NUMERICAL DIFFERENTIATION FOR UNEQUAL INTERVALS

When the x values are not equally spaced, then the interval of differencing is not constant. In such cases, we express y as a polynomial in x using Newton's divided difference formula or Lagrange's interpolation formula and then differentiating it w.r.t x, the derivatives at any x in the given range can be found.

PROBLEMS

1) Using Lagrange's formula, find y' and y'' at $x=2$ for the following data

x :	0	1	3	6
y :	18	10	−18	90

Solution

Since the x values are unequally spaced, first Lagrange's interpolation formula is used to find y as a polynomial in x.

Given $x_0=0$, $x_1=1$, $x_2=3$, $x_3=6$

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)} \times 18 + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10) + \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= -(x-1)(x-3)(x-6) + x(x-3)(x-6) + x(x-1)(x-6) + x(x-1)(x-3)$$

$$f(x) = 2x^3 - 10x^2 + 18$$

Hence $f'(x) = 6x^2 - 20x$

$\Rightarrow f'(2) = -16$

Also $f''(x) = 12x - 20$

$\Rightarrow f''(2) = 4$

$\therefore$ y' and y'' at $x=2$ are -16 and 4 respectively

NUMERICAL INTEGRATION

Let $x_0, x_1, x_2, \ldots, x_n$ be the values of x and $y_0, y_1, y_2, \ldots, y_n$ be the corresponding values of y, where the x values are equally spaced with a common interval of differencing h. Then $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots x_n = x_0 + nh$

DEFINITION

The process of computing $\int_a^b y\,dx$, where $y = f(x)$ is given by a set of tabulated values $[x_i, y_i]$, $i = 0, 1, 2, \ldots, n$ and $a = x_0$, $b = x_n$ is called numerical integration.

Geometrically, $\int_a^b y\,dx$ represents the area under the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$.

① TRAPEZOIDAL RULE

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$= \frac{h}{2}\left[(\text{sum of the first and last ordinates}) + 2(\text{sum of the remaining ordinates})\right]$$

where n = number of intervals

$h = \dfrac{x_n - x_0}{n}$ is the common difference in x values

Note:

(1) This rule is the simplest one but is the least accurate

(2) The error in the Trapezoidal rule is of order h^2

② SIMPSON'S ONE-THIRD RULE

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+y_5+\cdots)\right]$$

$= \frac{h}{3}$[(sum of the first and last ordinates) + 2(sum of ordinates with even suffixes) + 4(sum of ordinates with odd suffixes)]

Note:

(1) This rule is the most accurate of the three rules

(2) simpson's one-third rule can be applied only when n, the number of intervals is even

(3) The error in Simpson's one-third rule is of order h^4

③ SIMPSON'S THREE-EIGHTH RULE

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots)+2(y_3+y_6+y_9+\cdots)\right]$$

$= \frac{3h}{8}$[(sum of the first and last ordinates) + 2(sum of ordinates with suffixes as multiples of 3) + 3(sum of remaining ordinates)]

Note:
Simpson's three eigth rule can be applied only when n, the number of intervals is a multiple of 3.

REMARK:
In all the three rules, the accuracy of the result increases as the value of h decreases and the value of n increases

PROBLEMS

1) From the following table, find the area bounded by the curve and the x axis from $x = 7.47$ to $x = 7.52$

x:	7.47	7.48	7.49	7.50	7.51	7.52
$y = f(x)$:	1.93	1.95	1.98	2.01	2.03	2.06

Solution

Here $n=5$. Since Simpson's rules cannot be used, Trapezoidal rule is used.

$$Area = \int_{x=7.47}^{x=7.52} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$Area = \frac{h}{2}\left[(y_0+y_5) + 2(y_1+y_2+y_3+y_4)\right], \text{ where } h = 0.0$$

$$= \frac{0.01}{2}\left[(1.93+2.06) + 2(1.95+1.98+2.01+2.03)\right]$$

$$= 0.09965$$

2) Evaluate $\int_0^4 e^x dx$ taking $h=1$

Solution

Given $h=1$, $x_0=0$, $x_n=4$

Since the x values start from 0 and go till 4 with a common difference of 1, the x values are 0,1,2,3,4

x:	0	1	2	3	4
$y=e^x$:	1	2.7183	7.3891	20.0855	54.5982
	y_0	y_1	y_2	y_3	y_4

Since $n=4$, it is even. Hence Simpson's one third rule can be applied

By Simpson's one-third rule,

$$\int_0^4 e^x dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+y_5+\cdots)\right]$$

$$= \frac{h}{3}\left[(y_0+y_4)+2(y_2)+4(y_1+y_3)\right]$$

$$= \frac{1}{3}\left[(1+54.5982)+2(7.3891)+4(2.7183+20.0855)\right]$$

$$= 53.8639$$

3) Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places.

Hence evaluate $\log_e 2$.

Solution

To use Simpson's one third rule, n the number of intervals has to be even.

Let $n=6$. Then $h = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}$.

$\therefore$ The x values vary from 0 to 1 with a common difference of $\frac{1}{6}$.

x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y=\frac{1}{1+x}$:	1	0.8571	0.75	0.6667	0.6	0.5455	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By simpson's one third rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$= \frac{\frac{1}{6}}{3}\left[(1+0.5)+2(0.75+0.6)+4(0.8571+0.6667+0.5455)\right]$$

$$\int_0^1 \frac{dx}{1+x} = 0.6932 \simeq 0.693 \text{ correct to 3 decimal places}$$

To evaluate $\log_e 2$

By actual integration, we get

$$\left[\log_e(1+x)\right]_0^1 = 0.6932$$

$$\Rightarrow \log_e 2 - \log_e 1 = 0.6932$$

$$\Rightarrow \boxed{\log_e 2 = 0.6932}$$

4) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule, with $h=0.2$. Hence determine the value of π.

Solution

Here $h=0.2$, $x_0=0$, $x_n=1$

$\therefore$ x values vary from 0 to 1 with a common difference of 0.2

x:	0	0.2	0.4	0.6	0.8	1.0
$y=\frac{1}{1+x^2}$:	1	0.9615	0.8621	0.7353	0.6098	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[1.5 + 2(3.1687)\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.78374$$

To find the value of π

We know that, by actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

$$\Rightarrow 0.78374 = \frac{\pi}{4}$$

$$\Rightarrow \boxed{\pi = 3.13496}$$

5) By dividing the range into ten equal parts, evaluate $\int_0^\pi \sin x\,dx$ by Trapezoidal and simpson's rule. Verify your answer with integration.

Solution

Given $n=10$, $x_0=0$, $x_n=\pi$

$$\therefore h = \frac{x_n - x_0}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10} \Rightarrow \boxed{h = \frac{\pi}{10}}$$

$\therefore$ x values vary from 0 to π with a common difference of $\frac{\pi}{10}$

x:	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$
$y=\sin x$:	0.0 y_0	0.3090 y_1	0.5878 y_2	0.8090 y_3	0.9511 y_4	1.0 y_5

x:	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
$y=\sin x$:	0.9511 y_6	0.8090 y_7	0.5878 y_8	0.3090 y_9	0 y_{10}

Trapezoidal Rule

$$\int_0^\pi \sin x\,dx = \frac{h}{2}\left[(y_0+y_{10}) + 2(y_1+y_2+\cdots+y_9)\right]$$

$$= \frac{\pi}{20}\left[(0+0) + 2(0.3090+0.5878+0.8090+0.9511+1.0+0.9511+0.8090+0.5878+0.3090)\right]$$

$$= 1.9843$$

Simpson's one third rule

Since $n=10$ is even, Simpson's one-third rule can be applied.

By Simpson's one third rule,

$$\int_0^{\pi} \sin x\,dx = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{\pi}{30}\left[(0+0) + 2(0.5878+0.9511+0.9511+0.5878) + 4(0.3090+0.8090+1+0.8090+0.3090)\right]$$

$$= 2.0001$$

Actual Integration

By actual integration,

$$\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = 1+1 = 2$$

Hence Simpson's one third rule is more accurate than Trapezoidal rule.

6) The velocity v of a particle at distance s from a point on its path is given by the following table:

S (feet) :	0	10	20	30	40	50	60
V (feet/sec) :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet using Simpson's one third rule. Compare the result with Simpson's three eigth rule.

Solution

Velocity = rate of change of displacement

$$\Rightarrow v = \frac{ds}{dt} \quad \Rightarrow dt = \frac{1}{v}ds$$

∴ Time taken to travel 60 feet is given by

$$t = \int_0^{60} \frac{1}{v}\,ds \quad \left(\text{Here } x = s,\ y = \frac{1}{v}\right)$$

The new table is

$S(x)$:	0	10	20	30	40	50	60
$\frac{1}{V}(y)$:	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{1}{3}^{rd}$ Rule

$$t=\int_0^{60}\frac{1}{V}ds=\frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$=\frac{10}{3}\left[(0.0213+0.0263)+2(0.0156+0.0164)+4(0.0172+0.0154+0.0192)\right]$$

Time $=1.0635$ seconds

Simpson's $\frac{3}{8}^{th}$ Rule

$$t=\int_0^{60}\frac{1}{V}ds=\frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$=\frac{30}{8}\left[(0.0213+0.0263)+3(0.0172+0.0156+0.0164+0.0192)+2(0.0154)\right]$$

$=1.0644$ seconds

7) The table below gives the velocity v of a moving particle at time t seconds. Find the distance covered by the particle in 12 seconds and also the acceleration at t=2 seconds.

t :	0	2	4	6	8	10	12
v :	4	6	16	34	60	94	136

Solution

To find distance

we know $v=\frac{ds}{dt}$

$ds=v\,dt$

$\Rightarrow S=\int_0^{12}v\,dt$

$n=6$ is a multiple of 3. Hence Simpson's $\frac{3}{8}^{th}$ rule can be applied [As n is also even, $\frac{1}{3}^{rd}$ rule can also be applied]

$$\therefore \left.\begin{array}{l}\text{Distance covered}\\ \text{in 12 seconds}\end{array}\right\} = \int_0^{12} v\,dt$$

$$= \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3\times 2}{8}\left[(4+136)+3(6+16+60+94)+2(34)\right]$$

$$= 552 \text{ metres}$$

<u>To find acceleration</u>

We know $a = \left(\frac{dv}{dt}\right)_{t=2}$ = acceleration

Hence we use Newton's forward difference formula for derivatives. The forward difference table is

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$
0	4	2	8	0
2	6	10	8	0
4	16	18	8	0
6	34	26	8	0
8	60	34	8	
10	94	42		
12	136			

$$u = \frac{x - x_0}{h} = \frac{2-0}{2} = 1$$

$$\Rightarrow \boxed{u=1}$$

$$\therefore \text{Acceleration} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{2}\left[2+\frac{1}{2}(8)+0\right]$$

$$= 3 \text{ metre/sec}^2$$

1. Use lagrange's interpolation formula to fit a polynomial to the following data:

x	0	1	3	4
y	-12	0	6	12

2. Use lagrange's inverse interpolation method to find the value of x corresponding to $y=100$ from the following data

x	3	5	7	9	11
y	6	24	58	108	174