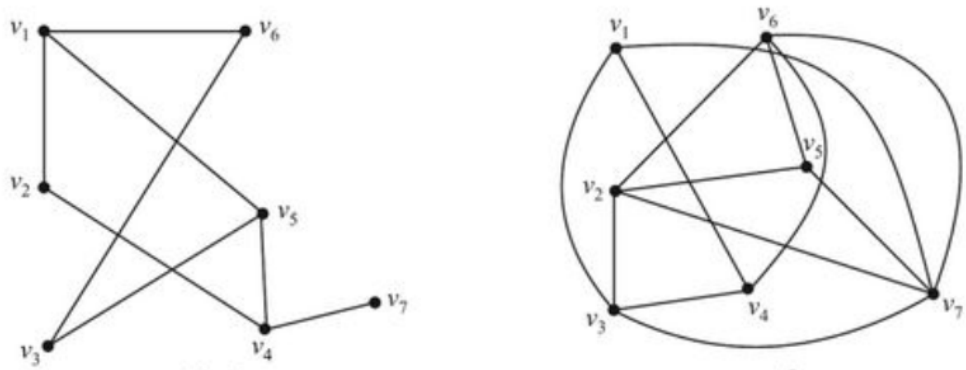
THE COMPLEMENT OF A GRAPH

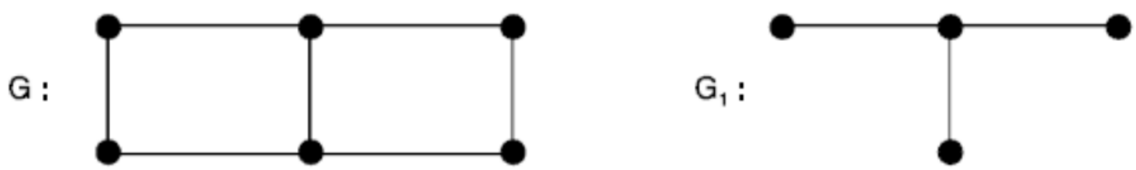
Let G be a simple graph. The complement of G denoted by G^c has the same vertex set as G. Two vertices in G^c are adjacent if and only if they are not adjacent in G.
A graph G and its complement G^c are depicted below



(a) G (b) G^c

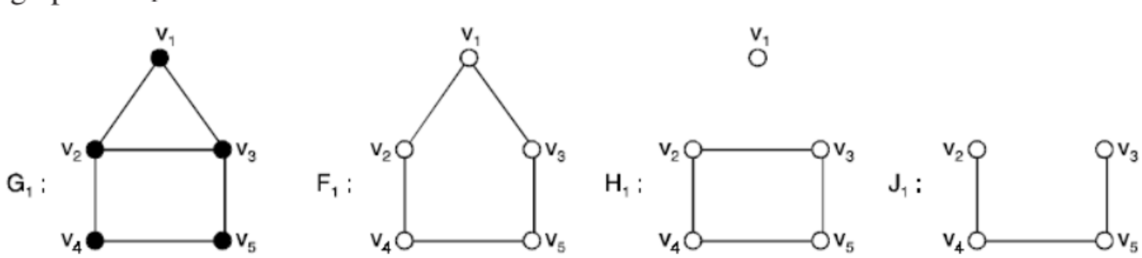
SUBGRAPH

If G and H are two graphs with vertex sets V(H), V(G) and edge sets E(H) and E(G) respectively such that V(H) ⊆V(G) and E(H) ⊆E(G) then we call H as a subgraph of G or G as a supergraph of H.
In the figure given below G_1 is a subgraph of graph G.



SPANNING SUBGRAPH

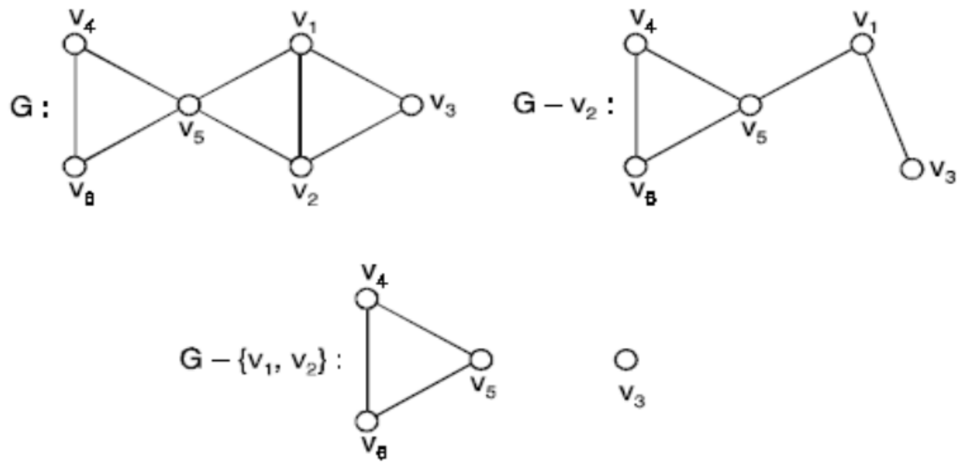
A graph H is called a subgraph of a graph G if V(H) ⊆V(G) and E(H) ⊆E(G.
If V(H) ⊂ V(G) and E(H) ⊂ E(G) then H is called a **proper subgraph** of G.
If V(H) = V(G) then we say that H is a **spanning subgraph** of G.
A spanning subgraph need not contain all the edges in G. The graphs F_1 and H_1 of the figure shown below are spanning subgraphs of G_1, but J_1 is not a spanning
subgraph of G_1.



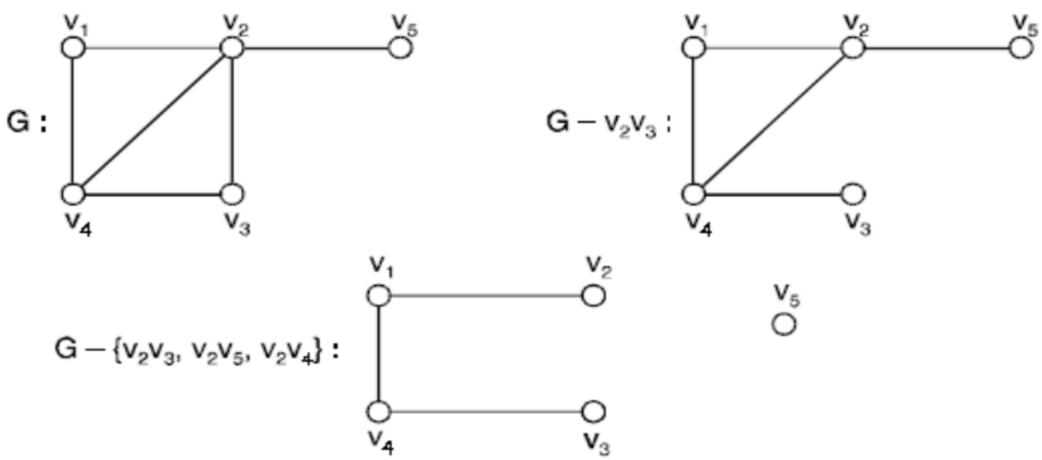
Removal of a vertex and an edge

The removal of a vertex vi from a graph G result in that subgraph G – vi of G containing of all vertices in G except vi and all edges not incident with vi. Thus G – v_i is the maximal subgraph of G not containing vi. On the otherhand, the removal of an edge x_j from G yields the spanning subgraph G – x_j containing all edges of G except x_j. Thus G – x_j is the maximal subgraph of G not containing edge x_j.

The following figure shows deletion of vertices and the corresponding edges from a graph



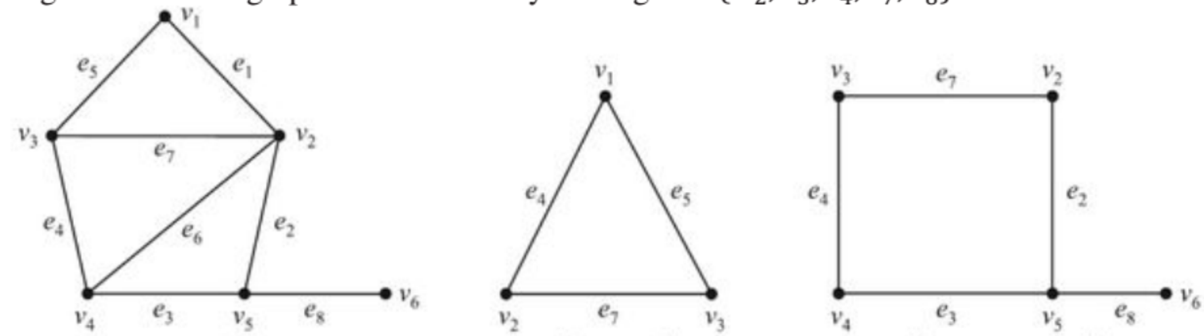
The following figure shows deletion of edges from a graph



INDUCED SUB GRAPH:

Let G be a graph with vertex set V(G), edge set E(G) and S be a non empty subset of V(G). A subgraph of G whose vertex set is S and all edges of G which have both their ends in S is known as the subgraph induced by S and is denoted by G[S] or $< S >$. Any subgraph induced by a set of vertices will be called a **vertex induced subgraph or simply an induced sub graph**. In other words a sub graph H of a graph G where V(H) ⊆V(G) and E(H) consists of only those edges that are incident on the elements of V(H), is called an **induced sub graph** of G.

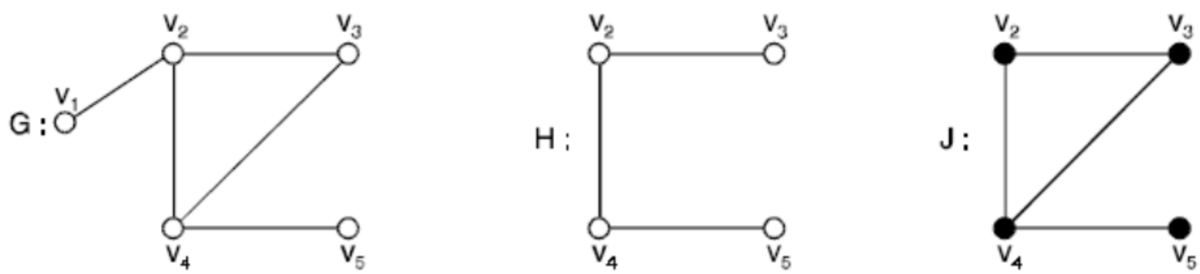
Let M be a non empty subset of E(G). A subgraph of G whose edge set is M and whose vertices are the ends of edges in M, is said to be a subgraph induced by M and is denoted by G[M] or $< M >$. The second figure below displays the vertex induced sub graph of graph G induced by vertex set $\{v_1, v_2, v_3\}$ and the third image in the figure shown below is the edge induced sub graph of G induced by the edge set $\{ e_2, e_3, e_4, e_7, e_8\}$



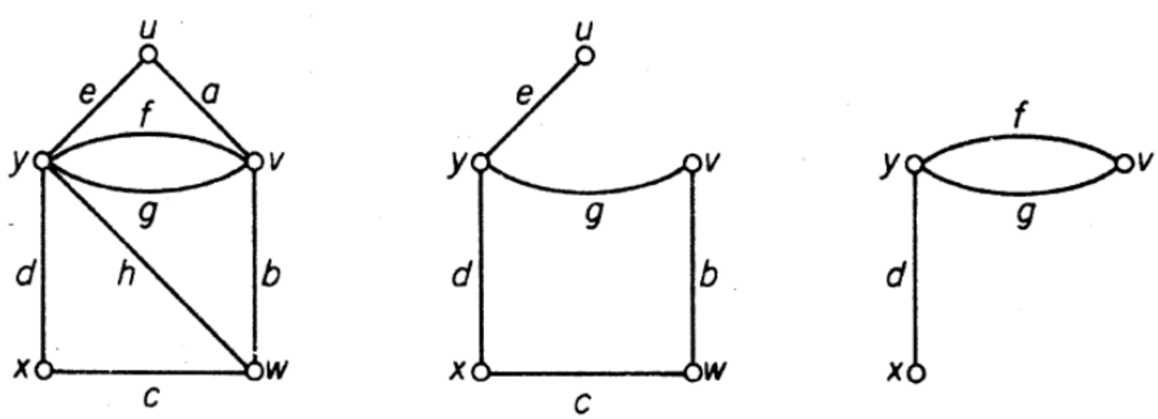
Graph G

Induced sub graphs of graph G

Example H is not an induced subgraph since v_4v_3 ∈E(G), but v_4v_3 ∉E(H).



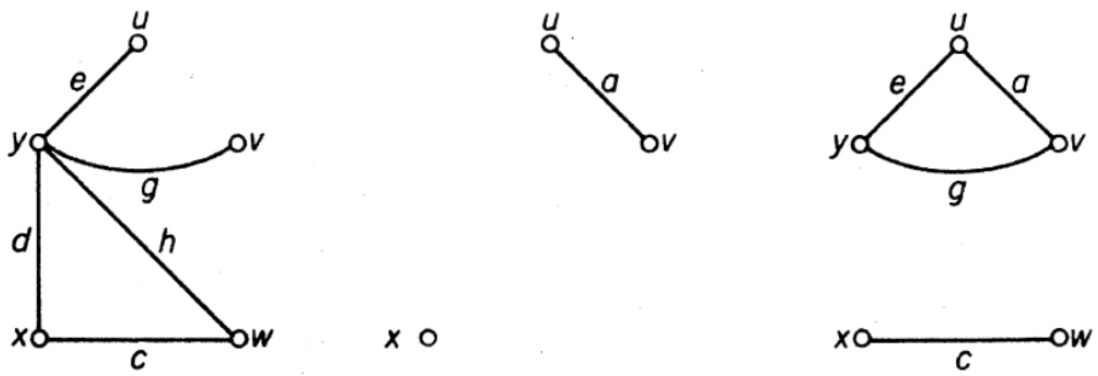
Example for spanning sub graph, vertex induced sub graph and edge induced sub graph



G

A spanning subgraph of G

$G-\{u, w\}$



$G - \{a, b, f\}$

The induced subgraph $G[\{u, v, x\}]$

The edge-induced subgraph $G[\{a, c, e, g\}]$

GRAPHS ISOMORPHISM

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. A function $f : V_1 \rightarrow V_2$ is called a graphs isomorphism if

(i) f is one-to-one and onto.

(ii) For all $a, b \in V_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$ when such a function exists, G_1 and G_2 are called isomorphic graphs and is written as $G_1 \cong G_2$.

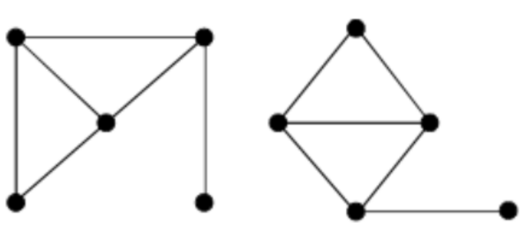
In other words, two graphs G_1 and G_2 are said to be isomorphic to each other if there is a one to-one correspondence between their vertices and between edges such that incidence relationship is preserved. It is written as $G_1 \cong G_2$ or $G_1 = G_2$.

The necessary conditions for two graphs to be isomorphic are

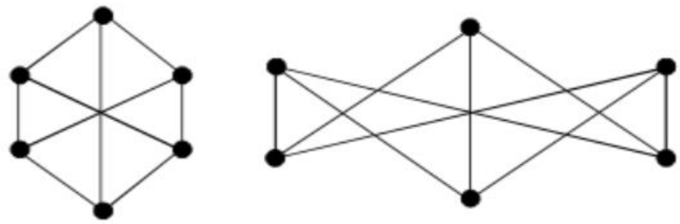
1. Both must have the **same number of vertices**
2. Both must have the **same number of edges**
3. Both must have **equal number of vertices** with the **same degree.**
4. They must have the same degree sequence and same cycle vector (c_1, ..., c_n), where c_i is the number of cycles of length i.

The isomorphic pair of graphs are shown below

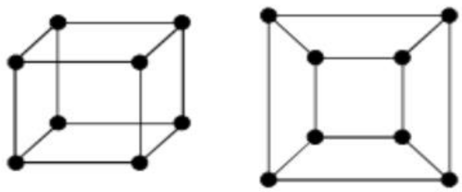
Example 1:



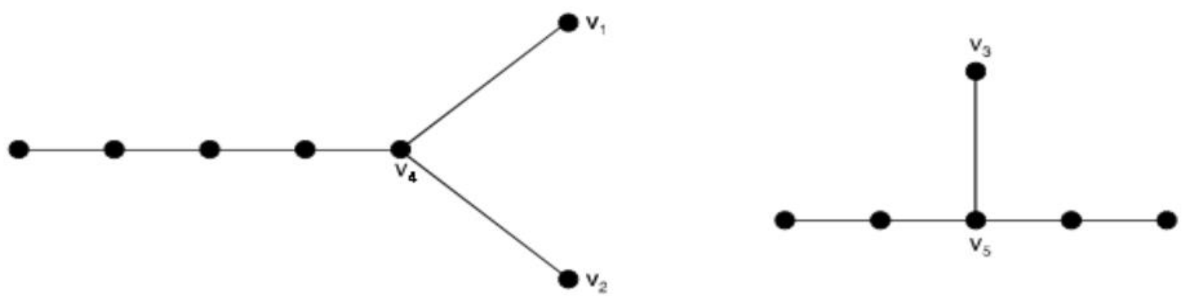
Example 2:



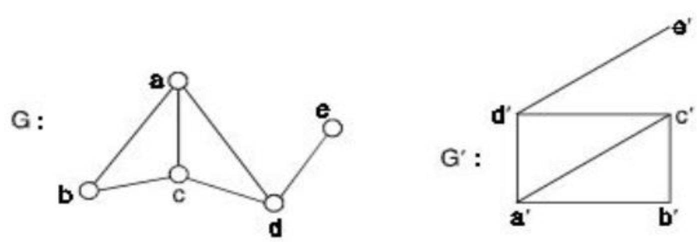
Example 3:



The following example shows two graphs that are not isomorphic



Problem 11. *Show that the following graphs are isomorphic*

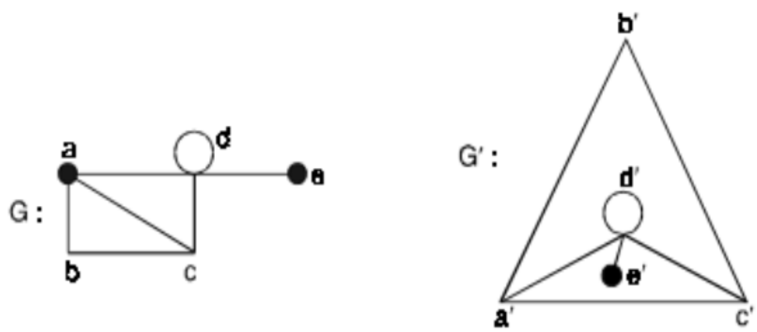


Solution. Let f : G →G′ be any function defined between two graphs degrees of the graph G and G′ are as follows :

deg (G) deg (G′)
deg (a) = 3 deg (a') = 3
deg (b) = 2 deg (b') = 2
deg (c) = 3 deg (c') = 3

deg (d) = 3 deg (d') = 3
deg (e) = 1 deg (e') = 1
Each has 5-vertices and 6-edges.
$d(a) = d(a') = 3$
$d(b) = d(b') = 2$
$d(c) = d(c') = 3$
$d(d) = d(d') = 3$
$d(e) = d(e') = 1$
Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.
Therefore, the given two graphs are isomorphic.

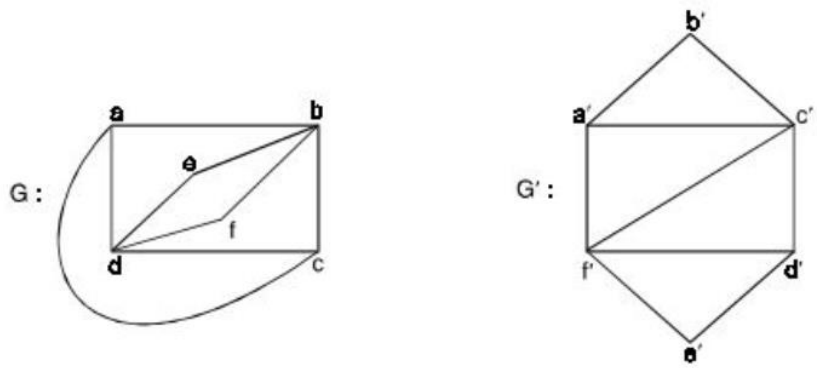
Problem 12. *Show that the following graphs are isomorphic.*



Solution. Let f : G →G′ be any function defined between two graphs degrees of the graphs G and G′ are as follows :
deg (G) deg (G′)
deg (a) = 3 deg (a') = 3
deg (b) = 2 deg (b') = 2
deg (c) = 3 deg (c') = 3

deg (d) = 5 deg (d') = 5
deg (e) = 1 deg (e') = 1
Each has 5-vertices, 6-edges and 1-circuit.
deg(a) = deg(a') = 3
deg(b) = deg(b') = 2
deg(c) = deg(c') = 3
deg(d) = deg(d') = 5
deg(e) = deg(e') = 1
Hence the correspondence is $a - a'$, $b - b'$, ..., $e - e'$.
Therefore, the given two graphs G and G′□are isomorphic.

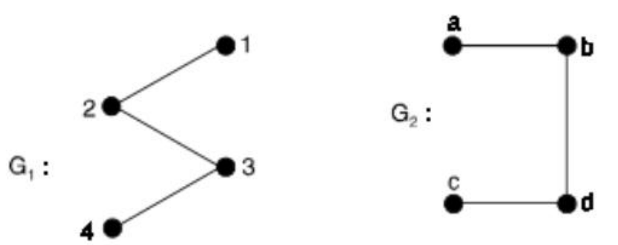
Problem 13. *Are the 2-graphs, given below, isomorphic ? Give a reason.*

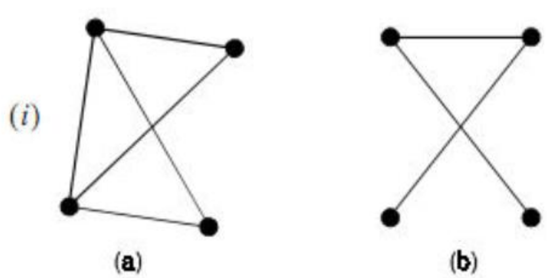


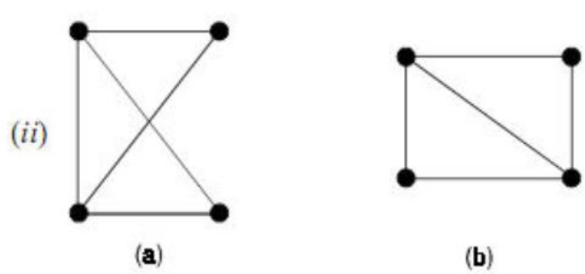
Solution. Let us enumerate the degree of the vertices
Vertices of degree 4 : $b - f'$, $d - c'$
Vertices of degree 3 : $a - a'$, $c - d'$
Vertices of degree 2 : $e - b'$, $f - e'$
Now the vertices of degree 3, in G are a and c and they are adjacent in G, while these are a' and d' which are not adjacent in G′.
Hence the 2-graphs are not isomorphic.

Problems for practice

1. *For each pair of graphs shown, either label the graphs so as to exhibit an isomorphism or explain why the graphs are not isomorphic.*



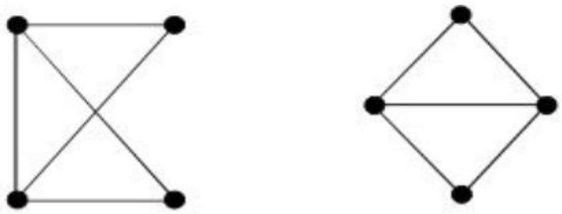




(*iii*)

(a) (b)

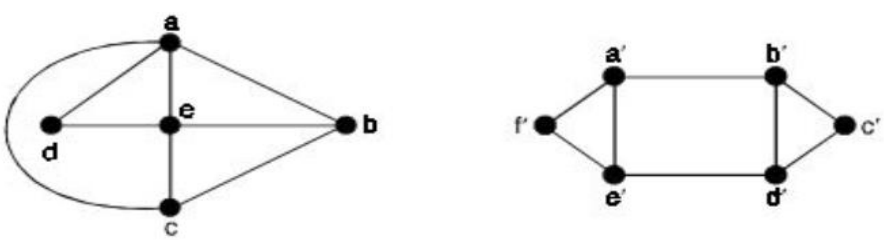
(*iv*)



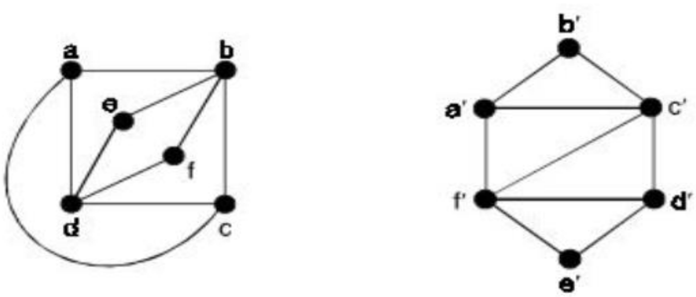
(a) (b)

2. *Are the 2-graphs, given below, isomorphic ? Give a reason.*

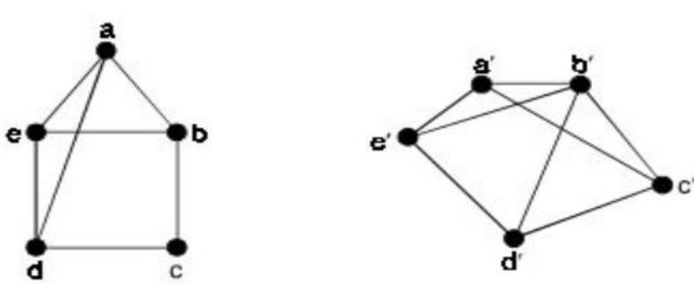
(*i*)



(*ii*)

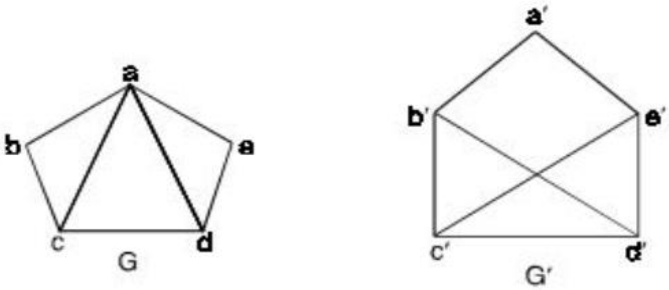


(*iii*)

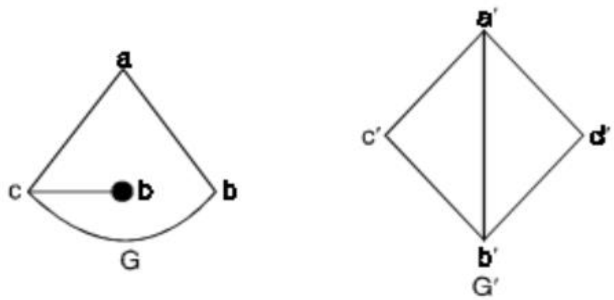


3. *Find whether the following pairs of graphs are isomorphic or not*

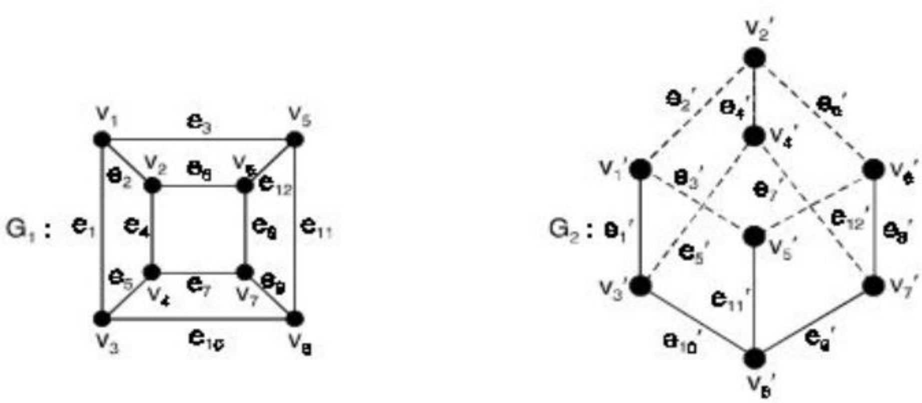
(*i*)



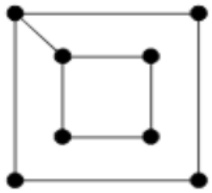
(*ii*)



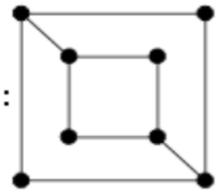
4. *Consider two graphs G1 and G2 as shown below, show that the graphs G1 and G2 are isomorphic.*



G_1 :

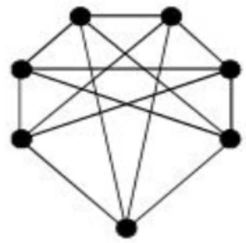


G_2 :



G_1 :

G_2 :



REPRESENTATION OF GRAPHS

Although a diagrammatic representation of a graph is very convenient for a visual study but this is only possible when the number of nodes and edges is reasonably small. Two types of representation are given below :

Matrix representation

Matrices are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. There are number of matrices which one can associate with any graph. We shall discuss adjacency matrix and the incidence matrix.

ADJACENCY MATRIX

Representation of undirected graph

The adjacency matrix of a graph G with n vertices and no parallel edges is an n by n matrix A $= (a_{ij})_{n \times n}$ whose elements are given by

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between } i\text{th and } j\text{th vertices} \\ 0 & \text{if there is no edge between } i\text{th and } j\text{th vertices} \end{cases}$$

Note that for a given graph, the adjacency matrix is based on ordering chosen for the vertices. Hence, there are as many as n ! different adjacency matrices for a graph with n vertices, since there are n ! different ordering of n vertices. However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns.

There are a number of observations that one can make about the adjacency matrix A of a graph G. They are

(i) A is symmetric *i.e.* $a_{ij} = a_{ji}$ for all i and j

(ii) The entries along the principal diagonal of A all zeros if and only if the graph has no self loops. A self loop at the vertex corresponding to $a_{ij} = 1$.

(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.
(*iv*) The (*i, j*) entry of A^m is the number of paths of length m from vertex v_i to vertex v_j.
(*v*) If G be a graph with n vertices v_1, v_2, ... v_n and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix

$$B = A + A^2 + A^3 + ... + A^{n-1}$$

Then G is a connected graph if B has no zero entries of the main diagonal.
This result can be also used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex $v1$ is represented by a 1 at the (*i, j*)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, since the (*i, j*)th entry equals the number of edges these are associated to $\{vi - vj\}$.

All undirected graphs, including multigraphs and pseudo graphs, have symmetric adjacency matrices.

Representation of directed graph

The adjacency matrix of a directed graph D, with n vertices is the matrix $A = (a_{ij})_{n \times n}$ in which

$$aij = \begin{cases} 1 & \text{if arc } \{v_i - v_j\} \text{ is in D} \\ 0 & \text{otherwise} \end{cases}$$

One can make a number of observations about the adjacency matrix of a diagonal.

Observations

(*i*) A is not necessary symmetric, since there may not be an edges from v_i to v_j when there is an edge from v_j to v_i.
(*ii*) The sum of any column of j of A is equal to the number of arcs directed towards v_j

(*iii*) The sum of entries in row i is equal to the number of arcs directed away from vertex v_i (out degree of vertex v_i)
(*iv*) The (*i, j*) entry of A^m is equal to the number of path of length m from vertex v_i to vertex v_j

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero-one matrices when there are multiple edges in the same direction connecting two vertices.

In the adjacency matrix for a directed multigraph a_{ij} equals the number of edges that are associated to (v_i, v_j).