Interpolation, Numerical Differentiation and Integration

Interpolation with Equal Intervals :

Defn: Interpolation

Interpolation is the process of finding the intermediate values of a function [which is not explicitly known] from a set of its values at specific points given in a tabulated form.

Suppose that the following table represents a set of corresponding values of x and $y = f(x)$:

x	x_0	x_1	x_2	$\dots x_n$
y	y_0	y_1	y_2	$\dots y_n$

The process of computing y corresponding to x where $x_i < x < x_{i+1}$, $i = 0, 1, 2 \dots n-1$ is interpolation

Defn: Extrapolation

If $x < x_0$ or $x > x_n$ then the process is called extrapolation.

Note :-

The term interpolation is used in both cases.

Defn :- Polynomial Interpolation

The process of representing $f(x)$ by a polynomial $p(x)$ called polynomial interpolation

Gregory Newton's Forward Interpolation Formula for Equal Intervals

If $y_0, y_1, y_2 \dots y_n$ are the values of $y=f(x)$ corresponding to equidistant values of $x_0, x_1, x_2 \dots x_n$ where $x_i - x_{i-1} = h$ for $i=1,2 \dots n$

then $$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \dots\dots + \frac{u(u-1)\dots(u-(n-1))}{n!}\Delta^n y_0$$

where $u = \frac{x-x_0}{h}$.

This result is known as Gregory-Newton forward interpolation (or) Newton's formula for equal intervals.

Notes :-

1. Since the formula derived involves the forward differences of y at y_0, it is called Newton's forward interpolation formula.

- If only 2 values of y, namely y_0 and y_1 corresponding to $x=x_0$ and x_1 are given, the above formula takes the form

$$y = y_0 + \frac{(x-x_0)}{h}(y_1 - y_0)$$

ie) $y - y_0 = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$ which is called linear interpolation formula

3. If 3 values of y, namely y_0, y_1 and y_2 corresponding to $x = x_0, x_1$ and x_2 are given then Newton's forward interpolation formula is called parabolic interpolation formula.

Gregory - Newton's Backward Interpolation Formula for Equal Intervals

If $y_0, y_1 \cdots y_n$ are the values of $y = f(x)$ corresponding to equidistant values of $x = x_0, x_1, \ldots x_n$ where $x_i - x_{i-1} = h$ for $i = 1, 2 \ldots n$

then $y = y_n + \frac{u}{1!} \nabla y_n + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_n$

$$+ \cdots\cdots + \frac{u(u+1)\cdots(u+n-1)}{n!} \nabla^n y_n$$

where $u = \frac{x - x_n}{h}$.

Note:-

Since this formula involves the backward differences of y at y_n, it is called Newton's backward interpolation formula.

WORKED EXAMPLES

1. If $y(10) = 35.3$, $y(15) = 32.4$, $y(20) = 29.2$, $y(25) = 26.1$, $y(30) = 23.2$ and $y(35) = 20.5$, Find $y(12)$ using Newton's forward interpolation formula.

Solution:

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
(x_0) 10	(y_0) 35.3					
		(Δy_0) −2.9				
15	32.4		$(\Delta^2 y_0)$ 0.3			
		−3.2		$(\Delta^3 y_0)$ 0.4		
20	29.2		0.1		$(\Delta^4 y_0)$ 0.3	
		−3.1		0.1		$(\Delta^5 y_0)$ 0.2
25	26.1		0.2		−0.1	
		−2.9		0.0		
30	23.2		0.2			
		−2.7				
35	20.5					

Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 \quad \text{—— (1)}$$

where $u = \frac{x - x_0}{h} = \frac{12-10}{5} = 0.4$

Using the values of y_0 and the forward differences from the difference table in (1), we have

$$y(x=12) = y(u=0.4)$$

$$= 35.3 + \frac{(0.4)(-2.9)}{1!} + \frac{(0.4)(-0.6)(-0.3)}{2!} + \frac{(0.4)(-0.6)(-1.6)(0.4)}{3!} + \frac{(0.4)(-0.6)(-1.6)(-2.6)(-0.3)}{4!} + \frac{(0.4)(-0.6)(-1.6)(-2.6)(-3.6)}{5!}(0.2)$$

$$= 35.3 - 1.16 + 0.036 + 0.0256 + 0.01248 + 0.0059904$$

$$= 34.2200704$$

$$\simeq 34.22$$

2. From the given table, compute the value of $\sin 38^\circ$.

x	0	10	20	30	40
$\sin x$	0	0.17365	0.34202	0.50000	0.64279

Solution :-

To determine the value of $y = \sin x$ near the lower end, we apply Newton's backward interpolation formula. The difference table is as given below.

x°	$y(x) = \sin x^\circ$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0°	0				
		0.17365			
10°	0.17365		−0.00528		
		0.16837		−0.00511	
20°	0.34202		−0.01039		0.00031 ($\nabla^4 y_0$)
		0.15798		−0.0048 ($\nabla^3 y_0$)	
30°	0.50000		−0.01519 ($\nabla^2 y_0$)		
		0.14279 (∇y_0)			
40° (x_0)	0.64279 (y_0)				

Newton's backward interpolation formula is

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n \quad \text{where } u = \frac{x - x_n}{h}.$$

$$u = \frac{x - x_n}{h} = \frac{38 - 40}{10} = -0.2$$

$$y(38) = y(u = -0.2).$$

$$= 0.64279 + (-0.2)(0.14279) + \frac{(-0.2)(-0.2+1)}{2!}(-0.01519)$$

$$+ \frac{(-0.2)(-0.2+1)(-0.2+2)}{3!}(-0.0048) + \cdots$$

$$= 0.64279 - 0.028558 + 0.0012152 + 0.0002304 + \cdots$$

$\sin 38^\circ = 0.61568$

Note:-

It is obvious that either of the two formulas may be used to interpolate (or) extrapolate y corresponding to any value of x, whatever be its position

3. The population of a town in the census is as given in the data. Estimate the population in the year 1996 using Newton's (i) forward interpolation and (ii) backward interpolation formula.

Year (x)	1961	1971	1981	1991	2001
Population (in 1000's)	46	66	81	93	101

Solution:- The difference table is

x	y	Δy (or) ∇y	$\Delta^2 y$ (or) $\nabla^2 y$	$\Delta^3 y$ (or) $\nabla^3 y$	$\Delta^4 y$ (or) $\nabla^4 y$
1961	46				
		20			
1971	66		−5		
		15		2	
1981	81		−3		−3
		12		−1	
1991	93		−4		
		8			
2001	101				

(i) Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \quad \text{—— (1)}$$

where $u = \frac{x - x_0}{h} = \frac{1996 - 1961}{10} = 3.5$

Using the values of y_0 and the forward differences from the difference table in (1), we have

$$y(x=1996) = y(u=3.5)$$

$$= 46 + \frac{3.5 \times 20}{1!} + \frac{(3.5)(2.5)(-5)}{2!} + \frac{(3.5)(2.5)(1.5)}{3!}(2)$$

$$+ \frac{(3.5)(2.5)(1.5)(0.5)}{4!}(-3)$$

$$= 46 + 70 - 21.875 + 4.375 - 0.8203125$$

$$= 97.6796875$$

(ii) Newton's backward interpolation formula is

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots \quad \text{——(2)}$$

where $u = \frac{x - x_n}{h} = \frac{1996 - 2001}{10} = -0.5$

Using the values of y_n and the backward differences from the difference table, we have

$$y(x=1996) = y(u=-0.5)$$

$$= 101 - \frac{0.5 \times 8}{1!} + \frac{(-0.5)(0.5)}{2!}(-4) + \frac{(-0.5)(0.5)(1.5)}{3!}(-1)$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)}{4!}(-3)$$

$$= 101 - 4.0 + 0.5 + 0.0625 + 0.1171875$$

$$= 97.6796875$$

4. Find $e^{-0.75}$ and $e^{-2.25}$ from the following data using both Newton's forward and backward formulas

x	1.00	1.25	1.50	1.74	2.00
$y = e^{-x}$	0.3679	0.2865	0.2231	0.1738	0.1353

(8)

Solution :- The difference table is

x	y	Δy (or) ∇y	$\Delta^2 y$ (or) $\nabla^2 y$	$\Delta^3 y$ (or) $\nabla^3 y$	$\Delta^4 y$ (or) $\nabla^4 y$
1.00	0.3679				
		−0.0814			
1.25	0.2865		0.0180		
		−0.0634		−0.0039	
1.50	0.2231		0.0141		0.0006
		−0.0493		−0.0033	
1.75	0.1738		0.0108		
		−0.0385			
2.00	0.1353				

when $x = 0.75$

(i) Newtons forward formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

where $u = \frac{x - x_0}{h} = \frac{0.75 - 1.00}{0.25} = -1$.

$$y(x = 0.75) = 0.3679 + \frac{(-1)(-0.0814)}{1!} + \frac{(-1)(-2)}{2!}(0.0180)$$

$$+ \frac{(-1)(-2)(-3)}{3!}(-0.0039) + \frac{(-1)(-2)(-3)(-4)}{4!}(0.0006)$$

$$= 0.3679 + 0.0814 + 0.0180 + 0.0039 + 0.0006$$

$$= 0.4718$$

Newton's backward formula is

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \cdots$$

where $u = \frac{x - x_n}{h} = \frac{0.75 - 2.00}{0.25} = -5$

$$y(x = 0.75) = 0.1353 + \frac{(-5)}{1!}(-0.0385) + \frac{(-5)(-4)}{2!}(0.0108)$$

$$+ \frac{(-5)(-4)(-3)}{3!}(-0.0033) + \frac{(-5)(-4)(-3)(-2)}{4!}(-0.0006)$$

$$= 0.1353 + 0.1925 + 0.1080 + 0.0330 + 0.0030$$

$$= 0.4718$$

(ii) when $x = 2.25$

Newton's forward formula is $y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \cdots$

where $u = \frac{x - x_0}{h} = \frac{2.25 - 1.00}{0.25} = 5$

$$y(x=2.25) = 0.3679 + \frac{5}{1!}(-0.0814) + \frac{(5)(4)}{2!}(0.0180) + \frac{(5)(4)(3)}{3!}(-0.0039) + \frac{(5)(4)(3)(2)}{4!}(0.0006)$$

$$= 0.3679 - 0.4070 + 0.1800 - 0.0390 + 0.0030$$

$$= 0.1049$$

Newton's backward formula is $y(x) = y_n + \frac{u}{1!}\nabla y_n + \cdots$

where $u = \frac{x - x_n}{h} = \frac{2.25 - 2.50}{0.25} = 1.$

$$y(x=2.25) = 0.1353 + \frac{(1)}{1!}(-0.0385) + \frac{(1)(2)}{2!}(0.0108) + \frac{(1)(2)(3)}{3!}(-0.0033) + \frac{(1)(2)(3)(4)}{4!}(0.0006)$$

$$= 0.1353 - 0.0385 + 0.0108 - 0.0033 + 0.0006$$

$$= 0.1049$$

5. Find the interpolating polynomial for y from the following data using both Newton's forward and backward formulae

x	4	6	8	10
y	1	3	8	16

Solution :-

The difference table is

x	y	Δy (or) ∇y	$\Delta^2 y$ (or) $\nabla^2 y$	$\Delta^3 y$ (or) $\nabla^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

(10)

(i) Newton's forward formula is $y = y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \cdots$

where $u = \frac{x - x_0}{h} = \frac{x-4}{2}$

ie) $y = y(x) = 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)\left(\frac{x-4}{2} - 2\right)}{3!}(0)$

ie) $y = 1 + (x-4) + \frac{3}{8}(x-4)(x-6)$

ie) $y = \frac{1}{8}\left[3x^2 - 22x + 48\right]$ which is the required interpolating polynomial for y.

(ii) Newton's backward formula is

$y = y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \cdots$

where $u = \frac{x - x_n}{h} = \frac{x-10}{2}$

ie) $y = y(x) = 16 + \frac{\left(\frac{x-10}{2}\right)}{1!}(8) + \frac{\left(\frac{x-10}{2}\right)\left(\frac{x-10}{2} + 1\right)}{2!}(3)$

$= 16 + 4(x-10) + \frac{3}{8}(x-10)(x-8)$

$y = \frac{1}{8}\left[3x^2 - 22x + 48\right]$ which is the required interpolating polynomial for y.