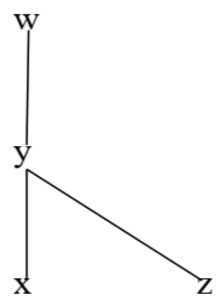
HASSE DIAGRAM

A partial ordering $\leq$ on a finite set P can be represented in a plane by means of a diagram called *Hasse diagram* or a *partially ordered set set diagram* of $\langle P, \leq \rangle$. If $x \ll y$, then we place y above x, and draw a line (edge) between them. The upward direction indicates successor and downward direction indicates the predecessor. And the incomparable elements are in the same horizontal line.



y is immediate successor of x (or) x is immediate predecessor of y.

z is immediate predecessor of y, and x and y are incomparable.

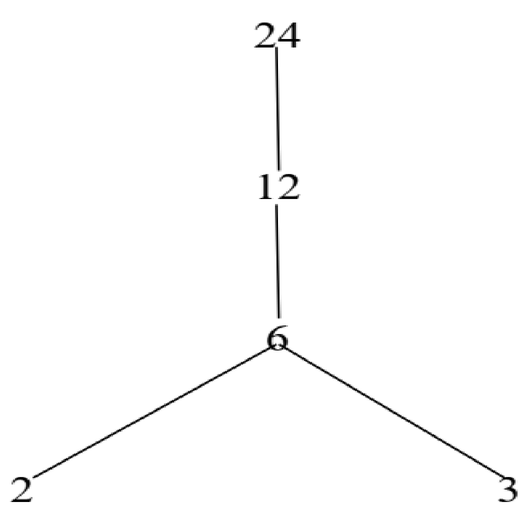
x is predecessor of w but not immediate predecessor.

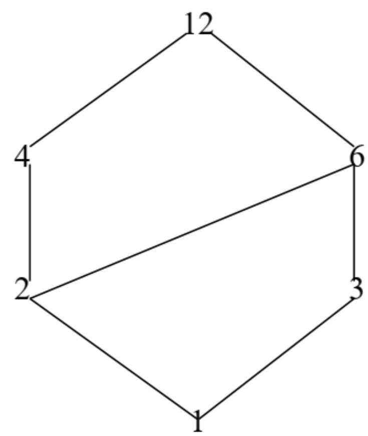
PROBLEMS

1.Let

$P_1 = \{2,3,6,12,24\}$

$P_2 = \{1,2,3,4,6,12\}$ and $\leq$ be a relation such that $x \leq y$ if and only if $x|y$.

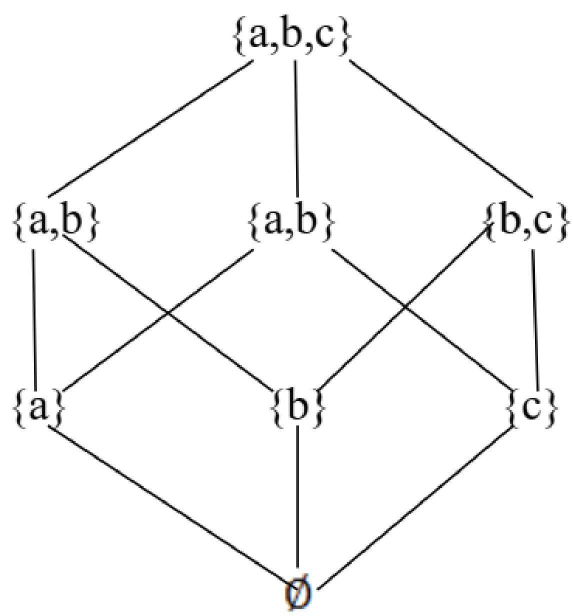




2.Let

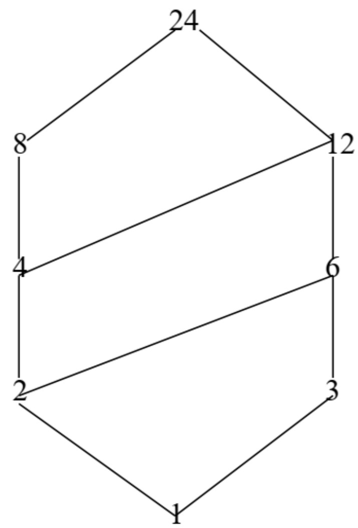
$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\},\}$ be the power set of $\{a,b,c\}$.

Consider the inclusion $(\subseteq)$ relation as the partial ordering on $\rho(A)$, then the Hasse diagram of $\langle \rho(A), \subseteq \rangle$ is



3.Let us consider the set of all divisor of 24, then it is a poset which is denoted by D_{24}

That is $D_{24} = \{1,2,3,4,6,8,12,24\}$ and let the divisor relation be partial ordering.



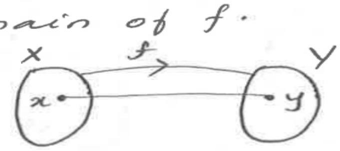
FUNCTIONS

FUNCTIONS OR MAPPINGS:-

A special type of relation is that which associates with each member of the first set, only one member of the second. Such a relation (or) correspondence is called a function from one set into the other. Thus a function is only a special type of relation or correspondence.

DEFINITION:-

Let X and Y are any two nonempty sets. A relation (or) function f from X to Y is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$. $y = f(x)$ is called the image of x. x is called the preimage or an argument. X is the domain of f denoted by D_f and Y is called the co-domain of f.



RANGE:

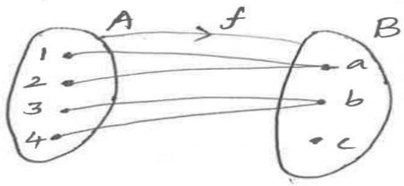
The set of images of all elements of X is called range of f and is denoted by R_f, which gives $R_f \subseteq Y$.

EXAMPLE:-

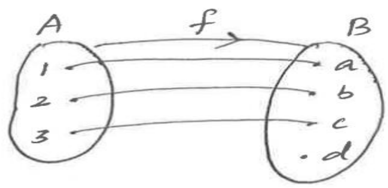
Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c\}$ and $f : A \to B$

such that $f(1) = a = f(2)$

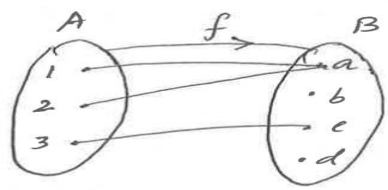
$f(3) = b = f(4)$



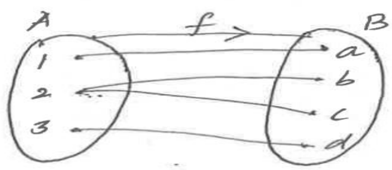
Here $\{a, b\}$ is the range set.



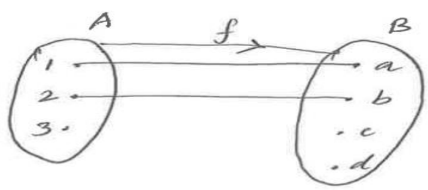
f is a function



f is a function



f is not a function since 2 has two images.



f is not a function since 3 has no image.

TYPES OF FUNCTIONS:-

INJECTIVE:-

A function $f : X \to Y$ is called one-to-one or injective if distinct elements of X are mapped into distinct elements of Y.

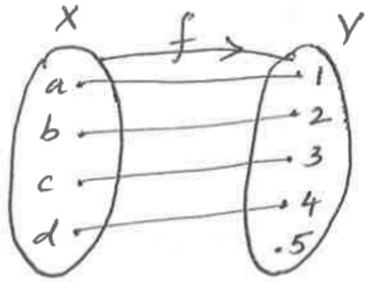
In otherwords, f is 1-1 iff [if and only if]

$$f(x) = f(y) \Rightarrow x = y, \text{ for all } x, y \in X$$

(or)

$$f(x) \neq f(y) \Rightarrow x \neq y$$

EXAMPLE:



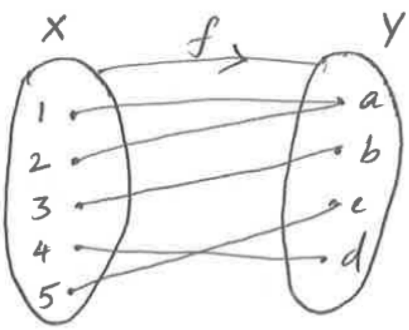
The function f is 1-1, as every element of X has distinct image in Y.

SURJECTIVE FUNCTION:-

A function $f: X \to Y$ is called onto (or) surjective if each element in Y is the image of atleast one element in X. (ie) $R_f = Y$. otherwise it is called **into** function.

In otherwords, a function f is onto iff for every element $y \in Y$, there is an element $x \in X$ such that $f(x) = y$.

EXAMPLE:



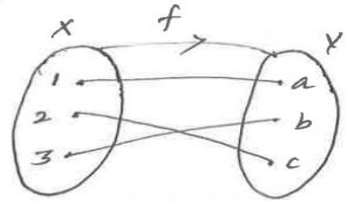
The function f is onto as every element of Y is the image of some element of X.

BIJECTIVE FUNCTION:-

A function $f: X \to Y$ is called bijective (or) bijection (or) one-to-one correspondence if it is both one-to-one and onto.

Obviously if X and Y are finite such that $f: X \rightarrow Y$ is bijective, then X and Y have the same number of elements.

EXAMPLE:-

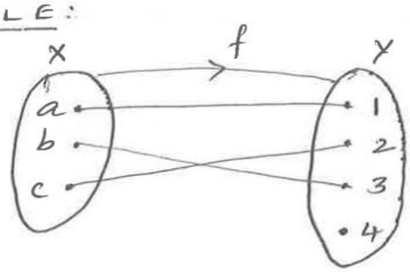


The function f is bijective as f is both one-to-one and onto.

INTO FUNCTION:

A function $f: X \rightarrow Y$ is called an into function if the range of f is not equal to the co-domain Y.

EXAMPLE:

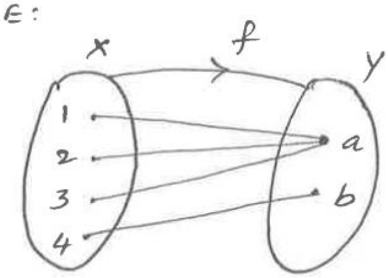


The function f is into function as the range of f $\{1, 2, 3\}$ is not equal to the co-domain $\{1, 2, 3, 4\}$

MANY - ONE FUNCTION:

A function $f: X \rightarrow Y$ is called many-one function if there exists two or more different elements in X having same image in Y.

EXAMPLE:



The function f is many-one function as the elements 1, 2, 3 have the same image a.

COMPOSITION OF FUNCTIONS:-

If $f: A \rightarrow B$ and $g: B \rightarrow C$ then the composi-tion of f and g is a new function from $A \rightarrow C$ denoted by gof is given by

$$(gof)x = g[f(x)], \text{ for all } x \in A.$$

PROPERTIES:

Composition of functions is associative. (i) $f: A \rightarrow B$ $g: B \rightarrow C$ and $h: C \rightarrow D$ are functions then

$$ho(gof) = (hog)of.$$

PROOF:

Since $f: A \rightarrow B$ and $g: B \rightarrow C$, then $gof: A \rightarrow C$

Since $gof: A \rightarrow C$ and $h: C \rightarrow D$, then $ho(gof): A \rightarrow D$

Now $f: A \rightarrow B$ and $(hog): B \rightarrow D$, then $(hog)of: A \rightarrow D$

Thus the domain and co-domain of $ho(gof)$ and those of $(hog)of$ are the same.

Let $x \in A$, $y \in B$ and $z \in C$ such that

$$y = f(x), \quad z = g(y)$$

Then $(g\circ f)x = g[f(x)] = g(y) = z$

$h\circ(g\circ f)x = h(z) \longrightarrow ①$

Also $[(h\circ g)\circ f]x = (h\circ g)f(x)$

$= h[g[f(x)]]$

$= h[g(y)]$

$= h(z) \longrightarrow ②$

$\therefore$ from ① and ② $h\circ(g\circ f) = (h\circ g)\circ f$

If $f : A \to B$ and $g : B \to C$ are functions, then $g\circ f : A \to C$ is an injection, surjection or bijection accordingly as f and g are injections, surjection or bijection.

PROOF:

(i) Let $a_1, a_2 \in A$

then $(g\circ f)(a_1) = (g\circ f)(a_2)$

$g[f(a_1)] = g[f(a_2)]$

$\Rightarrow f(a_1) = f(a_2)$ [$\because$ g is injective

$\Rightarrow a_1 = a_2$ [$\because$ f is injective

$\therefore$ $g\circ f$ is injective.

(ii) Let $c \in C$. Since g is onto, there is an element $b \in B$ such that $c = g(b)$.

Since f is onto there is an element $a \in A$ such that $f(a) = b$.

$$\text{Now } (g \circ f)\, a = g[f(a)] = g(b) = c$$

$\therefore\ g \circ f : A \to C$ is onto

from (i) and (ii) it follows that $g \circ f$ is bijective when f and g are bijective.

IDENTITY FUNCTION:-

The function $f : A \to A$ where $f(x) = x,\ x \in A$ is called the identity function on A.

In otherwords, the identity function is the function that assigns each element of A to itself and is denoted by I_A or simply I.

I_A is bijection.

INVERSE OF A FUNCTION:

The function $g : B \to A$ is called the inverse of $f : A \to B$ if $x = g(y)$ whenever $y = f(x)$. The inverse of f is denoted by f^{-1}. Thus if f^{-1} is the inverse of f then $x = f^{-1}(y)$ where $y = f(x)$.

PROPERTIES:

1. The inverse of a function f, if exists is unique.

PROOF:

Let g and h are inverses of f.

By definition, $g \circ f = I_A$ and $f \circ g = I_B \longrightarrow$ ①

$h \circ f = I_A$ and $f \circ h = I_B \longrightarrow$ ②

Now $h = h \circ I_B = h \circ (f \circ g)$ by ①

$$= (h \circ f) \circ g$$
$$= I_A \circ g$$
$$h = g.$$

Hence the result.

2\. If $f: A \to B$ and $g: B \to C$ are invertible function, then $g \circ f: A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(ie) the inverse of the composition of two functions is equal to the composition of the inverse of the function in the reverse order.

PROOF:

Since $f: A \to B$ and $g: B \to C$ are invertible, they are bijective. Hence $g \circ f: A \to C$ is also bijective.

$\therefore$ $g \circ f$ is also invertible.

$(g \circ f)^{-1}: C \to A$ exists.

Since $g^{-1}: C \to B$ $f^{-1}: B \to A$

$f^{-1} \circ g^{-1}: C \to A$ can be formed.

Thus both $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are functions from $C \to A$.

Now for any $a \in A$, let $b = f(a)$ and $c = g(b)$ $\longrightarrow (1)$

$(g \circ f)a = g[f(a)] = g(b) = c$

$(g \circ f)^{-1}(c) = a \longrightarrow (2)$

By (1) $a = f^{-1}(b)$ and $b = g^{-1}(c)$

$\therefore (f^{-1} \circ g^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a \longrightarrow (3)$

$\therefore$ from (2) and (3) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

PROBLEMS:-

1. Let $f: R \to R$ defined by $f(x) = |x|$, $x \in R$ (R is set of real numbers). Show that

(i) f is not one-to-one (ii) f is not onto.

SOLUTION:

Given $f: R \to R$ such that $f(x) = |x|$

$$\therefore f(x) = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$$

(i) f is not one-to-one since 2, −2 have the same image 2.

(ii) f is not onto, since −1 in co-domain R has no pre-image.

2. Prove that the mapping $f: R \to R$ (R is the set of all real numbers) defined by $f(x) = \frac{1}{x}$ is one-to-one and onto.

SOLUTION:

Let $x, y \in R$

If $f(x) = f(y)$

$$\Rightarrow \frac{1}{x} = \frac{1}{y}$$

$$\Rightarrow x = y$$

$\Rightarrow$ f is one-to-one

Let y be any non-zero real number in codomain R, then $\frac{1}{y}$ is a non-zero number in domain R.

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

3. Check whether the function $f(x) = x^2 - 11$ from R to R is 1-1 ? onto or both?

SOLUTION:

Given $f(x) = x^2 - 11$, $x \in R$

If $f(x) = f(y)$

$\Rightarrow x^2 - 11 = y^2 - 11$

$x^2 = y^2$

$\Rightarrow x = \pm y$

$\therefore f$ is not 1-1

For onto, $\exists\, y \in R$ such that $f(x) = y$

$x^2 - 11 = y$

$x^2 = y + 11$

$x = \pm\sqrt{y+11} \notin R$ for various values of y. $\therefore f$ is not onto.

4. Let $S = \{0,1,2,3,4,5\}$ and $f: S \to S$ defined by $f(x) = 4x \pmod 6$, write f as the set of ordered

pairs and check whether f is 1-1, onto or both.

SOLUTION:

Given $S = \{0, 1, 2, 3, 4, 5\}$

$f(x) = 4x \pmod 6$

$f(x)$ is the remainder when $4x$ is divided by 6, $x \in S$.

Hence $f = \{(0,0)\ (1,4)\ (2,2)\ (3,0)\ (4,4)\ (5,2)\}$

Since $f(0) = 0 = f(3)$, $f(1) = f(4) = 4$

f is not 1-1.

Also the range of $f = \{0, 2, 4\} \neq S$

Hence f is not onto.

5. If $f: R \to R$ and $g: R \to R$ are functions defined by $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$, find $f \circ g$, $g \circ f$, $f \circ f$ & $g \circ g$.

SOLUTION:

$$
\begin{aligned}
(f \circ g)x &= f[g(x)] \\
&= f[2x-3] \\
&= (2x-3)^2 + 3(2x-3) + 1 \\
&= 4x^2 - 12x + 9 + 6x - 9 + 1 \\
&= 4x^2 - 6x + 1
\end{aligned}
$$

$$
\begin{aligned}
(g \circ f)x &= g[f(x)] \\
&= g[x^2 + 3x + 1]
\end{aligned}
$$

$$= 2(x^2+3x+1) - 3$$

$$(g\circ f)x = 2x^2+6x-1$$

$$(f\circ f)x = f[f(x)]$$
$$= f[x^2+3x+1]$$
$$= (x^2+3x+1)^2 + 3(x^2+3x+1) + 1$$
$$= x^4+9x^2+1+6x^3+6x+2x^2 + 3x^2+9x+3+1$$
$$= x^4+6x^3+14x^2+15x+5$$

$$(g\circ g)x = g[g(x)]$$
$$= g[2x-3]$$
$$= 2(2x-3)-3$$
$$= 4x-9$$

6. If $f: R \to R$ and $g: R \to R$ are defined as $f(x) = x^2-2$ and $g(x) = x+4$. Find $g\circ f$ and $f\circ g$, and state whether the functions f&g are injective, surjective and bijective.

SOLUTION:

$$(g\circ f)x = g[f(x)] = g[x^2-2]$$
$$= x^2-2+4$$
$$= x^2+2$$

$$(f\circ g)x = f[g(x)] = f(x+4) = (x+4)^2-2$$

$$= x^2 + 8x + 16 - 2$$

$$(f \circ g)x = x^2 + 8x + 14$$

$$\therefore f \circ g \neq g \circ f.$$

Given that $f(x) = x^2 - 2$

If $f(x) = f(y)$

$$\Rightarrow x^2 - 2 = y^2 - 2$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = \pm y$$

$\therefore f$ is not $1-1$

For onto, for every element $y \in R$, $\exists$ an element $x \in R$ such that $f(x) = y$.

$$x^2 - 2 = y$$

$$\Rightarrow x^2 = y + 2$$

$$\Rightarrow x = \pm\sqrt{y+2} \notin R$$ for various values of y. $\therefore f$ is not onto.

Similarly, given $g(x) = x + 4$

If $g(x) = g(y)$

$$\Rightarrow x + 4 = y + 4$$

$$\Rightarrow x = y$$

$$\Rightarrow g \text{ is } 1-1$$

Let $y \in R$. Suppose $x \in R$ such that

$$g(x) = y$$
$$x + 4 = y$$
$$x = y - 4 \in R$$
$\Rightarrow g$ is onto.

7. If $f : R \to R$ defined by $f(x) = x^3 - 2$ find f^{-1}.

SOLUTION:

Assume $f(x_1) = f(x_2)$, $x_1, x_2 \in R$

$$\Rightarrow x_1^3 - 2 = x_2^3 - 2$$
$$\Rightarrow x_1^3 = x_2^3$$
$$\Rightarrow x_1 = x_2$$
$\Rightarrow f$ is 1-1

Let $y = f(x) = x^3 - 2$

$$\Rightarrow x^3 - 2 = y$$
$$x^3 = y + 2$$
$$x = (y+2)^{1/3}$$

So for every y in co-domain R, $\exists$ an x in domain R such that $x = (y+2)^{1/3}$

then $f(x) = x^3 - 2 = \left[(y+2)^{1/3}\right]^3 - 2 = y$

$\therefore f$ is onto

Hence f is bijective. $\Rightarrow f^{-1}$ exists and

$f^{-1}(y) = (y+2)^{1/3}$.

Hence $f^{-1} : R \to R$ defined by $f^{-1}(y) = (y+2)^{1/3}$.

8. Show that $f: R - \{1\} \to R - \{3\}$ defined by $f(x) = \frac{3x-2}{x-1}$ is bijective. find also its inverse.

SOLUTION:

Given $f(x) = \frac{3x-2}{x-1}$

Let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{3x_1 - 2}{x_1 - 1} = \frac{3x_2 - 2}{x_2 - 1}$$

$$\Rightarrow (3x_1 - 2)(x_2 - 1) = (3x_2 - 2)(x_1 - 1)$$

$$\Rightarrow 3x_1x_2 - 3x_1 - 2x_2 + 2 = 3x_1x_2 - 3x_2 - 2x_1 + 2$$

$$\Rightarrow -3x_1 - 2x_2 = -3x_2 - 2x_1$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is 1-1

Let $y \in R - \{3\}$. We have to find an $x \in R - \{1\}$ such that $f(x) = y$.

$$\Rightarrow \frac{3x-2}{x-1} = y \Rightarrow x = \frac{y-2}{y-3}$$

Thus, for $y \in R - \{3\}$, $\exists$ $x = \frac{y-2}{y-3}$ in $R - \{1\}$ such that $f(x) = \frac{3x-2}{x-1} = \frac{3\left(\frac{y-2}{y-3}\right) - 2}{\frac{y-2}{y-3} - 1} = y$

$\therefore f$ is onto $\Rightarrow f$ is bijection.

Hence f^{-1} exists and is defined by $f^{-1}(y) = \frac{y-2}{y-3} = x$.

Also $f^{-1}: R - \{3\} \to R - \{1\}$

9. Show that $f: Z \to Z$ defined by $f(x) = 3x$ has no inverse.

SOLUTION:-

Here 1 is not the image of any element of Z. suppose, if we take 1 as image then $\exists\, x \in Z$ such that $f(x) = 1$. (i.e) $3x = 1 \Rightarrow x = \frac{1}{3}$ which is contradiction. Since $\frac{1}{3}$ is not an integer.

Hence f is not onto.

Hence f is not bijection

$\Rightarrow f$ has no inverse.

10. Let Z^+ denote the set of all positive integers and Z denote the set of integers. Let $f: Z^+ \to Z$ be defined by $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{1-n}{2} & \text{if } n \text{ is odd} \end{cases}$. prove that f is a bijection and find f^{-1}.

SOLUTION:

Let $f(n_1) = f(n_2)$

If n_1 and n_2 are even

$$\frac{n_1}{2} = \frac{n_2}{2} \Rightarrow n_1 = n_2$$

If n_1 and n_2 are odd.

$$\frac{1-n_1}{2} = \frac{1-n_2}{2}$$

$$\Rightarrow n_1 = n_2$$

If one member is even say n_1, then the other member is odd say n_2.

$$f(n_1) = f(n_2)$$

$$\Rightarrow \frac{n_1}{2} = \frac{1-n_2}{2}$$

$$n_1 = 1 - n_2 \Rightarrow n_1 + n_2 = 1$$

Since $n_1, n_2 \in Z^+$ is in positive integers.

$$\Rightarrow n_1 + n_2 \geq 2$$

$\therefore$ n_1 even and n_2 odd cannot occur.

Hence $n_1 = n_2 \Rightarrow f$ is 1-1

Now let m be an integer in codomain Z, we must find a positive integer n in domain Z^+ such that $f(n) = m$.

If $m > 0$, we have $n = 2m$ (even).

If $m \leq 0$ we have $n = 1 - 2m$ (odd)

Hence f is onto

$\therefore$ f is a bijection.

$$\text{and } f^{-1}(n) = \begin{cases} 2n, & \text{if } n > 0 \\ 1-2n, & \text{if } n \leq 0 \end{cases}$$

11. If $f: R \to R$ and $g: R \to R$ be two functions defined by $f(x) = 2x+1$ for all $x \in R$, $g(y) = \frac{y}{3}$ for all $y \in R$. Verify $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

SOLUTION:

Given $f(x) = 2x+1$ for all $x \in R$

$$g(y) = \frac{y}{3} \text{ for all } y \in R$$

$$(g \circ f)x = g[f(x)] = g(2x+1)$$

$$= \frac{2x+1}{3}$$

Take $(g \circ f)x = y$

$$\Rightarrow y = \frac{2x+1}{3} \Rightarrow \frac{3y-1}{2} = x$$

$$(g \circ f)^{-1}x = \frac{3x-1}{2} \text{ for any } x \in R \longrightarrow (1)$$

Given $f(x) = 2x+1 = y$, for any $y \in R$

$$\Rightarrow x = \frac{y-1}{2}$$

$$f^{-1}(x) = \frac{x-1}{2} \text{ for any } x \in R$$

Also given $g(x) = \frac{x}{3} = y$

$$\Rightarrow x = 3y$$

(ie) $g^{-1}(x) = 3x$, for any $x \in R$

Now $(f^{-1} \circ g^{-1})x = f^{-1}[g^{-1}(x)]$

$$= f^{-1}(3x)$$

$$= \frac{3x-1}{2} \longrightarrow (2)$$

$\therefore$ from (1) and (2) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

12. Let $f : N \times N \rightarrow N \times N$ defined by $f(x,y) = (2x-y, x-2y)$ for all $(x,y) \in N \times N$. Show that f is bijective and find f^{-1}.

(a) verify whether $f \circ g = g \circ f$

(b) Explain why f and g have inverses but h does not.

(c) find f^{-1} and g^{-1}

(d) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

SOLUTION:-

(a) $(f \circ g)(1) = f[g(1)] = f(3) = 4$

$(f \circ g)(2) = f[g(2)] = f(5) = 3$

$(f \circ g)(3) = f[g(3)] = f(1) = 2$

$(f \circ g)(4) = f[g(4)] = f(2) = 1$

$(f \circ g)(5) = f[g(5)] = f(4) = 5$

$\therefore f \circ g = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$

Similarly $g \circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$

$\therefore f \circ g \neq g \circ f$

(b) Both f and g are 1-1 and onto

$\therefore$ They are invertible

$h(1) = h(2) = 2$

But $1 \neq 2$ $\therefore$ h is not 1-1

Also range$(h) = \{1,2,3,4\} \neq S$ $\therefore$ h is not onto

$\therefore h^{-1}$ does not exist.

(c) f^{-1} is obtained by reversing the elements in all the ordered pairs of f.

SOLUTION:

Suppose $f(x_1, y_1) = f(x_2, y_2)$

$$(2x_1 - y_1,\ x_1 - 2y_1) = (2x_2 - y_2,\ x_2 - 2y_2)$$

Then $2x_1 - y_1 = 2x_2 - y_2 \longrightarrow (1)$

$$x_1 - 2y_1 = x_2 - 2y_2 \longrightarrow (2)$$

solving (1) and (2) we get

$$x_1 = x_2\ ;\ y_1 = y_2$$

$$\therefore (x_1, y_1) = (x_2, y_2)$$

$$\Rightarrow f \text{ is } 1-1$$

Let $f(x,y) = (u, v) = (2x - y,\ x - 2y)$

$$\Rightarrow u = 2x - y\ ;\ v = x - 2y$$

Solving we get $x = \frac{2u - v}{3}$, $y = \frac{u - 2v}{3}$ for every

$(u, v) \in N \times N$

$\therefore f$ is onto

$\Rightarrow f$ is bijective

Hence f^{-1} exists.

$$\therefore f^{-1}(x,y) = \left(\frac{2x - y}{3}, \frac{x - 2y}{3} \right)$$

PROBLEMS FOR PRACTICE:-

1. Show that the function $f: R \to R$ defined by $f(x) = 3x-1$ for $x \in R$ is a bijection.

   Ans: f is bijective

2. Let $f: Z \to Z$ (Z, the set of integers) be the function defined by $f(x) = 2x-3$, for $x \in Z$. Check whether f is one-to-one and onto.

   Ans: f is 1-1 and not onto

3. Verify which of the following are 1-1, onto or both. (i) $f: R \to R$ such that $f(x) = 2x+1$, $x \in R$

   (ii) $f: Z^+ \to Z^+$ such that $f(x) = x^2+2$, $x \in Z^+$

   (iii) $f: R \to R$ such that $f(x) = -4x^2+12x-9$, $x \in R$

Ans: (i) f is 1-1 and onto

(ii) f is 1-1 and not onto

(iii) f is not 1-1 and not onto.