

## Equations

- Slice Selection :  $ideal \ O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z)\delta(z-z_s)dz$
- Better Approximation :  $O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z)TopHat\left(\frac{z-z_s}{\frac{1}{2}\Delta z}\right)dz$
- Radon Transform :  $P(\theta,z) = \int \int O(x,y)\delta(x\cos(\theta)+y\sin(\theta)+z)$
- Fourier Transform :  $\tilde{g}(k) = \int G(x)e^{-ikx} dx$
- Central Slice Theorem :  $k_z = k_x \cos(\theta) + k_y \sin(\theta)$   
 $z = x \cos(\theta) + y \sin(\theta)$
- Back Projection :  $O_r(x,y) = \frac{1}{\pi} \int_0^{\pi} P(\theta,z)d\theta$
- Map :  $\tilde{O}_r(k_x, k_y) = \tilde{P}(\theta, k_z)$   
where  $k_z = k_x \cos(\theta) + k_y \sin(\theta)$

## More Organized Proof of The Central Slice Theorem

The PSF associated with the simple Back projection is :

$$PSF|_{BF} = \frac{1}{r}$$

$$\therefore O_r(x,y) = O(x,y) \otimes \frac{1}{\sqrt{x^2+y^2}}$$

where  $O_r(x,y) = B\{P(\alpha,z)\}$

$$\text{and } B = \frac{1}{\pi} \int_0^{\pi} P(\alpha, x\cos(\alpha) + y\sin(\alpha))d\alpha$$

$$\therefore \underbrace{O_r(x,y)}_{\updownarrow} = \underbrace{O(x,y)}_{\updownarrow} \otimes \underbrace{\frac{1}{\sqrt{x^2+y^2}}}_{\updownarrow}$$

$$\tilde{O}_r(k_x, k_y) = \tilde{O}(k_x, k_y) \cdot \frac{1}{|k|}$$

so

$$\tilde{O}(k_x, k_y) = |k| \tilde{O}_r(k_x, k_y)$$

$$1. \quad P(\alpha, z) = \iint O(x, y) \delta(x \cos(\alpha) + y \sin(\alpha) - z) dx dy$$

2. Equate the  $z$ -axis with a tilted reference frame

$$x' \parallel z, \quad y' \perp z$$

$$\therefore \quad x = x' \cos(\alpha) - y' \sin(\alpha)$$

$$y = x' \sin(\alpha) + y' \cos(\alpha)$$

and

$$x' = x \cos(\alpha) + y \sin(\alpha)$$

3. Substitute #2 into #1 and change integral to  $dx' dy'$  (still over all space)

$$P(\alpha, z) = \iint O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) \delta(x' - z) dx' dy'$$

4. Integrate along  $x'$  and note that  $z$  is only a point along the  $x'$  axis.

$$P(\alpha, x') = \iint O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) dy'$$

5. Fourier Transform along  $x'$

$$\tilde{p}(\alpha, k_x) = \iint O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) e^{-ix' k_x} dx' dy'$$

6. Transform back to the  $(x, y)$  coordinate system

$$\tilde{p}(\alpha, k_x) = \iint O(x, y) e^{-i(x \cos(\alpha) + y \sin(\alpha)) k_x} dx dy$$

7. Define the tilted  $k$ -space coordinate system.

$$k_x = k_x \cos(\alpha) - k_y \sin(\alpha)$$

$$k_y = k_x \sin(\alpha) - k_y \cos(\alpha)$$

8. Rewrite #6 as

$$\tilde{p}(\alpha, k_x) = \iint O(x, y) e^{-i(k_x \cos(\alpha) - k_y \sin(\alpha))x} e^{-i(k_x \sin(\alpha) + k_y \cos(\alpha))y} dx dy \Big|_{k_y=0}$$

$$\tilde{p}(\alpha, k_x) = \iint O(x, y) e^{-ik_x x} e^{-ik_y y} dx dy \Big|_{k_y=0}$$

$$= F_{2D} \{O(x, y)\} \Big|_{k_y=0}$$

## The Central Slice Theorem

Consider a 2-dimensional example of an emission imaging system.  $O(x,y)$  is the object function, describing the source distribution. The projection data, is the line integral along the projection direction.

$$P(0^\circ, y) = \int O(x, y) dx$$

The Central Slice Theorem can be seen as a consequence of the separability of a 2-D Fourier Transform.

$$\tilde{o}(k_x, k_y) = \int O(x, y) e^{-ik_x x} e^{-ik_y y} dx dy$$

The 1-D Fourier Transform of the projection is,

$$\begin{aligned} \tilde{p}(k_y) &= \int P(0^\circ, y) e^{-ik_y y} dy \\ &= \int O(x, y) e^{-ik_y y} dx dy \\ &= \int O(x, y) e^{-ik_y y} e^{-i0x} dx dy \\ &= \tilde{o}(0, k_y) \end{aligned}$$

The one-dimensional Fourier transformation of a projection obtained at an angle  $J$ , is the same as the radical slice taken through the two-dimensional Fourier domain of the object at the same angle.

