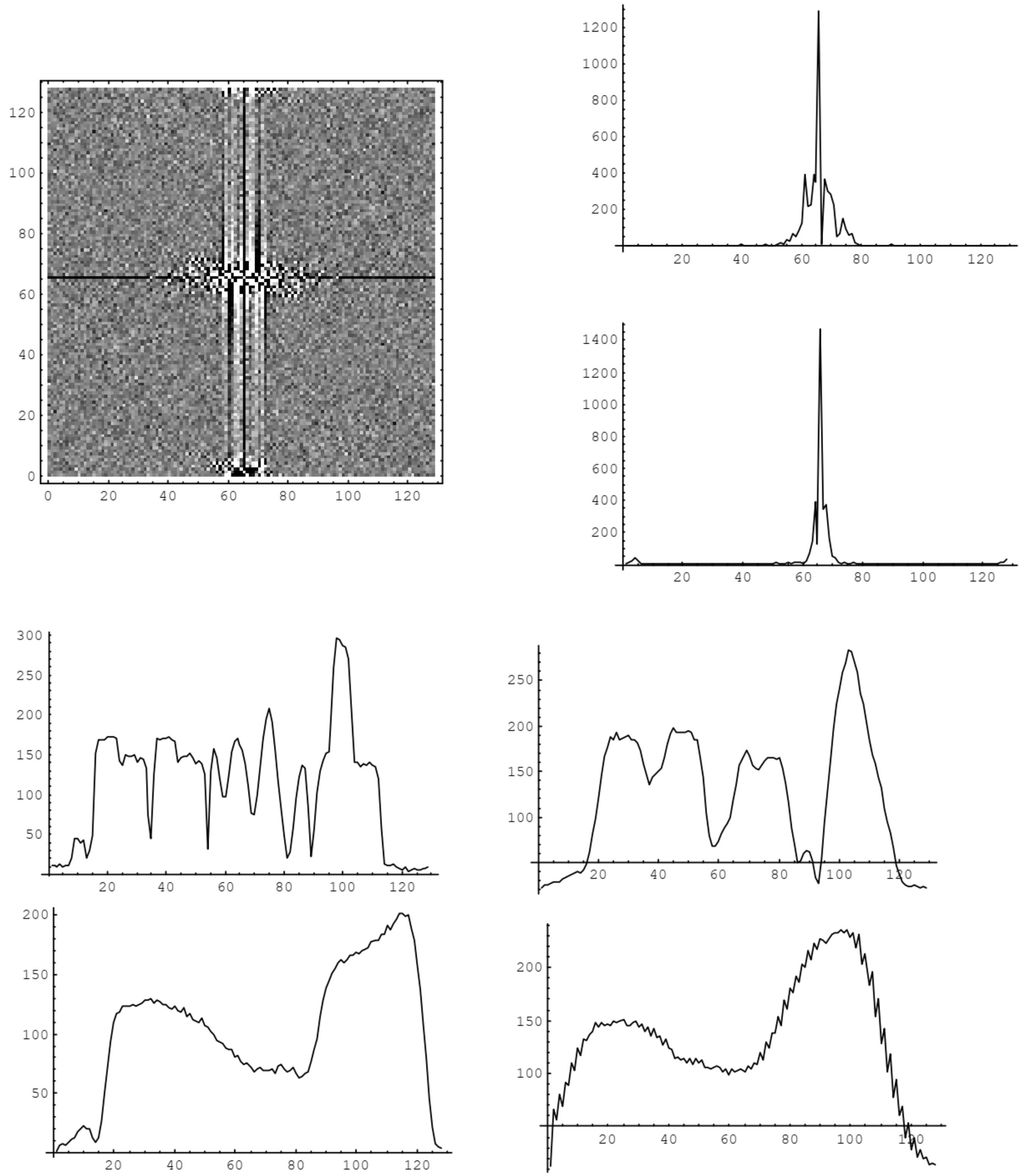


2D FT

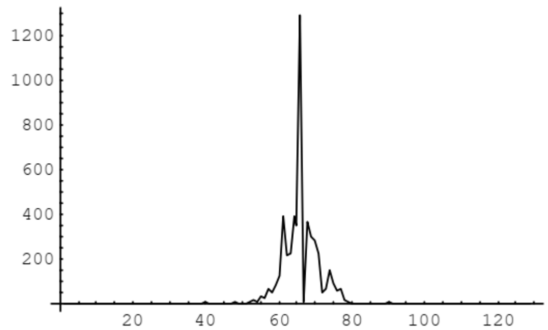


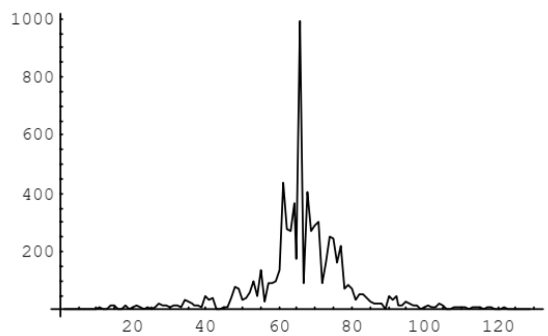
Optics with pinhole

Projections



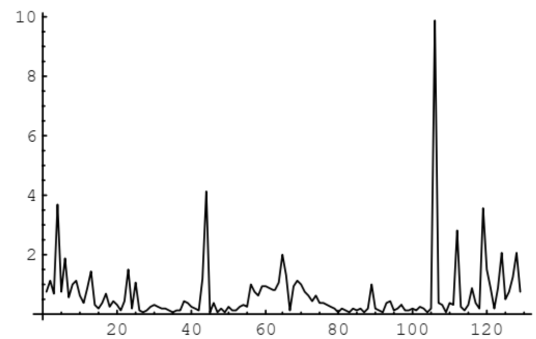
Deconvolution to determine MTF of Pinhole





MTF= Abs @pinhole Dê Abs @image D;

ListPlot @MTF, 8PlotRange ÆAll , PlotJoined ÆTrue

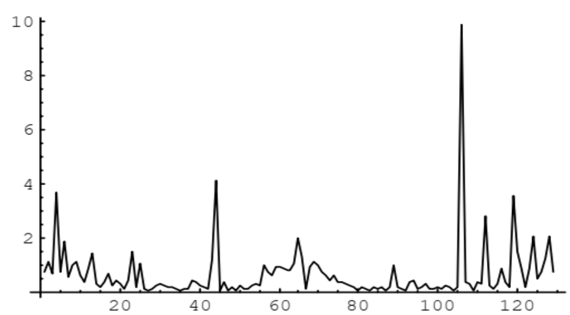


ÖGraphicsÖ

FT to determine PSF of Pinhole

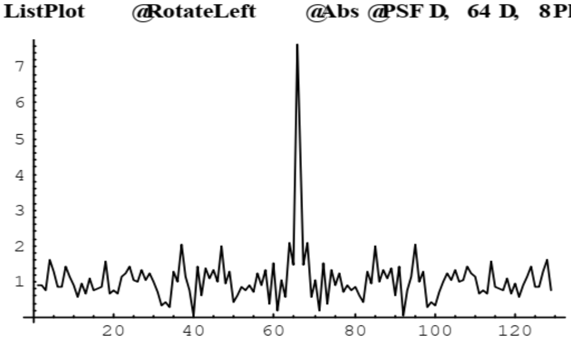
MTF= Abs @pinhole Dê Abs @image D;

ListPlot @MTF, 8PlotRange ÆAll , PlotJoined ÆTrue <D



ÖGraphicsÖ **PSF = Fourier @RotateLeft @MTF, 64 DD;**

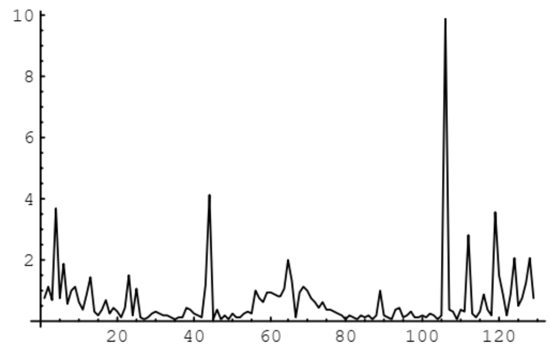
ListPlot @RotateLeft @Abs @PSF D, 64 D, 8PlotRange ÆAll , PlotJoined ÆTrue <D

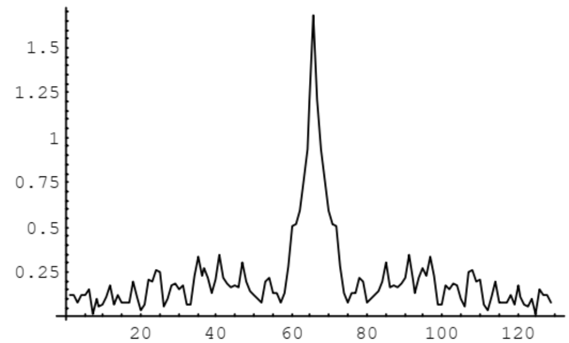


Filtered FT to determine PSF of Pinhole

MTF= Abs @pinhole Dê Abs @image D;

ListPlot @MTF, 8PlotRange ÆAll , PlotJoined ÆTrue <D





Filter = Table @Exp @- Abs @x - 64 Dê 15 D, 8x, 0, 128 <D êê N;

ListPlot @Filter , 8PlotRange ÆAll , PlotJoined ÆTrue <D