NUMERICAL METHODS FOR SOLVING EQUATIONS

COURSE OBJECTIVE: Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

Numerical Solution of algebraic and transcendental equations: Regula Falsi method, Newton Raphson method & Secant method. Numerical Solution of simultaneous linear algebraic equations: Gauss elimination method, Gauss Jordan method, Gauss Jacobi method & Gauss Seidel method.

Regula - Falsi Method

This method is the oldest method for finding the real roots of an algebraic and transcendental equations. consider the equation $f(x)=0$ and let 'a' & 'b' be two values of x such that $f(a)$ and $f(b)$ are of opposite signs. Hence a root of $f(x)=0$ lies between a & b Let it be c. To find c, let us calculate x_n values $(n=1,2,\dots)$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Now we find the sign of $f(x_1)$ and re assign a and b to those values for which the signs of the functional values are different. then

x_2 is calculated using the same formula, this process is repeated until two consecutive values of x_n is the same or equal upto the preferred no. of decimal places. Then that value of x_n is the required root.

order of convergence of this method is 1.618

Problem

Obtain a real root of $x^3 - x - 1 = 0$ correct to three decimal places by regula falsi method

$f(x) = x^3 - x - 1$

$f(1) = -1, \quad f(2) = 5 \quad \Rightarrow \quad a = -1, \ b = 5$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{1 \times 5 - 2 \times (-1)}{5 - (-1)} = 1.1667$$

$f(1.1667) = -0.57867 \Rightarrow$ root lies between 1.1667 & 2

$\therefore$ $a = 1.1667$, $b = 2$

$f(a) = -0.57869$, $f(b) = 5$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = 1.25311 \Rightarrow f(x_2) = -0.15757$$

Proceeding in the same way the iteration values are tabulated below.

$x_1 = 1.1667$	Root lies between x_1, b.
$x_2 = 1.25311$	Root lies between x_2, b.
$x_3 = 1.27593$	" x_3, b
$x_4 = 1.30361$	" x_4, b
$x_5 = 1.31569$	" x_5, b
$x_6 = 1.32088$	" x_6, b
$x_7 = 1.32309$	" x_7, b
$x_8 = 1.32403$	" x_8, b
$x_9 = 1.32443$	

Hence the root is 1.324.

Practice Problems

Solve the foll. equations by regula falsi method

1. Find a root between 2 and 3 for $x \log_{10} x = 1.2$

   Ans 2.741

2. Find a root between 0 & 1 for $x e^x = 2$

   Ans 0.85260.

Newton's Method (or) Newton-Raphson Method.

To solve $f(x) = 0$, the iterative formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided $|f(x) \cdot f''(x)| < |f'(x)|^2$

order of convergence of this method is 2

Problem Find the positive root of $f(x) = x^3 - 3x^2 + 7x - 8 = 0$ by Newton-Raphson's method

$$f(x) = x^3 - 3x^2 + 7x - 8$$

$$f'(x) = 3x^2 - 6x + 7$$

$$f(0) = -8, \quad f(1) = -3, \quad f(2) = 2$$

$\Rightarrow$ Root lies between 1 and 2.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 3x_n^2 + 7x_n - 8}{3x_n^2 - 6x_n + 7}$$

$$x_{n+1} = \frac{2x_n^3 + 3x_n^2 + 8}{3x_n^2 - 6x_n + 7}$$

put $x_0 = 2$ ($\because f(2)$ is closer to zero than $f(1)$).

$$x_1 = \frac{2 \cdot 2^3 + 3 \cdot 2^2 + 8}{3 \cdot 2^2 - 6 \cdot 2 + 7} = 1.71428$$

proceeding the same way we the iterative values

$x_2 = 1.67422$

$x_3 = 1.67416$

$\therefore$ The root is 1.674

Problems for Practice

1. Find a +ve roots of $f(x) = \cos x - xe^x = 0$

   Ans 0.51776

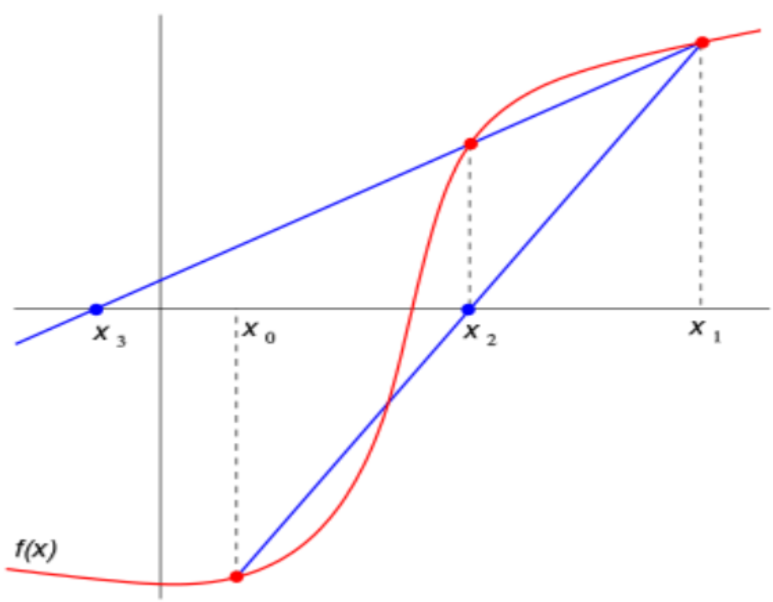
2. Find a +ve root of $x \tan x = 1.28$

   Ans 0.93826.

SECANT METHOD

We assume we have two estimates of the root α, say $x_0$and x_1. Then we produce a linear function $q(x)=a_0 + a_1x$ with $q(x_0)=f(x_0)$, $q(x_1)=f(x_1)$ (*) This line is sometimes called a *secant line*. Its equation is given by

$q(x)= \{(x_1-x)f(x_0)+(x-x_0)f(x_1)\}/(x_1-x_{0)}$---------(2) which is linear in x.



We solve the equation $q(x) = 0$,let $x_2$be the root of -------(2).

Which implies $x_2 = x_1 - f(x_1)\{(x_1 - x_0)/(f(x_1)-f(x_0))\}$

Repeating the process, with x_1 and x_2 to form another secant line,and then use its root to approximate α.

The general iteration formula

$x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, ..$ is called *secant method* for solving f(x) = 0.

Convergence[

The iterates x_n of the secant method converge to a root αof f(x) = 0 , if the initial values x_0 and x_1 are sufficiently close to the root. The order ofconvergence is α, where $\alpha = \frac{1}{2}\left(1 + \sqrt{5}\right) \cong 1.618$ is the golden ratio.

1. Solve for a positive root of $x^3 - 4x + 1$ by using secant method.

Solution; let $f(x) = x^3 - 4x + 1 = 0$.

f(0)= 1 +ve

f(1)= -2 - ve

since f(0) and f(1) are of opposite sign there exists a positive root lies between 0 and 1 .

let $x_0 = 0.3$ and $x_1 = 0.4$, $f(x_0) = -0.17, \; f(x_1) = -0.5360$

By secant method $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, ..$

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$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$
$$x_2 = 0.2536$$
$$f(x_2) = 0.0019$$
$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$
$$x_3 = 0.2541$$
$$f(x_3) = 0$$

Hence the root is 0.2541.

2. Solve for a positive root of x – cosx by using secant method.

Solution; let f(x) = x– cos x = 0.

f(0) = - 1 - ve

f(1) = 0.4597 + ve

since f(0) and f(1) are of opposite sign there exists a positive root lies between 0 and 1 .

let $x_0 = 0.7$ and $x_1 = 0.9$, $f(x_0) = -0.17,\ f(x_1) = -0.5360$

By secant method $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})},\quad n = 1, 2, ..$

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$
$$x_2 = 0.7378$$
$$f(x_2) = -0.0022.$$
$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$
$$x_3 = 0.7391$$
$$f(x_3) = 0$$

Hence the root is 0.7391.

3.Solve for a positive root of $x\,e^x = 2$ by using secant method.

Solution; let $f(x) = x\,e^x - 2 = 0$.

f(0) = - 2 - ve

f(1) = 0.718 + ve

since f(0) and f(1) are of opposite sign there exists a positive root lies between 0 and 1 .

let $x_0 = 0.6$ and $x_1 = 0.8$, $f(x_0) = -0.9067,\ f(x_1) = -0.2196$

By secant method $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, ..$

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$x_2 = 0.8529$

$f(x_2) = -0.029.$

$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

$x_3 = 0.8529$

$f(x_3) = 0.0013$

$$x_4 = x_3 - f(x_3)\frac{x_3 - x_2}{f(x_3) - f(x_2)}$$

$x_4 = 0.8526$

$f(x_4) = 0$

Hence the root is 0.8526

PROBLEMS FOR PRACTICE

Solve for a positive root of $x\log_{10}x - 1.2 = 0$ by using secant method

(Solution.root is 2.7045.)

Find a positive root of $x^3 = 2x + 5$ by using secant method

(Solution.root is 2.0945.)

Solution to Simultaneous Linear Equations

(i) Gauss Elimination Method

To solve $AX = B$, A is a square matrix of order 'n' and X and B are column matrices with n elements.

In this method the co-efficient matrix A is reduced to a upper Triangular matrix, thereby the values of $x_1, x_2, \dots x_n$ are found one by one.

Example :- Using Gauss - Elimination method solve

$2x + y + 4z = 12$, $8x - 3y + 2z = 20$, $4x + 11y - z = 33$

Matrix form of the above system of equation is

$$\begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \\ 33 \end{pmatrix}$$

Consider the Augmented Matrix.

$$(A, B) = \begin{pmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{pmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_2 - 4R_1 \\ R_3 \leftrightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -27 & -27 \end{pmatrix} \quad R_3 \leftrightarrow R_3 + \frac{9}{7} R_2$$

$-27z = -27 \Rightarrow \boxed{z = 1}$, $-7y - 14z = -28 \Rightarrow -7y - 14 = -28 \Rightarrow -7y = -14 \Rightarrow \boxed{y = 2}$

$2x + y + 4z = 12 \Rightarrow 2x + 2 + 4 = 12 \Rightarrow 2x = 6 \Rightarrow x = 3$

$\Rightarrow x = 3, \ y = 2, \ z = 1$

Practice Problems

Solve the foll. system of equation by Gauss - Elimination Method

1. $2x + y + 4z = 4$, $x - 3y - z = -5$, $3x - 2y + 2z = -1$

   Ans $x = 1$, $y = 2$, $z = 0$

2. $2x_1 + 4x_2 + x_3 = 3$, $3x_1 + 2x_2 - 2x_3 = -2$, $x_1 - x_2 + x_3 = 6$

   Ans $x_1 = 2$, $x_2 = -1$, $x_3 = 3$

(ii) Gauss - Jordon Method

To solve the system of equation represented in matrix form $AX = B$, the coefficient matrix is reduced to a diagonal matrix by means linear transformations. thereby the values of the variable are found one after the other.

Example:- Solve : $x+2y+3z=6$, $2x+4y+z=7$, $3x+2y+9z=14$ by Gauss - Jordon method.

Matrix form of the equation is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 14 \end{pmatrix}$$

Augmented Matrix is

$$(A, B) = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 4 & 1 & 7 \\ 3 & 2 & 9 & 14 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & -5 & -5 \\ 0 & -4 & 0 & -4 \end{pmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & -5 & -5 \end{pmatrix} \quad R_2 \rightleftharpoons R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & -5 & -5 \end{pmatrix} \quad R_1 \leftrightarrow R_1 + \tfrac{1}{2} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & -5 & -5 \end{pmatrix} \quad R_1 \leftrightarrow R_1 + 3/5\, R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_2/-4 \\ R_3 \leftrightarrow R_3/-5 \end{matrix} \quad \Rightarrow \boxed{x=1,\ y=1,\ z=1}$$

Problems for Practice

Solve the following system of equation by Gauss - Jordon Method

1. $2x+3y-z=3$, $x+y+3z=-2$, $x+y+z=0$

   Ans $x=1$, $y=0$, $z=-1$

2. $2x+y+4z=12$, $8x-3y+2z=20$, $4x+11y-z=33$

   Ans $x=3$, $y=2$, $z=1$

Gauss – Jacobi Method.

Aim: To to solve the system of equation

$$AX = B \quad \text{——— (1)} \quad \text{where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \; B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The system of eqn (1) will be solvable by iteration (Gauss – Jacobi or Gauss – seidel) method if the coefficient matrix A is diagonally dominant.

The solution of (1) will exist (iteration will converge) if the absolute value of the leading diagonal element of the coefficient matrix are greater than the sum of absolute values of the other coefficients.

Solve for x, y and z

$$x = \frac{1}{a_1}\left[d_1 - b_1 y - c_1 z\right]$$

$$y = \frac{1}{b_2}\left[d_2 - a_2 x - c_2 z\right]$$

$$z = \frac{1}{c_3}\left[d_3 - a_3 x - b_3 y\right]$$

Initial estimates $x^{(0)} = 0$, $y^{(0)} = 0$ and $z^{(0)} = 0$,

I^{st} iteration

$$x^{(1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right]$$

$$y^{(1)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(0)} - c_2 z^{(0)}\right]$$

$$z^{(1)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(0)} - b_3 y^{(0)}\right]$$

Proceeding in the same way, the r^{th} iteration is

$$x^{(r+1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(r)} - c_1 z^{(r)}\right]$$

$$y^{(r+1)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(1)} - c_2 z^{(r)}\right]$$

$$z^{r+1} = \frac{1}{c_3}\left[d_3 - a_3 x^{(r)} - b_3 y^{(r)}\right]$$

. continue the procedure still the convergence is assured.

Solve the system of equation using Gauss – Jacobi method. $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$, $10x - 5y - 2z = 3$.

Solu Given equation can be rearranged asuch that they are diagonally dominant as follows

$$10x - 5y - 2z = 3 \Rightarrow x = \frac{1}{10}\left[3 + 5y + 2z\right]$$

$$4x - 10y + 3z = -3 \Rightarrow y = -\frac{1}{10}\left[3 + 4x + 3z\right]$$

$$x + 6y + 10z = -3 \Rightarrow z = -\frac{1}{10}\left[3x + x + 6y\right]$$

Taking the initial values $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$

The 1st iteration Values are

$$x^{(1)} = \frac{1}{10}\left[3 + 5y^{(0)} + 2z^{(0)}\right] = 0.3$$

$$y^{(1)} = \frac{1}{10}\left[3 + 4x^{(0)} + 3z^{(0)}\right] = 0.3$$

$$z^{(1)} = -\frac{1}{10}\left[3 + x^{(0)} + 6y^{(0)}\right] = 0.3.$$

II iteration

$$y\ x^{(2)} = \frac{1}{10}\left[3 + 5y^{(1)} + 2\ z^{(1)}\right] = 0.39$$

$$y^{(2)} = \frac{1}{10}\left[3 + 4\ x^{(1)} + 3\ z^{(1)}\right] = 0.33$$

$$z^{(2)} = -\frac{1}{10}\left[3 + x^{(1)} + 6\ y^{(1)}\right] = -0.51$$

III iteration

$$x^{(3)} = \frac{1}{10}\left[3 + 5\ y^{(2)} + 2\ z^{(2)}\right] = 0.363$$

$$y^{(3)} = \frac{1}{10}\left[3 + 4\ x^{(2)} + 3\ z^{(2)}\right] = 0.303$$

$$z^{(3)} = -\frac{1}{10}\left[3 + x^{(2)} + 6\ y^{(2)}\right] = -0.537$$

IV iteration

$$x^{(4)} = \frac{1}{10}\left[3 + 5\ y^{(3)} + 2\ z^{(3)}\right] = 0.344$$

$$y^{(4)} = \frac{1}{10}\left[3 + 4x^{(3)} + 3z^{(3)}\right] = 0.2844$$

$$z^{(4)} = -\frac{1}{10}\left[3 + x^{(3)} + 6\ y^{(3)}\right] = -0.518$$

Repeating the iteration process, the values are tabulated as follows

iteration	$x = \frac{1}{10}(3+5y+2z)$	$y = \frac{1}{10}(3+4x+3z)$	$z = -\frac{1}{10}(3+x+6y)$
0	0	0	0
1	0.3	0.3	−0.3
2	0.39	0.33	−0.51
3	0.363	0.303	−0.537
4	0.344	0.284	−0.518
5	0.338	0.282	−0.505
6	0.34	0.284	−0.503
7	0.341	0.285	−0.504
8	0.342	0.285	−0.505
9	0.342	0.285	−0.505

soln is $x = 0.342$, $y = 0.285$, $z = -0.505$.

(1)

Solve the following system of equations by Gauss-Seidel method.

$4x+2y+z=14$, $x+5y-z=10$, $x+y+8z=20$.

Here the diagonal elements are dominant, hence we apply Gauss-Seidel method.

Let the initial values of $y=0$ and $z=0$.

I Iteration

$x^{(1)} = \frac{1}{4}[14-2y-z] = \frac{14}{4} = 3.5$

$y^{(1)} = \frac{1}{5}[10-x+z] = 1.3$.

$z^{(1)} = \frac{1}{8}[20-x-y] = \frac{1}{8}(20-3.5-1.3) = 1.9$.

II Iteration:

$x^{(2)} = \frac{1}{4}[14-2y^{(1)}-z^{(1)}] = 2.375$

$y^{(2)} = \frac{1}{5}[10-x^{(2)}-z^{(1)}] = 1.905$

$z^{(3)} = \frac{1}{8}[20-x^{(2)}-y^{(2)}] = 1.965$

III Iteration

$x^{(3)} = \frac{1}{4}[14-y^{(2)}-z^{(2)}] = 2.05625$

$y^{(3)} = \frac{1}{15}[10-x^{(3)}-z^{(2)}] = 1.98175$

$z^{(3)} = \frac{1}{8}[20-x^{(3)}-y^{(3)}] = 1.99525$

IV iteration;

$x^{(4)} = \frac{1}{4}[14-y^{(3)}-z^{(3)}] = 2.010312$

$y^{(4)} = \frac{1}{5}[10-x^{(4)}-z^{(3)}] = 1.996988$

$z^{(4)} = \frac{1}{8}[20-x^{(4)}-y^{(4)}] = 1.999087$

(2)

Continuing the iteration process, the values are tabulated as follows.

Iteration	$x = \frac{1}{4}(14-2y-z)$	$y = \frac{1}{5}[10-x+z]$	$z = \frac{1}{8}[20-x-y]$
0		0	0
1	3.5	1.3	1.9
2	2.375	1.905	1.965
3	2.05625	1.98175	1.99525
4	2.0103125	1.996988	1.999087
5	2.001734	1.99947	1.999849
6	2.00030	1.99991	1.99997

The values of solution correct upto 4 decimal places are $x = 2.0000$, $y = 1.9999$, $z = 1.9999$

—o—

Solve the following system of equations using Gauss-Seidel method.

1) $10x-5y-2z=3$, $4x-10y+3z=-3$, $x+6y+10z=-3$

Ans $x = 0.342$, $y = 0.285$, $z = -0.505$

2) $x+y+54z=110$, $27x+6y-z=85$, $6x+15y+2z=72$

Soln $x = 2.425$, $y = 3.573$, $z = 1.926$.

Solve the following system of equation using Gauss-Jacobi method.

(1) $10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$

$x = 0.342$, $y = 0.285$, $z = -0.505$

(2) $8x - 3y + 2z = 20$, $4x + 11y - z = 33$, $6x + 3y + 12z = 35$

$x = 3.0168$ $y = 1.9858$, $z = 0.9117$