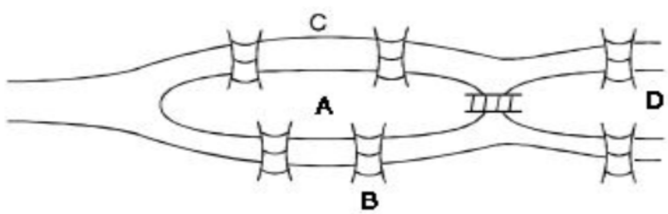
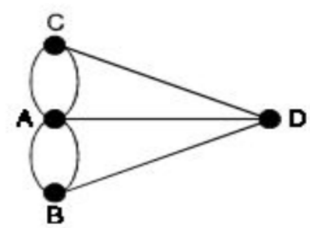
GRAPH THEORY

INTRODUCTION

The concept of graph theory is considered to have originated in 1736 with the publication of Euler's solution of the Konigsberg bridge problem. Euler (1707–1782) is regarded as the father of graph theory.

The Konigsberg Bridge Problem: The city of Konigsberg was located on the Pregel river in Prussia. The river divided the city into four separate landmasses, including the island of Kneiphopf. These four regions were linked by seven bridges as shown in the diagram. Residents of the city wondered if it were possible to leave home, cross each of the seven bridges exactly once, and return home. The Swiss mathematician Leonhard Euler thought about this problem and gave a solution.





The key to Euler's solution was in a very simple abstraction of the puzzle. Let us redraw our diagram of the city of Konigsberg by representing each of the land masses as a vertex and representing each bridge as an edge connecting the vertices corresponding to the land masses. We now have a graph that encodes the necessary information. The problem reduces to finding a "closed walk" in the graph which traverses each edge exactly once, this is called an Eulerian circuit. Euler proved such a circuit does not exist.

Graph theory is the study of points, lines and the ways in which sets of points can be connected by lines or arcs. Graphs in this context differ from the more familiar coordinate plots that portray mathematical relations and functions.

Graph theory has many colourful applications in many branches such as Physics, Chemistry, Communication Science, Computer technology, Electrical and Civil engineering, Architecture, Operations research, Genetics, Sociology, Economics etc.. It has proven useful in the design of integrated circuits (IC s) for computers and other electronic devices. These components more often called chips, contain complex, layered microcircuits that can be represented as sets of points interconnected by lines or arcs. Using graph theory, engineers develop chips with maximum component density and minimum total interconnecting conductor length. This is important for optimizing processing speed and electrical efficiency.

BASIC TERMINOLOGIES OF GRAPHS

A graph is usually denoted as G = (V, E), where V is called the **vertex set** of G and E is the **edge set** of G**.** The elements of the set V are called **vertices** or **points** or **nodes** and the members of the set E are called **edges** or lines or **arcs**.

The number of vertices in a graph G is called the **order of the graph** and is denoted by $|V|$. The number of edges in a graph is called the **size of the graph** and is denoted by $|E|$.

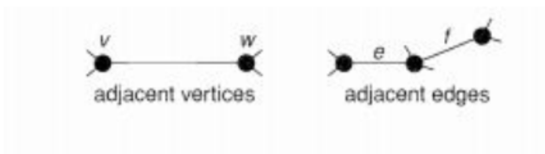
A graph is **finite** if both its vertex set and edge set are finite. Otherwise it is an **infinite graph.** We study only finite graphs, so the term **graph** means only **finite graphs**.

A graph with p vertices and q edges is called a (p, q) graph. A graph with one vertex i.e., a (1, 0) graph is called **trivial graph** and all other graphs are non-trivial. A graph with zero edges i.e., a (p, 0) graph is called **empty or null or void graph**.

Each graph has a diagram associated with it. These diagrams are useful for understanding problems involving such graphs.

Adjacency

Two vertices v and w of a graph G are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. Similarly, two distinct edges e and *f* are adjacent if they have a vertex in common

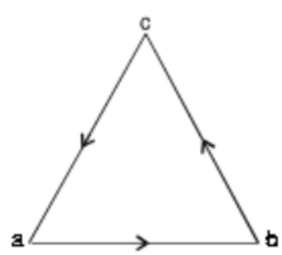


DIRECTED AND UNDIRECTED GRAPHS

Directed graph

A directed graph G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction then the graph is called **directed graph or digraph.**

In the diagram of directed graph, each edge is represented by an arrow or directed curve from initial point to the terminal point.



Suppose $e = (a, b)$ is a directed edge in a digraph, then

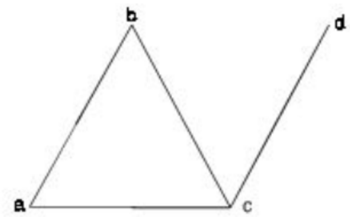
(i) a is called the initial vertex of e and b is the terminal vertex of e

(ii) e is said to be incident from vertex to vertex b.

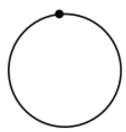
Un-directed graph

An un-directed graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices. In other words, if each edge of the graph G has no direction then the graph is called un-directed graph.

Figure given below is an example of an undirected graph. An edge joining the vertex pair a and b can be referred as either (a, b) or (b, a).



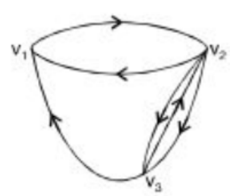
Loop : An edge of a graph that joins a vertex to itself is called loop.
Example:



Multigraph: Two or more edges of a graph G joining the same pair of vertices are called multiple edges or parallel edges. The corresponding graph is called multigraph. In a multigraph no loops are allowed.

Un-directed multigraph

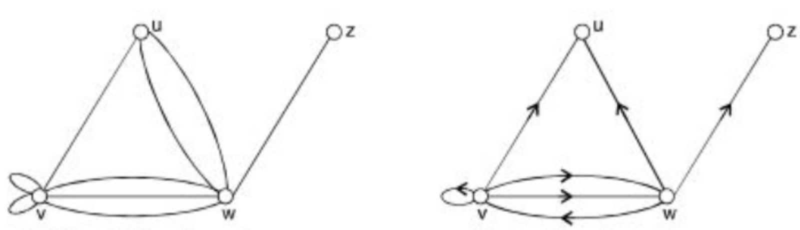
In the above figure there are two parallel edges joining nodes v_2, v_3.



Directed multigraph

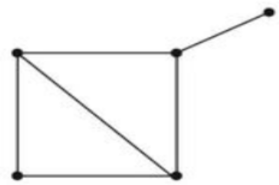
In the above figure there are two parallel edges associated with vertices v_2 and v_3

Pseudo graph: A graph, in which loops and multiple edges are allowed, is called a pseudo graph.



Un-directed Pseudo graph Directed Pseudo graph

Simple graph: A graph with no loops and multiple edges is called a simple graph.

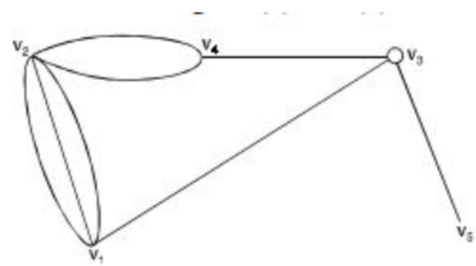


(a) Simple graph

DEGREE OF A VERTEX:

For an undirected graph, the number of edges incident on a vertex v_i with self-loops counted twice is called the degree of a vertex v_i and is denoted by deg (v_i) or deg v_i or d(v_i). The degree of a vertex is also referred to as its **valency.**

For example let us consider the graph G given below. The degrees of vertices are deg (v_1) = 4, deg (v_2) = 5, deg (v_3) = 3, deg (v_4) = 3, and deg (v_5) = 1.



Isolated vertex: A vertex having no incident edge on it is called an isolated vertex. In other words vertex with zero degree is called an isolated vertex.

Pendent vertex or end vertex: A vertex of degree one, is called a pendent vertex or an end vertex and the corresponding edge is called the pendant edge. The vertex to which an end vertex is adjacent is called **support vertex**. In the above Figure, v_5 is a pendent vertex.

Degree Sequence: The vertex degrees of a graph arranged in non-increasing order is called degree sequence of the graph G. The degree sequence of the above graph is 5, 4, 3, 3, 1

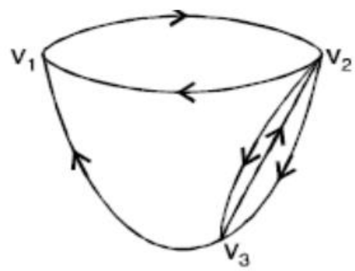
IN DEGREE and OUT DEGREE of a Vertex

In a digraph G, the number of edges beginning at vertex v_i, is called the out degree of a vertex v_i, denoted by $deg_G^+(v_i)$ or out deg (v_i).

In a digraph G, the number of edges ending at vertex v_i, is called the in degree of a vertex v_i, denoted by $deg_G^-(v_i)$ or in deg (v_i).

A vertex with zero in degree is called a **source** and a vertex with zero out degree is called a **sink**.

The sum of the in degree and out degree of a vertex is called the total degree of the vertex.

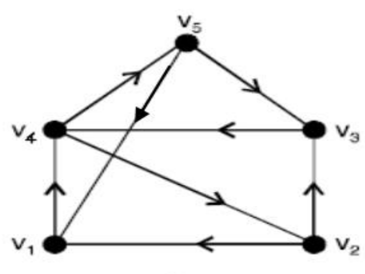


$deg_G^-(v_1) = 2, deg_G^+(v_1) = 1, deg_G^-(2) = 2, deg_G^+(v_2) = 3, deg_G^-(v_3) = 2, deg_G^+(v_3) = 2$

Note: For any directed graph the following property is true

$\sum_{v\in V} deg^-(v) = \sum_{v\in V} deg^+(v) = |E|$

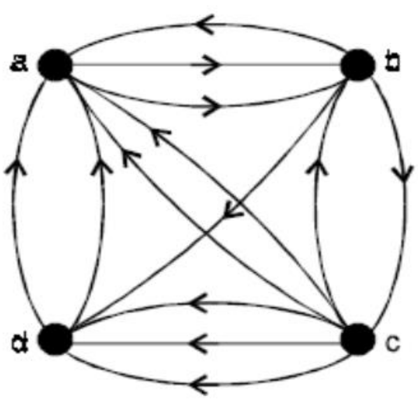
Problem 1. *Find the in-degree and out-degree of each vertex of the following directed graph*



Solution.

in-degree $v1$ = 2, out-degree $v1$ = 1
in-degree $v2$ = 1, out-degree $v2$ = 2
in-degree $v3$ = 2, out-degree $v3$ = 1
in-degree $v4$ = 2, out-degree $v4$ = 2
in-degree $v5$ = 1, out-degree $v5$ = 2

Problem 2. *Find the in-degree and out-degree of each vertex of the following directed graph*



Solution.

in-degree a = 5, out-degree a = 2
in-degree b = 3, out-degree b = 3
in-degree c = 1, out-degree c = 6
in-degree d = 4, out-degree d = 2.

Theorem 1: (THE HANDSHAKING THEOREM)

Statement: If G = (V, E) be an undirected graph with e edges, then $\sum_{v \epsilon V} deg_G(v) = 2e$. i.e., the sum of degrees of the vertices is an undirected graph is even.
(or)
If V = {v_1, v_2, ... v_n} is the vertex set and E is the edge set of a non directed graph G then $\sum_{i=1}^{n} deg_G(v_i) = 2|E|$

Proof :
Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degrees equals twice the number of edges.
Thus $\sum_{i=1}^{n} deg_G(v_i) = 2|E|$

Note : This theorem applies even if multiple edges and loops are present. The above rule can be applied when several people shake hands. The total number of hands shaken must be even that is why the theorem is called handshaking theorem.

Corollary 1: In a non directed graph, the total number of odd degree vertices is even.
Proof :
Let G = (V, E) a non directed graph. Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.
Then $\sum_{v_i \epsilon V} deg_G(v_i) = \sum_{v_i \epsilon U} deg_G(v_i) + \sum_{v_i \epsilon W} deg_G(v_i)$
$\Rightarrow 2e - \sum_{v_i \epsilon U} deg_G(v_i) = \sum_{v_i \epsilon W} deg_G(v_i)$
$\Rightarrow \sum_{v_i \epsilon W} deg_G(v_i)$ is also even
∴ The number of odd vertices in G is even.

Theorem 2: If G is a directed graph, then $\sum_{i=1}^{n} deg_G^+(v_i) = \sum_{i=1}^{n} deg_G^-(v_i) = |E|$
Proof: Since when the degrees are summed, each edge contributes a count of one to the degree of each of the two vertices on which the edge is incident, we get the result.

Corollary 2: If k = δ(G) is the minimum degree of all the vertices of a non directed graph G, then

$$k|V| \leq \sum_{v \epsilon V} deg_G(v) = 2|E|$$

In particular, if G is a k-regular graph, then

$$k|V| = \sum_{v \epsilon V} deg_G(v) = 2|E|$$

Problem 3. Show that the total number of odd degree vertices of a (p, q)-graph is always even.
Solution. Let v_1, v_2 ...... v_k be the odd degree vertices in G.
Then, we have $\sum_{i=1}^{p} deg_G(v_i) = 2q$= even number
$\Rightarrow \sum_{i=1}^{k} deg_G(v_i) + \sum_{i=k+1}^{p} deg_G(v_i) =$ even number
$\Rightarrow \sum_{i=1}^{k} deg_G(v_i)$ = even number $- \sum_{i=k+1}^{p} deg_G(v_i)$
$\Rightarrow \sum_{i=1}^{k} deg_G(v_i)$ = even number – even number
= even number.

⇒ This implies that number of terms in the left-hand side of the equation is even.
Therefore, k is an even number.

Problem 4. Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2.
Solution. Suppose the graph with 6 vertices has e number of edges. Therefore by Handshaking lemma. $\sum_{i=1}^{6} deg_G(v_i) = 2|e|$
⇒ $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$
Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.
Hence the above equation becomes, $(4 + 4) + (2 + 2 + 2 + 2) = 2e$
⇒ $16 = 2e$ ⇒ $e = 8$.
Hence the number of edges in a graph with 6 vertices with given condition is 8.

Problem 5. How many vertices are needed to construct a graph with 6 edges in which each vertex is of degree 2?
Solution. Suppose these are n vertices in the graph with 6 edges. Also given the degree of each vertex is 2.
By handshaking lemma, $\sum_{i=1}^{n} deg_G(v_i) = 2|e| = 2\times 6 = 12$
⇒ $d(v_1) + d(v_2) + \ldots\ldots + d(v_n) = 12$
⇒ $\underbrace{2 + 2 + \cdots + 2}_{n\ times} = 12$
⇒ $2n = 12$
⇒ $n = 6$ vertices are needed.

Problem 6. Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.
Solution: By the handshaking theorem,
$\sum_{i=1}^{n} deg_G(v_i) = 2|e|$ where e is the number of edges with n vertices in the graph G.
⇒ $d(v_1) + d(v_2) + \ldots\ldots + d(v_n) = 2e$ .... (1)
We know that the maximum degree of each vertex in the graph G can be $(n - 1)$.
Therefore, equation (1) reduces $\underbrace{(n-1) + (n-1) + \cdots + (n-1)}_{n\ times} = 2e$
⇒ $n(n - 1) = 2e$
⇒ $e = \frac{n(n-1)}{2}$.
Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

Problem 7. Is it possible to draw a simple graph with 4 vertices and 7 edges ? Justify.
Solution. In a simple graph with n-vertices, the maximum number of edges will be $\frac{n(n-1)}{2}$.
Hence a simple graph with 4 vertices will have at most $\frac{4\times 3}{2} = 6$ edges.
Therefore, a simple graph with 4 vertices cannot have 7 edges.
Hence such a graph does not exist.

Problem 8. Show that there exists no simple graph which has the following degree sequence (i) 0, 2, 2, 3, 4 (ii) 1, 1, 2, 3 (iii) 2, 2, 3, 4, 5, 5 (iv) 2, 2, 4, 6.
Solution. (i) to (iii) : There are odd number of odd degree vertices in the graph. Hence there exists no graph corresponds to this degree sequence.

(iv) Number of vertices in the graph is four and the maximum degree of a vertex is 6, which is not possible as the maximum degree cannot exceed one less than the number of vertices.

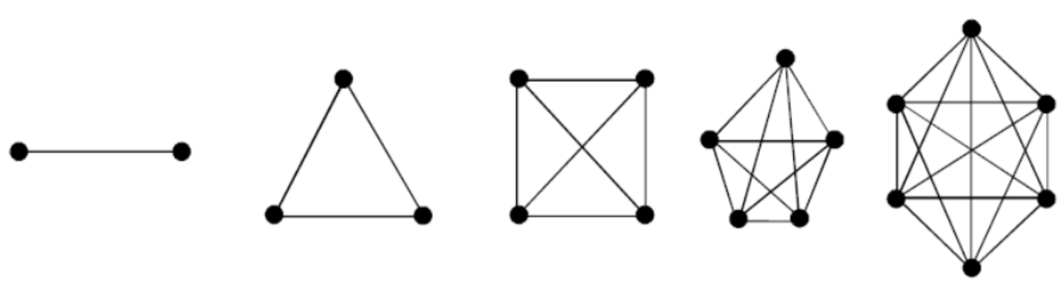
Problem 9. Show that the following sequence 6, 5, 5, 4, 3, 3, 2, 2, 2 is graphical.
Solution:We can reduce the sequence as follows :
Given sequence 6, 5, 5, 4, 3, 3, 2, 2, 2
Reducing first 6 terms by 1 counting from second term 4, 4, 3, 2, 2, 1, 2, 2.
Writing in decreasing order 4, 4, 3, 2, 2, 2, 2, 1
Reducing first 4 terms by 1 counting from second 3, 2, 1, 1, 2, 2, 1
Writing in decending order 3, 2, 2, 2, 1, 1, 1
Reducing first 3 terms by 1, counting from second 1, 1, 1, 1, 1, 1
Sequence 1, 1, 1, 1, 1, 1 is graphical.
Hence the given sequence is also graphical

Problem 10. Show that the sequence 6, 6, 6, 6, 4, 3, 3, 0 is not graphical.
Solution: To prove that the sequence is not graphical.
The given sequence is 6, 6, 6, 6, 4, 3, 3, 0
Resulting the sequence 5, 5, 5, 3, 2, 2, 0
Again consider the sequence 4, 4, 2, 1, 1, 0
Repeating the same 3, 1, 0, 0, 0
Since there exists no simple graph having one vertex of degree three and other vertex of degree one. The last sequence is not graphical.
Hence the given sequence is also not graphical.

SOME SPECIAL GRAPHS:

COMPLETE GRAPH

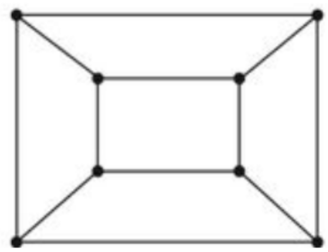
A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex.
i.e., if G contains exactly one edge between each pair of distinct vertices.
A complete graph is usually denoted by **K*n*.** It should be noted that K*n* has exactly $\frac{n(n-1)}{2}$ edges.
The figure given below shows complete graphs K_2 to K_6

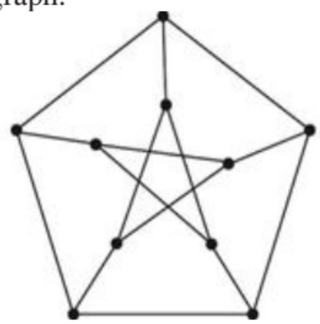


REGULAR GRAPH

A graph in which all vertices are of **equal degree,** is called a **regular graph.**
If the degree of each vertex is *r*, then the graph is called a regular **graph of degree *r*.**
Note 1: Every null graph is regular of degree zero.

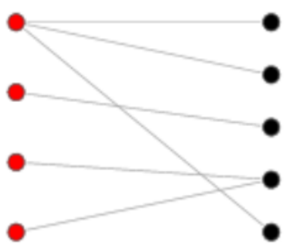
Note 2: The complete graph K_n is a regular of degree $n-1$.
Note 3: If G has n vertices and is regular of degree r, then G has $nr/2$ edges.
Note 4: The figure given below shows regular graphs which are also called as cubic graphs. The second graph is also known as Petersen graph.





BIPARTITE GRAPH

A graph G is said to be **bipartite** if its vertex set can be partitioned into two subsets such that no two vertices in the same partition are adjacent. In other words if the simple graph G(V, E) can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 to a vertex in V_2 and no edge in G connects either two vertices in V_1 or V_2 then G is called a **bipartite graph.** The following figure shows bipartite graph



If each vertex of V_1 is connected with every vertex of V_2 by an edge, Then G is said to be a **complete bipartite graph.** If V_1 contains m vertices and V_2 contains n vertices then the complete bipartite graph is denoted by $K_{m,n}$. The following figure shows bipartite $K_{3,2}$ graph