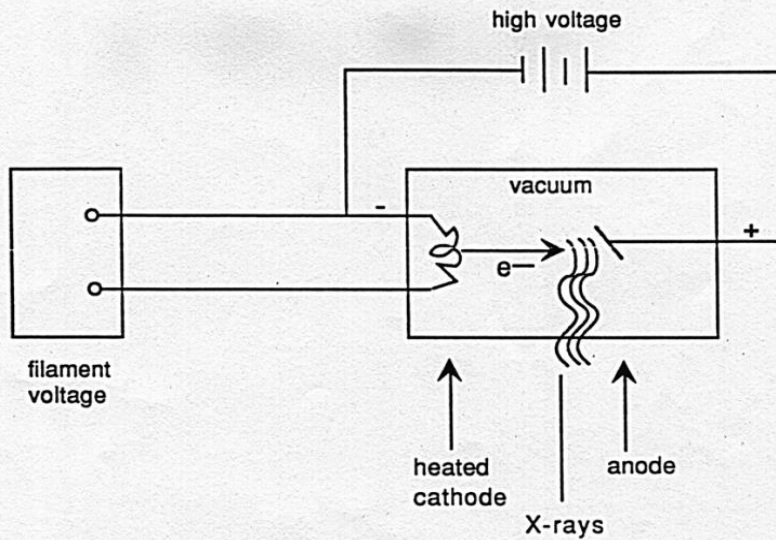


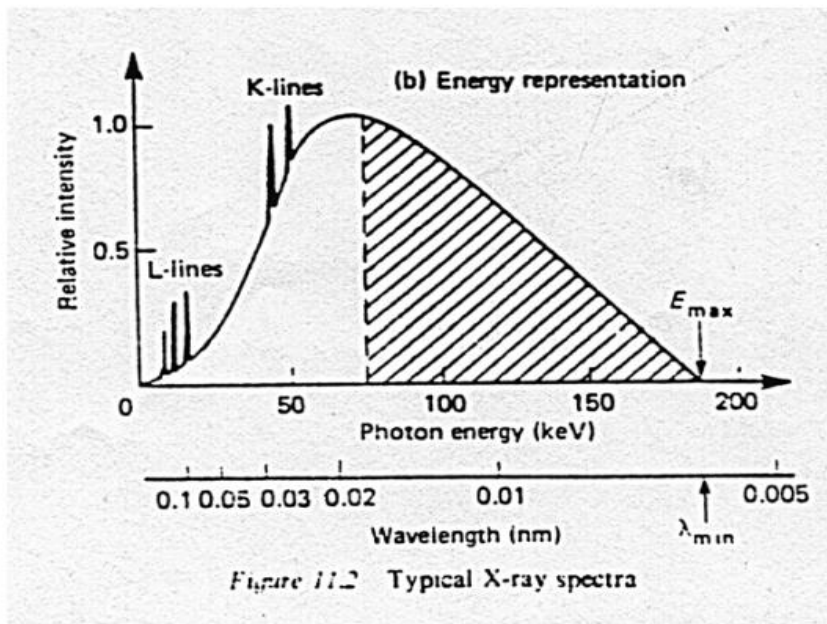
The Origins of X-Rays

X-rays are high energy ($> 1\text{keV}$) electromagnetic radiation. They are often produced by bombarding a metal target with high-speed electrons.



A heated cathode emits electrons by thermionic emission. These are accelerated to the anode and the target. The electrons lose about 99 percent of their energy in lowenergy collisions (producing mostly heat), and about one percent reappears as X-rays.

The X-Ray Spectrum



The X-Ray Spectrum (Changes in Voltage)

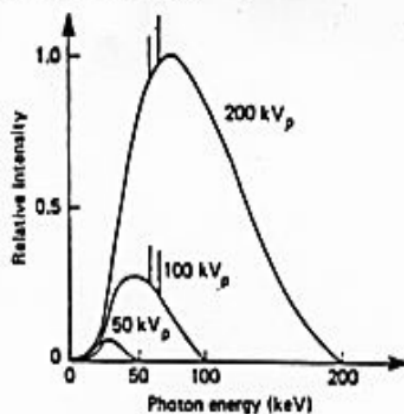
The continuous spectrum is from electrons decelerating rapidly in the target and transferring their energy to single photons, Bremsstrahlung.

$$E_{\max} = eV_p$$

$V_p \equiv$ peak voltage across the X-ray tube

The characteristic lines are a result of electrons ejecting orbital electrons from the innermost shells. When electrons from outer shells fall down to the level of the inner ejected electron, they emit a photon with an energy that is characteristic to the atomic transition.

Changes in the voltage:



When the voltage is increased:

- 1) $E_{\max} \propto V_p$
- 2) peak of continuous spectrum moves to higher energy
- 3) total output intensity $\propto V_p^2$
- 4) more characteristic lines may appear

The X-Ray Spectrum (Changes in Tube)

Changes in tube current:

(change in thermionic emission rate)

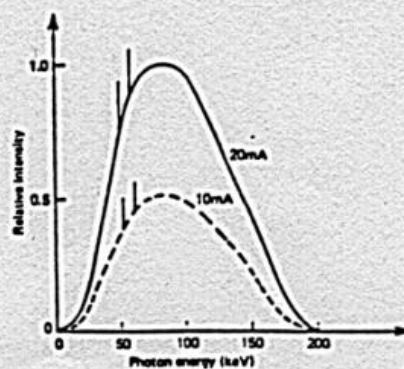


Figure 11.4 Effect of tube current on spectrum

- 1) shape remains the same
- 2) E_{\max} remains unchanged
- 3) total output intensity $\propto I$

The X-Ray Spectrum (Changes in Target Material)

Increase in Z :

1. Increase in X-ray intensity since greater mass and positive charge of the target nuclei increase the probability of X-ray emission total output intensity of Z
2. Characteristic lines shift to higher energy, K and L electrons are more strongly held
3. No change in E_{\max}

The X-Ray Spectrum

Filtrations typically one wishes to remove low-energy X-rays from the beam. This is accomplished by placing a sheet of metal in the path of the X-ray beam.

1. Changes the X-ray spectrum shape by removing low-energy electrons
2. Shifts the spectrum peak to higher energies
3. Reduces the overall X-ray output
4. Shifts E_{\min} to higher energies
5. No change in E_{\max} .

Beam Hardening

The beam from an X-ray source is not mono-energetic and the lower energy photons will be more attenuated than the higher energy ones.

Source Considerations In X-ray Imaging

Notice that as θ is reduced the loading efficiency increases, but the angular width of the beam decreases.

Width $\theta = 16^\circ$, the effective spot size is reduced

$$\therefore S(x,y) = \text{TopHat}[x] \otimes [\delta(x+1) + \delta(x-1)] \bullet \text{TopHat}[y]$$

What is the FT of $\delta(x,y)$? $\text{sinc}\left(\frac{k_x}{2}\right) \bullet \cos(k_x) \bullet \text{sinc}\left(\frac{k_y}{2}\right)$

Heel Effect

The true spot on an anode is inside the anode.

Why not use larger angles? Greater spot size.

What about X-ray spectrum vs. angle?

Scatter Analysis #1

The incremental density of the scattered photons generated in the plane at height z is: $dn_s(z) = n(z)\mu_s dz$

where μ_s is the linear attenuation coefficient for Compton scatter

and $n(z)$ is the number of photons read at z

$$n(z) = n_0 e^{-\mu z}; \quad \mu = \text{total linear attenuation coefficient}$$

Scatter Analysis #2

It is not enough to know the number of photons scattered, we also need to know how many are scattered towards the detector.

- at diagnostic energy ???, the fraction forward scattered, k

$$k = 0.52 + \frac{0.07 E (keV)}{80}$$

- the number that reaches the detector is

$$F(z) = k \underbrace{e^{-\mu(L-z)}}_{\substack{\text{those} \\ \text{attenuated} \\ \text{by remainder} \\ \text{of path}}} \underbrace{\frac{\Omega(z)}{2\pi}}_{\substack{\text{solid} \\ \text{angle}}}$$

- if only 1 scatter event per photon

$$n_s = \int n(z) \mu_s F(z) dz$$

Scatter Analysis #3

$$\Omega(z) = 2\pi \left[1 - \frac{L-z}{\sqrt{r^2 + (L-z)^2}} \right]$$

$$n_s = \int n_o e^{-\mu z} \mu_s k e^{-\mu(L-z)} \left[1 - \frac{L-z}{\sqrt{r^2 + (L-z)^2}} \right] dz$$

$$n_s = n_o \mu_s k e^{-\mu L} \underbrace{\int \left[1 - \frac{L-z}{\sqrt{r^2 + (L-z)^2}} \right] dz}_{L+r-\sqrt{L^2+r^2}}$$

Scatter Analysis #4

But this is not the entire picture, we know that there are multiple scatter events for individual photons.

The mean distance traveled along z for forward directed particles before a scatter event is: $1/2\mu_s$

$$T(\theta) = \frac{1}{S} \left\{ [(n+1)s - h \tan(\theta)] e^{-n\mu t / \sin(\theta)} + (h \tan(\theta) - nS) e^{-(n+1)\mu t / \sin(\theta)} \right\}$$

$$\tan^{-1}\left(\frac{nS}{h}\right) < \theta < \tan^{-1}\left[\frac{(n+1)S}{h}\right] \quad ; \quad n = 0 \rightarrow \infty$$

Poisson Density Function

As we have seen, X-rays are discrete photons.

The probability that exactly k photons will be emitted over a definite period in time is given by the Poisson density function.

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

where $\lambda \equiv$ the average number of photons
during the time interval of interest

A defining feature of the Poisson distribution is that the variance, σ^2 , (or the central second moment - width) is equal to the mean.

$$\sigma^2 = \left\{ \int \frac{k^2 \lambda^2 e^{-\lambda}}{k!} dk - \left[\int \frac{k \lambda e^{-\lambda}}{k!} dk \right]^2 \right\}^{1/2}$$

$$= \lambda \quad \text{In a Poisson process of mean } \lambda,$$

the variance is λ and the standard deviation is $\sqrt{\lambda}$.

The signal-to-noise of a measurement X-ray photons is then:

signal $\propto \lambda E_o$ where $\lambda =$ average # of photons

Noise \propto root mean square deviation from kE_o

standard deviation $\sigma = \sqrt{\lambda}$

$$\therefore \frac{S}{N} = \frac{\lambda E_o}{\sqrt{\lambda E_o}} = \sqrt{\lambda}$$

Consider the effect of an energy spectrum for the S/N.

$$\frac{S}{N} = \frac{\lambda_1 E_1 + \lambda_2 E_2}{\sqrt{\lambda_1 E_1} + \sqrt{\lambda_2 E_2}} \neq \sqrt{\lambda_1 + \lambda_2}$$

Detection efficiency generally goes as the stopping power, therefore lower for higher energy photons .

Types of Noise (Additive Noise)

Additive Noise - When the energy photons is low then there are many photons and they may be thought of as arriving continuously. There are virtually no statistical fluctuations in the arrival rate, only Johnson type noise added by the measurement system.

$$N = \sqrt{4kTRB}$$

where k = Boltzmann's Constant

R = resistance

B = bandwidth

Types of Noise (Quantum Noise)

Quantum noise (“counting” noise) - high energy per photon, therefore only a few photons are required but now since each photon can be detected individually and the counting rate is low, there are statistics associated with the arrival of the photon at the detector.

$$\frac{S}{N} = \frac{Nh\nu}{\underbrace{\sqrt{Nh\nu}}_{\text{quantum}} + \underbrace{4kT}_{\text{additive}}}$$

where $N \equiv$ number of photons per time element

$T \equiv$ temperature

power $S/N \therefore 4kT$ instead $\sqrt{4kT}$

B vanishes since per unit time

$Nh\nu \equiv$ signal intensity

| ν | $4kT/h\nu$ | |
|--------------|----------------------|--|
| 10^6 Hz | 2.5×10^7 | Radio waves |
| 10^{11} Hz | 2.5×10^2 | Microwaves |
| 10^{19} Hz | 2.5×10^{-6} | X-rays ($\lambda = 0.2 \text{ \AA}$) |

Photon Statistics

So for X-rays $\frac{S}{N} = \sqrt{N}$

So S/N depends on the counting statistics of photons reading the detector. Outline of proof that photons energy from a material continue to follow Poisson statistics. The emission of X-ray from a source follow Poisson statistics.

$$P_k = \frac{N_o^k e^{-N_o}}{k!}$$

\equiv probability in a given time interval of emitting k photons, where N_o is the average number emitted during each interval

Interactions of photons with matter is a binary process. They interact or not (ideal case), therefore it is a binomial process.

$$p^k = \text{probability of transmission} = e^{-\int \mu dz}$$

$$q^k = \text{probability of being stopped} = (1 - e^{-\int \mu dz})$$

Put these two together to find the probability of sending k photons through an object, Q(k).

$$Q(k) = P(k) \binom{k}{k} p^k + P(k+1) \binom{k+1}{k} p^k q^1 + P(k+2) \binom{k+2}{k} p^k q^2 + \dots$$

$\binom{l}{m} = \frac{l!}{m!(l-m)!} = \text{binomial coefficient}$

probability of photon source generating k+n photons
 # of permutations of such an event
 probability of k photons being transmitted
 probability of n photons being transmitted

Poisson Distribution

$$P(k+n) \binom{k+n}{k} p^k q^n = \frac{N_o^{k+n} e^{-N_o}}{(k+n)!} \frac{(k+n)!}{k!n!} p^k q^n$$

$$= e^{-N_o} \frac{(pN_o)^k}{k!} \frac{(qN_o)^n}{n!}$$

$$\therefore Q(k) = e^{-N_o} \frac{(pN_o)^k}{k!} \underbrace{\sum_{n=0}^{\infty} \frac{(qN_o)^n}{n!}}_{e^{qN_o}}$$

Notice $e^{-N_o} e^{qN_o} = e^{-pN_o}$ when $q = 1 - p$

$$\therefore Q(k) = \frac{e^{-pN_o} (pN_o)^k}{k!}$$

This is a Poisson process of rate pN_o

Poisson Distribution

Photons emerging from an attenuating object continue to follow a Poisson distribution, however with the rate scaled by the attenuation.

$$p = e^{-\int \mu dz}$$

Note: True for an all or nothing process. The photons emitted have a mean value.

$$N = N_o e^{-\int \mu dz}$$

$$\frac{S}{N} = \frac{\Delta N}{\sqrt{N}} = C\sqrt{N}$$

$\Delta N \equiv$ variation in number per element
defining the structure of interest

$\sqrt{N} \equiv$ noise . standard deviation in
number of photons

Clearly S/N is increased at the cost of dose.