LAGRANGE'S INTERPOLATION FORMULA FOR UNEQUAL INTERVALS

If $y_0, y_1 \cdots y_n$ are the values of a function $y = f(x)$ corresponding to the arguments $x_0, x_1 \cdots x_n$ which are not necessarily equally spaced then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \cdots\cdots \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

1. Determine by lagrange's method the percentage number of patients over 40 years, using the following data:

Age over (x) years :	30	35	45	55
% number (y) of patients :	148	96	68	34

Solution:- By lagrange's formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 \quad \text{—— (1)}$$

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Put $x_0=30$, $x_1=35$, $x_2=45$, $x_3=55$, $y_0=148$, $y_1=96$, $y_2=68$ and $y_3=34$ in ①

$$y=\frac{(x-35)(x-45)(x-55)}{(-5)(-15)(-25)}\times 148+\frac{(x-30)(x-45)(x-55)}{(5)(-10)(-20)}\times 96+$$
$$\frac{(x-30)(x-35)(x-55)}{(15)(10)(-10)}\times 68+\frac{(x-30)(x-35)(x-45)}{(25)(20)(10)}\times 34$$

$$[y]_{x=40}=-\frac{148}{5}+\frac{3}{4}\times 96+\frac{68}{2}-\frac{34}{20}$$
$$=74.7$$

2. Apply Lagrange's interpolation formula to find $f(x)$ if $f(1)=2$, $f(2)=4$, $f(3)=8$, $f(4)=16$ and $f(7)=128$. Hence find $f(5)$ and $f(6)$.

Solution

Lagrange's interpolation formula is

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}f(x_0)+\cdots\cdots$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4) \quad \text{——— ①}$$

Putting $x_0=1$, $x_1=2$, $x_2=3$, $x_3=4$, $x_4=7$ and the given values of $f(x)$ in ①, we have

$$f(x)=\frac{(x-2)(x-3)(x-4)(x-7)}{(-1)(-2)(-3)(-6)}\times 2+\frac{(x-1)(x-3)(x-4)(x-7)}{(1)(-1)(-2)(-5)}\times 4+$$
$$\frac{(x-1)(x-2)(x-4)(x-7)}{(2)(1)(-1)(-4)}\times 8+\frac{(x-1)(x-2)(x-3)(x-7)}{(3)(2)(1)(-3)}\times 16+$$
$$\frac{(x-1)(x-2)(x-3)(x-4)}{(6)(5)(4)(3)}\times 128$$

$$f(x)=\frac{1}{90}\left[11x^4-80x^3+295x^2-310x+264\right] \quad \text{——— ②}$$

Put $x=5$ and $x=6$ in ②, we get $f(5)=32.93$ & $f(6)=66.67$

INVERSE LAGRANGE'S INTERPOLATION FORMULA

To find the values of y corresponding to some x. Here we treat y as a function of x. The process of finding x given y is called the inverse interpolation.

In such a case, we will take y as independent variable and x as dependent variable and use lagrange's interpolation formula.

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} \cdot x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1 +$$

$$\cdots\cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n$$

This formula is called formula of inverse interpolation.

1. Find the age corresponding to the annuity value 13.6 given the table

Age(x)	30	35	40	45	50
Annuity Value (y)	15.9	14.9	14.1	13.3	12.5

Solution:

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} x_0 + \cdots + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} x_4$$

$$x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)} \times 30 +$$

$$\frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)} \times 35 +$$

$$\frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)} \times 40 +$$

$$\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)} \times 45 +$$

$$\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)} \times 50$$

$x[y=13.6] = 43$

1. Define interpolating polynomial
2. State Gregory-Newton forward interpolation formula
3. State Gregory-Newton backward interpolation formula.

1. Find the value of $f(1.02)$ from the following data correct to 5 places of decimals using Newton's
(i) forward interpolation formula &
(ii) backward interpolation formula

x	1.0	1.1	1.2	1.3	1.4	1.5
$f(x)$	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749

2. Use both Newton's forward and backward interpolation formulas to find $\tan 17^\circ$ from the following data

x	0	4	8	12	16	20
$\tan x^\circ$	0	0.0699	0.1405	0.2126	0.2167	0.3640

3. Find $y(5)$ using both Newton's forward and backward interpolation formulas if $y(10)=35.3$, $y(15)=32.4$, $y(20)=29.2$, $y(25)=26.1$, $y(30)=23.2$ & $y(35)=20.5$

NUMERICAL DIFFERENTIATION

Consider a set of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of a function. The process of computing the derivative of the function y at a particular value of x from the given set of values is called Numerical differentiation. This may be done by first approximating the function by a suitable interpolation formula and then differentiating it as many times as desired.

Numerical Differentiation can be done for equal and unequal intervals.

DIFFERENTIATION FOR EQUAL INTERVALS

Let $x_0, x_1, x_2, \ldots, x_n$ be the values of x and $y_0, y_1, y_2, \ldots, y_n$ be the corresponding values of y, where the x values are equally spaced with a common interval of differencing h. Then

$x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $\ldots$, $x_n = x_0 + nh$

GREGORY-NEWTON'S FORWARD DIFFERENCE FORMULA FOR DERIVATIVES

At $x = x_0 + uh$

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$y''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \cdots\right]$$

$$y'''(x) = \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \left(\frac{12u-18}{12}\right)\Delta^4 y_0 + \cdots\right]$$

... and so on

where $u = \frac{x - x_0}{h}$, x is the value at which the derivative needs to be found

x_0 is the first value of x

h is the common difference in x values

Particular case

At $x = x_0$

Then $u = \frac{x - x_0}{h} = 0$

Then the derivative formulae reduce to

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

GREGORY – NEWTON'S BACKWARD DIFFERENCE FORMULA FOR DERIVATIVES

At any $x = x_n + vh$

$$\frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{24}\right)\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \left(\frac{6v^2+18v+11}{12}\right)\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = y'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \left(\frac{12v+18}{12}\right)\nabla^4 y_n + \cdots\right]$$

where $v = \frac{x - x_n}{h}$, x is the value at which the derivative needs to be found, x_n is the last value of x, h is the common difference in the x values

Particular Case

At $x = x_n$

Then $v = \frac{x - x_n}{h} = 0$

Then the derivative formulae reduce to

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Note:

(1) If the derivative is required at a point nearer to the starting value in the table, Newton's forward difference formula for derivatives is used

(2) If the derivative at a point which is in the end of the table is required, then Newton's backward formula for derivatives is used.

PROBLEMS

1) Find the first two derivatives of $x^{1/3}$ at $x=50$ and $x=56$ given the table below.

x:	50	51	52	53	54	55	56
$y=x^{1/3}$:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution

To find the derivatives at $x=50$, Newton's forward formula for derivatives is used and to find the derivatives at $x=56$, Newton's backward formula for derivatives is used.

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

At $x=50$

$$u=\frac{x-x_0}{h}=\frac{50-50}{1} \Rightarrow \boxed{u=0}$$

By Newton's forward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$$

$$= \frac{1}{1}\left[0.0244 - \frac{1}{2}(0.0003) + \frac{1}{3}(0)\right]$$

$$= 0.02455$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[-0.0003\right]$$

$$= -0.0003$$

At $x=56$

$$V = \frac{x-x_n}{h} = \frac{56-56}{1} \Rightarrow \boxed{V=0}$$

By Newton's backward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + 0\right]$$

$$= 0.02275$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}(-0.0003)$$

$$= -0.0003$$

2) The population of a certain town is shown in the following table

Year :	1931	1941	1951	1961	1971
Population : (In Thousands)	40.6	60.8	79.9	103.6	132.7

Find the rate of growth of the population in the year 1961

<u>Solution</u>

Since we have to find rate of change of population, we have to find the first derivative. As 1961 lies in the end of the table, Newton's backward formula for derivatives is used.

Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1931	40.6				
		20.2			
1941	60.8		-1.1		
		19.1		5.7	
1951	79.9		4.6		$\boxed{-4.9}$ $\nabla^4 y_n$
		23.7		$\boxed{0.8}$ $\nabla^3 y_n$	
1961	103.6		$\boxed{5.4}$ $\nabla^2 y_n$		
		$\boxed{29.1}$ ∇y_n			
1971	132.7				

Here $h=10$ $\quad x_n=1971$

$$v=\frac{x-x_n}{h}=\frac{1961-1971}{10}=-1$$

By Newton's backward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_n+vh}=\frac{1}{h}\left[\nabla y_n+\left(\frac{2v+1}{2}\right)\nabla^2 y_n+\left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n+\left(\frac{2v^3+9v^2+11v+3}{12}\right)\nabla^4 y_n+\cdots\right]$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1961}=\frac{1}{10}\left[29.1+\left(\frac{-1}{2}\right)(5.4)+\left(\frac{-1}{6}\right)(0.8)+\left(\frac{-1}{12}\right)(-4.9)\right]$$

$$=2.6775$$

3) A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t (seconds). Calculate the angular velocity and angular acceleration of the rod at $t=0.6$ seconds.

t:	0	0.2	0.4	0.6	0.8	1.0
θ:	0	0.12	0.49	1.12	2.02	3.20

Solution

Since $x=0.6$ is towards the end, backward difference formula for derivatives is used.

Difference table

t	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.90		0.01	
0.8	2.02		0.28		
		1.18			
1.0	3.20				

Here $h = 0.2,\ x_n = 1.0$

$$\Rightarrow \quad v = \frac{x - x_n}{h} = \frac{0.6 - 1.0}{0.2} = -2$$

To find angular velocity

By Newton's backward difference formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x = x_n + vh} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{24}\right)\nabla^4 y_n + \cdots\right]$$

$$\Rightarrow \left[\frac{d\theta}{dt}\right]_{t=0.6} = \frac{1}{0.2}\left[1.18 - \frac{3}{2}(0.28) + \frac{1}{3}(0.01)\right]$$

$$= 3.81665 \text{ radians/sec}.$$

To find angular acceleration

By Newton's backward difference formula for derivatives,

$$\left[\frac{d^2y}{dx^2}\right]_{x = x_n + vh} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \cdots\right]$$

$$\Rightarrow \left[\frac{d^2\theta}{dt^2}\right]_{t=0.6} = \frac{1}{0.04}\left[0.28 - 0.01\right]$$

$$= 6.75 \text{ radians/sec}^2$$

4) From the values in the table given below, find the value of $\sec 31^\circ$

θ (in degrees) :	31	32	33	34
$\tan\theta$:	0.6008	0.6249	0.6494	0.6745

Solution

Since $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$, we first find the first order derivative. As $\theta = 31$ lies in the beginning of the table, Newton's forward interpolation formula for derivatives is used.

Difference table

$\theta(x)$	$y = \tan\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008	0.0241	0.0004	0.0002
32	0.6249	0.0245	0.0006	
33	0.6494	0.0251		
34	0.6745			

Here $h = 1^\circ$, $x_0 = 31^\circ$

$$u = \frac{x - x_0}{h} = \frac{31^\circ - 31^\circ}{1^\circ} = 0$$

By Newton's forward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$$

$$\Rightarrow \left[\frac{d}{d\theta}(\tan\theta)\right]_{\theta=31^\circ} = \frac{1}{1^\circ}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{1}{0.01745}(0.0240) \quad (\text{since } 1^\circ = 0.01745 \text{ radians})$$

$$= 1.3754$$

$$\Rightarrow \sec^2 31^\circ = 1.3754$$

Thus $\sec 31^\circ = \sqrt{1.3754} = 1.1728$