CARTESIAN PRODUCT OF SETS

The *Cartesian product* of the sets A and B, is written an $A \times B$, is the set of all ordered pairs in which the first elements are in A and the second elements are in B.

$i.e.\ A \times B = \{\langle x, y\rangle | x \in A \text{ and } x \in B\}$

For example

Let $A = \{1,2\}, B = \{a, b, c\},\ c = \{\alpha, \beta\}$

Now

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 1, c\rangle\langle 2, a\rangle, \langle 2, b\rangle, \langle 3, c\rangle\}$

$A \times C = \{\langle 1, \alpha\rangle, \langle 1, \beta\rangle, \langle 2, \alpha\rangle, \langle 2, \beta\rangle\}$

$A \times B = \{\langle \alpha, a\rangle, \langle \alpha, b\rangle, \langle \alpha, c\rangle\langle \beta, a\rangle, \langle \beta, b\rangle, \langle \beta, c\rangle\}$

It is clear from the definition

$A \times B \neq B \times A$ and $\langle\langle a, b\rangle, c\rangle \in (A \times B) \times C,$ is an ordered triple then $\langle a, b\rangle \in A \times B$ and $c \in C$.

Now , $A \times (B \times C) = \{\langle a, \langle b, c\rangle\rangle\ | a \in A \text{ and } \langle b, c\rangle \in \langle B, C\rangle\}$

Note that $\langle a, \langle b, c\rangle\rangle$ is not an ordered triple.

This fact show that $(A \times B) \times C \neq A \times (B \times C)$

i.e. Cartesian product is not associative.

Now

$A \times A = A^2 = \{\langle x, y\rangle, \forall x, y \in A\}$ and $A^n = A^{n-1} \times A$.

Note that if A has *n* elements and B has *m* elements $A \times B$ has *nm* elements.

PROBLEMS

1. If $A = \{1,2,3\}$, $B = \{a, b\}$. Find $A \times B, B \times A$ and $A \times A$ and $A^2 \times B$

Solution :

$A \times B = \{\langle 1,a\rangle,\langle 1,b\rangle,\langle 2,a\rangle,\langle 2,b\rangle,\langle 3,a\rangle,\langle 3,b\rangle\}$

$B \times A = \{\langle a,1\rangle,\langle a,2\rangle,\langle a,3\rangle,\langle b,1\rangle,\langle b,2\rangle,\langle b,3\rangle\}$

$A^2 = A \times A = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 1,3\rangle,\langle 2,1\rangle,\langle 2,2\rangle,\langle 2,3\rangle,\langle 3,1\rangle,\langle 3,2\rangle,\langle 3,3\rangle\}$

$A^2 \times B = \{\langle 1,1,a\rangle,\langle 1,1,b\rangle,\langle 1,2,a\rangle,\langle 1,2,b\rangle,\langle 1,3,a\rangle,\langle 1,3,b\rangle,\langle 2,1,a\rangle,\langle 2,1,b\rangle,$

$\langle 2,2,a\rangle,\langle 2,2,b\rangle,\langle 2,3,a\rangle,\langle 2,3,b\rangle,\langle 3,1,a\rangle,\langle 3,1,b\rangle,\langle 3,2,a\rangle,\langle 3,2,b\rangle,\langle 3,3,a\rangle,\langle 3,3,b\rangle\}$

2. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (B \cap C) \Leftrightarrow x \in A \; and \; y \in B \cap C$

$\Leftrightarrow x \in A \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{x \in A \; and \; y \in B\} \; and \; \{y \in B \; and \; y \in C\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times B\} \; and \; \{\langle x, y\rangle \in A \times C\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times B) \cap (A \times C)\}$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

3. Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (A \cap B) \times (C \cap D) \Leftrightarrow x \in (A \cap B)\ and\ y \in (C \cap D)$

$\Leftrightarrow\ \{x \in A\ and\ x \in B\}\ and\ \{y \in C\ and\ y \in D\}$

$\Leftrightarrow \{x \in A\ and\ y \in C\}\ and\ \{x \in B\ and\ y \in D\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times C\}\ and\ \{\langle x, y\rangle \in B \times D\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times C) \cap (B \times D)\}$.

ASSIGNMENT PROBLEMS

Part A

1. Define Cartesian product of sets? Given an example?
2. If $A = \{0,1\}$, find A^2.
3. If $A = \{1,2,3\}$ and $B = \{a, b\}$, find $A \times B, B \times A, A^2$.
4. True or False
   I. If $A = \{1,3,5,7,9\}$, the $\{\forall x \in A, x + 2\ is\ a\ prime\ number\}$
   II. If $A = \{1,2,3,4,5\}$, the $\{\exists\ x \in A, x + 3 = 10\}$
5. If $A \times B = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 5,2\rangle, \langle 5,3\rangle\}$

Part B

6. If A,B and C are sets, prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
7. Prove that $(A \times C) - (B \times C) = (A - B) \times C$.
8. If $A = \{a, b\}$ and $B = \{1,2\}$,and $C = \{2,3\}$, find
   I. $A \times (B \cup C)$
   II. $(A \times B) \cup (A \times C)$
   III. $A \times (B \cap C)$
   IV. $(A \times B) \cap (A \times C)$
9. Show that the Cartesian product is not commutative? It is commutative only for equality of sets?

RELATIONS

Binary relation

Any set of ordered pairs defines a *binary relation.*

If x and y are binary related, under the relation R, the we write $\langle x, y\rangle \in R$ or xRy. If not the case we write $\langle x, y\rangle \notin R$.

1. **Example** $F = \{\langle x, y\rangle \mid x \; is \; the \; father \; of \; y\}$

$L = \{\langle x, y\rangle \mid x \; and \; y \; are \; real \; number \; and \; x < y\}$

Then F, L are binary relations.

2.Example Let A and B be any two sets, then any non empty subset R of $A \times B$ is called a *binary relation.*

Now

$A = \{1,2,3\}$

$B = \{a, b\}$ then

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

Let

$R_1 = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

$R_2 = \{\langle 1, b\rangle, \langle 3, a\rangle\}$

$R_3 = \{\langle 2, a\rangle\}$

Then R_1, R_2 and R_3 are binary relations A to B.

Let S be any binary relation. The *domain* of S is the set of all elements x such that for some $y, \langle x, y\rangle \in S$.

$D(S) = \{x \mid \langle x, y\rangle \in S, for \; some \; y\}$

Similarly, the *range* of S is the set of all elements y such that, for some x, $\langle x, y\rangle \in S$

$i.e.$ $\quad R(S) = \{y \mid \langle x, y\rangle \in S, for \; some \; x\}$

Let

$S = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,b\rangle, \langle 3,a\rangle\}$

$D(S) = \{1,2,3\}$

$R(S) = \{a, b\}$

If $S \subseteq X \times Y$, then clearly $D(S) \subseteq X$ and $R(S) \subseteq Y$.

In case of $X = Y$, then the relation defined on $X \times X$ is called *an universal relation* in X.

If $X = \emptyset$, then a relation on $X \times X$ is called *void relation* in X.

Since relations are sets, then we can have their union and intersection and so on.

$R \cup S = \{\langle x, y\rangle \mid xRy \text{ or } xSy\}$

$R \cap S = \{\langle x, y\rangle \mid xRy \text{ and } xSy\}$

$R - S = \{\langle x, y\rangle \mid xRy \text{ and } \langle x, y\rangle \notin S\}$

$R + S = \{\langle x, y\rangle \mid \langle x, y\rangle \text{ is either in } R \text{ or in } S \text{ but not in both}\}$

Properties of Binary relations

1. Reflexive

Let R be a binary relation defined on X.

Then R is *reflexive* if, for every $x \in X, \langle x, y\rangle \in R$.

Example:

Let

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 2,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$ are defined on X.

Then R is reflexive, but S is not reflexive. Since $\langle 2,2\rangle \notin S$ and $2 \in X$.

2. Symmetric

A relation R from X to Y is *symmetric* if every $x \in X$ and $y \in Y$, whenever $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.

That is, if $xRy \Rightarrow yRx$, then R is symmetric

Example:

Let

$X = \{1,2\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$ are defined on X.

Then R is symmetric, but S is not symmetric. Since $\langle 1,2\rangle \in S$ but $\langle 1,2\rangle \notin S$.

3. Transitive

A relation R is *transitive* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$.

That is, if $xRy \wedge yRz$, then R is transitive.

Example:

Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,3\rangle, \langle 2,1\rangle\}$

Then R is transitive, but S is not transitive. Since $\langle 2,1\rangle \in S$ and $\langle 1,2\rangle \in S$ but $\langle 2,2\rangle \notin S$.

4.Irreflexive

A relation R in a set X is *irreflexive* if, for every $x \in X, \langle x, x\rangle \notin R$.

Example:

Let

$A = \{1,2,3\}$

$R = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 2,2\rangle, \langle 1,3\rangle\}$

Then R is irreflexive, but S is not reflexive. Since $\langle 3,3\rangle \notin S$ and $\langle 1,1\rangle \in S$.

5. Antisymmetric

A relation R in a set X is *antisymmetric* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then$x = y$.

That is, if $xRy \wedge yRx \Rightarrow x = y$, then R is antisymmetric.

Example:

Let

X be the set of all subsets of E.

R be the inclusion relation ($\subseteq$) defined on X.

$A \subseteq B \wedge B \subseteq A \Rightarrow A = B$

Therefore R is antisymmetric in X.

6. Relation matrix

Let $X = \{x_1, x_2, \dots x_m\}$, $Y = \{y_1, y_2, \dots y_m\}$ are ordered sets, R be a relation defined from X to Y, then the *relation matrix* of R, is defined as

$M_R = (r_{ij})\ i : 1 \to m, j: 1 \to n$

Example 1:

Let $X = \{1,2,3\}\ Y = \{a, b\}$

$R = \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 2,a\rangle, \langle 3,b\rangle\}$ be a relation from X to Y. Then $M_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2: Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$ be a relation on $X = \{1,2,3\}$.

Then $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

7. Composition of Binary Relations

The concept of composition of relation is different from union and intersection of two relations.

Definition:

Let R be a relation from X to Y and S be a relation from Y to Z. Then the composite $R \circ S$ is a relation from X to Z defined by

The operation $\circ$ in $R \circ S$ is called *“ composition of relations”*.

Example.

Let

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 2,2\rangle\}$

$S = \{\langle 2,3\rangle, \langle 4,1\rangle, \langle 4,3\rangle, \langle 2,1\rangle\}$. Then

$R \circ S = \{\langle 1,3\rangle, \langle 1,1\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$

$S \circ R = \{\langle 2,4\rangle, \langle 4,2\rangle, \langle 4,4\rangle, \langle 2,2\rangle\}$

Note that

$R \circ R = R^2$

$R \circ R \circ R = R^2 \circ R = R^3$

$R^{n-1} \circ R = R^n$ etc.,

Definition:

The relation matrix for $R \circ S$ is given by $M_{R \circ S} = M_R \odot M_S$ where $\odot$ is defined as follows.

$M_R \odot M_S = \langle m_{ij} \rangle$ where $m_{ij}(\langle i, j \rangle th\ \ element)$ is 1 if and only if row i of M_R and column j of M_S have a 1 in the same relative position k, for some k.

Example:

Let

$R = \{\langle 1,2 \rangle, \langle 1,5 \rangle, \langle 2,2 \rangle, \langle 3,4 \rangle, \langle 5,1 \rangle, \langle 5,5 \rangle\}$

$S = \{\langle 1,3 \rangle, \langle 2,5 \rangle, \langle 3,1 \rangle, \langle 4,2 \rangle, \langle 4,4 \rangle, \langle 5,2 \rangle, \langle 5,3 \rangle\}$. Then

$$\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
M_S &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
M_{R \circ S} &= M_R \odot M_S \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
\text{and} \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$R^2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,5\rangle, \langle 2,2\rangle, \langle 5,1\rangle, \langle 5,2\rangle, \langle 5,5\rangle\}$

Definition

Let R be a relation from X to Y. The *converse* of R, is written as $\tilde{R}$, is a relation from Y to X such that$xRy \Leftrightarrow x\tilde{R}y$.

Example:

If $R = \{\langle 1,a\rangle, \langle 2,b\rangle, \langle 2,a\rangle, \langle b,3\rangle$

$\tilde{R} = \{\langle a,1\rangle, \langle b,2\rangle, \langle a,2\rangle, \langle b,3\rangle$

Also it is clear that

1. $R = S \Leftrightarrow \tilde{R} = \tilde{S}$
2. $R \subseteq S \Leftrightarrow \tilde{R} \subseteq \tilde{S}$
3. $\widetilde{R \cup S} = \tilde{R} \cup \tilde{S}$

Result: The relation matrix $M_{\tilde{R}}$ is the transpose of the relation M_R.

$i.e. M_{\tilde{R}} = transpose\ of\ M_R$

Example:

Let

$R = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle$

$\tilde{R} = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 1,3\rangle, \langle 3,3\rangle$

We have

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{\tilde{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[M_R]^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{\tilde{R}}$$

EQUIVALENCE RELATION

Definition:

A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 1:

Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,4\rangle, \langle 4,1\rangle, \langle 4,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$ is an equivalence relation on X.

Example 2:

Equality of subsets on a universal set is an equivance relation.

Example 3:

Let

$X = \{1,2,3, \dots 7\}$

$R = \{\langle x, y\rangle \mid x - y \; is \; divisible \; by \; 3\}$

Now, $\forall x \in X, x - x = 0$ is divisible by 3.

Therefore $\forall x \in X, \langle x, x\rangle \in R$ (reflexive)

For any $x,y \in X$

Let $\langle x, x\rangle \in R \Rightarrow x - y$ is divisible by 3 we have $-(x - y) = y - x$ is also divisible by 3.

$\langle y, x\rangle \in R$ (symmetric)

Let $\langle x, y\rangle \in R \wedge \langle y, z\rangle \in R$

$\Rightarrow x - y$ is divisible by 3 and $y - z$ is divisible by 3.

$\Rightarrow (x - y) + (y - z)$ is divisible by 3.

$\Rightarrow x - z$ is divisible by 3.

Therefore $\langle x, y \rangle \in R$ (Transitive)

Therefore R is an equivalence relation on X.

EQUIVALENCE CLASSES

Definition:

Let R be an equivalence relation on a set X. For any $x \in X$, the set $[x]_R \subseteq X$ given by

$[x]_R = \{y \mid xRy \; for \; y \in X\}$

is called an R-*equivalence class* generated by $x \in X$.

Therefore, an equivalence class $[x]_R$ of $x \in X$ is the set of all elements which are related to *x* by an equivalence relation R on X.

Example:

Let Z be the set of all integers and R be the relation called *"congruence modulo 4"* defined by

$R = \{\langle x, y \rangle \mid (x - y) \; is \; divisible \; by \; 4, for \; x \; and \; y \in Z\}$ (or $x \equiv y (mod \; 4)$)
Now, we determine the equivalence classes generated by R.

$[0]_R = \{\ldots - 8, -4, 0, 4, 8 \ldots\}$

$[1]_R = \{\ldots - 7, -3, 1, 5, 9 \ldots\}$

$[2]_R = \{\ldots - 6, -2, 2, 6, 10 \ldots\}$

$[3]_R = \{\ldots - 5, -1, 3, 7, 11 \ldots\}$

Note that

$[0]_R = [4]_R, [1]_R = [5]_R, \ldots etc.$

Therefore $\frac{Z}{R} = \{[0]_R, [1]_R, [2]_R, [3]_R\}$

In a similar manner, we get the equivalence classed generated by the relation *"congruence modulo m"* for any integer *m*.

Therefore, an equivalence relation R on X, will divide the set X into an *equivalence classes,* and they are called *portion* of X.

PARTIAL ORDERED RELATION

A relation R on a set X is said to be a partial ordered relation, if R satisfies reflexive, antisymmetric, and transitive.

Example:

Let $\rho(A)$ be the power set of a set A.

Define a subset relation $(\subseteq)$ on $\rho\ (A)$, then $\subseteq$ is a partial ordered relation.

Usually we denote the partial ordered relations as $' \leq '$ is said to be *partially ordered set* (or) *poset,* which is denoted by $\langle X, \leq \rangle$. We will study more about posets in the subsequent sections.

1. Closures of a relation

Let R be a relation on the set X.

2. Reflexive closure

We have the relation R is reflexive if and only if the relation.

$R = \{\langle x, y \rangle \mid \forall\ x \in X\}$ is contained in R.

i.e. R is reflexive $\Leftrightarrow I \subset R$.

Definition:

Let R be a relation on X, then the smallest reflexive relation on X, containing R, is called *reflexive closure* of R.

Therefore $R_1 = R \cup I$ is the reflexive closure of R.

3. Symmetric closure

We have, the relation R is symmetric if $\langle x, y\rangle \in R \Leftrightarrow \langle y, x\rangle \in \tilde{R}$

$i.e. \tilde{R} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R\}$

Definition:

Let R be a relation X, then smallest symmetric relation on X, containing R, is called the *symmetric closure* of R.

Therefore $R \cup \tilde{R}$ is the symmetric of R.

4. Transitive closure

We have, the relation R is transitive, if $\langle x, y\rangle \in R \; and \; \langle y, z\rangle \in R$ then $\langle x, z\rangle \in R$.

Definition:

A relation R^+ is said to be the *transitive closure* of the relation R on X if R^+ is the smallest transitive relation on X, containing R,

i.e R^+ is the transitive closure of R, if

I. $R \subseteq R^+$
II. R^+ is transitive on X
III. There is no transitive relation R_1 on X, such that $R \subset R_1 \subset R^+$

Remarks:

1. The transitive closure of R can be obtained by

$$R^+ = R \cup R^2 \cup R^3 \cup \ldots = \bigcup_{i=1}^{\infty} R^i$$

2. We know that $\langle x, z \rangle \in R^2$ if and only if there is an element y such that $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$.

   Therefore $\langle a, b \rangle \in R^n$ if and only if we can find a sequence $x_1, x_2, \ldots . x_{n-1}$ in X such that $\langle a, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots \langle x_{n-1}, b \rangle$ are all in R.

   The sequence $a, x_1, x_2, \ldots . x_{n-1}, b$ is said to be a *chain* of length *n* from a to b in R. Here $x_1, x_2, \ldots . x_{n-1}$ are called interval vertices of the chain in R. Note that the interval vertices need not be distinct.

PROBLEMS

1. If $P = \{\langle 1,2 \rangle, \langle 2,4 \rangle, \langle 3,4 \rangle\}$, $Q = \{\langle 1,3 \rangle, \langle 2,4 \rangle, \langle 4,2 \rangle\}$

Find (i) $P \cup Q, P \cap Q, \tilde{P}, \tilde{P} \cup Q$ (ii) domains of $P, P \cup Q, P \cap Q$ and (iii) ranges of $Q, P \cup Q, P \cap Q$.

Solution:

$P \cup Q = \{\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,4 \rangle, \langle 3,4 \rangle, \langle 4,2 \rangle\}$

$P \cap Q = \{\langle 2,4 \rangle\}$

$\tilde{P} = \{\langle 2,1 \rangle, \langle 4,2 \rangle, \langle 4,3 \rangle\}$

$\tilde{P} \cup Q = \{\langle 1,3 \rangle, \langle 2,4 \rangle, \langle 4,2 \rangle, \langle 2,1 \rangle, \langle 4,3 \rangle\}$

Domain of $P = \{1,2,3\}$

Domain of $(P \cup Q) = D(P \cup Q) = \{1,2,3,4\}$

Domain of $(P \cap Q) = D(P \cap Q) = \{2\}$

Range of $Q = R(Q) = \{2,3,4\}$

Range of $(P \cup Q) = R(P \cup Q) = \{2,3,4\}$

Range of $(P \cap Q) = R(P \cap Q) = \{4\}$

It is clear that

$D(P \cup Q) = D(P) \cup D(Q)$ and

$R(P \cap Q) \subseteq R(P) \cap R(Q)$

In general $D(P) = R(\tilde{P})$ and $R(P) = D(\tilde{P})$.

2.Let $X = \{1,2,3,4\}$ and $R = \{\langle x,y\rangle \mid x,y \in X \; and \; (x-y) \; is \; an \; integeral$

$non \; zeromultiple \; of \; 2\}$ $S = \{\langle x,y\rangle \mid x,y \in X \; and \; (x-y) \; is \; an \; integeral$

$non \; zeromultiple \; of \; 3\}$. Find $R \cup S$ and $R \cap S$?

Solution:

Given that $R = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$ and

$S = \{\langle 1,4\rangle, \langle 4,1\rangle\}$ $R \cup S = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 4,1\rangle, \langle 4,2\rangle\}$

$R \cap S = \emptyset$

Remarks:

$D(R) = \{1,2,3,4\}$

$R(R) = \{1,2,3,4\}$

$D(S) = \{1,4\}$

$R(S) = \{1,4\}$

3.Let $S = \{\langle x, x^2\rangle \mid x \in N\}$ and $T = \{\langle x, 2x\rangle \mid x \in N\}$, where $= \{0,1,2,\ldots.\}$. Find the range of S and T, find $S \cup T$ and $S \cap T$?

Solution:

$S = \{\langle x, x^2 \rangle \mid x \in N\}$

$= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,4\rangle, \langle 3,9\rangle, \langle 4,16\rangle, \dots\dots\}$ and

$T = \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle 0,0\rangle, \langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 4,8\rangle, \dots\dots\}$

$R(S) = \{x^2 \mid x \in N\}$

$= \{0,1,4,9,16,25 \dots\dots\}$

$R(T) = \{2x \mid x \in N\}$

$= \{0,2,4,6,8,10, \dots\dots\}$

$S \cup T = \{\langle x, x^2 \rangle \mid x \in N\} \cup \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = x^2\ (or) 2x\}$

$= \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 3,9\rangle, \dots\dots\}$

$S \cap T = \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = 2x\ and\ y = x^2\}$

(Now $y = 2x$ and $y = x^2 \Rightarrow 2x = x^2\ i.e.\ x = 0\ or\ x = 2$

$x = 0\ y = 0\ and\ x = 2 \Rightarrow y = 4$)

$S \cap T = \{\langle 0,0\rangle, \langle 2,4\rangle\}$

4. Given an example which is neither reflexive nor irreflexive?

Solution:

Let $X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,4\rangle\}$

Then R is not reflexive, since $\langle 2,2\rangle \notin R$, for $2 \in X$ and R is not irreflexive, since $1 \in X$, and $\langle 1,1\rangle \in R$.

5. Test whether the following relations are transitive or not on

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle\}$

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$

$T = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$.

Solution: The relation R and T are transitive.

Since, in R, we have $\langle 1,1\rangle \in R$, then check any other pair starting with $\langle 1,z\rangle \in R$, then we must have $1R1 \wedge 1Rz \Rightarrow 1Rz$ i.e., $\langle 1,z\rangle \in R$, but there is no pair staring with 1. So, pass on to next pair $\langle 2,2\rangle$ then we check any other pair starting with 2, and so on.

In T, we have $\langle 1,1\rangle \in T$, then there are two pairs $\langle 1,2\rangle$ and $\langle 1,3\rangle$ must be the transitive of $\langle 1,1\rangle \in T$, then we must have $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in T. Then pass to $\langle 1,2\rangle \in T$ the transitive pairs are $\langle 2,1\rangle, \langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must have the pairs $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle$ in T.

Then pass to $\langle 1,3\rangle \in T$, find the transitive pairs of $\langle 1,3\rangle$ and so on, for all pairs in T. Hence T is a transitive relation.

The relation S is not transitive, since for $\langle 1,2\rangle \in S$, the transitive pairs are $\langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in S but $\langle 1,3\rangle \notin S$.

6. Let R denotes a relation on the set of pairs of positive $N \times N$ integers such that $\langle x,y\rangle R\, \langle u,v\rangle$ if and only if $xv = yu$. Show that R is an equivalence relations.

Solution:

Let

$P = \{\langle x,y\rangle \mid x \text{ and } y \text{ are positive integer}\}$

Now R is a relation defined on P as

$\langle x,y\rangle R \langle u,v\rangle \Leftrightarrow xv = yu$ for $\langle x,y\rangle, \langle u,v\rangle \in P$.

Let $\langle x,y\rangle, \langle u,v\rangle$ and $\langle m,n\rangle \in P$.

I. R is reflexive:

We have

$\langle x,y\rangle R \langle x,y\rangle \Leftrightarrow xy = yx$ (RHS) is true.

II. R is symmetric:

Let $\langle x,y\rangle R \langle u,v\rangle \Leftrightarrow xv = yu$

$\Leftrightarrow yu = xv$

$\Leftrightarrow uy = vx$

$\Leftrightarrow \langle u,v\rangle R \langle x,y\rangle$

III. R is transitive:

Let $\langle x,y\rangle R \langle u,v\rangle$ and $\langle u,v\rangle R \langle m,n\rangle$

$\Leftrightarrow (xv = yu)$ and $(un = vm)$

$\Leftrightarrow (xv = yu)$ and $(u = \frac{vm}{n})$

$\Leftrightarrow xv = y(\frac{vm}{n})$

$\Leftrightarrow xn = ym$

$\Leftrightarrow \langle u,v\rangle R \langle m,n\rangle$

Therefore R is reflexive, symmetric , and transitive.

Hence R is an equivalence relation.

7. Let R and S are equivalence relations on X, show that $R \cap S$ also equivalent? Whether $R \cup S$ is also an equivalent relation. If not given an example.

Solution:

Given let R and S are equivalence relations on X.

Let x, y and $z \in X$.

(i) We have $\langle x, x\rangle \in R$ and $\langle x, x\rangle \in S \Rightarrow \langle x, x\rangle \in R \cap S,\ \forall\, x \in X$.

Therefore $R \cap S$ is reflexive.

(ii)Let $\langle x, y\rangle \in R \cap S \Rightarrow \langle x, y\rangle \in R$ and $\langle x, y\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R$ and $\langle y, x\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R \cap S$

Therefore $R \cap S$ is symmetric.

(iii) Let $\langle x, y\rangle \in R \cap S$ and $\langle y, z\rangle \in R \cap S$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle x, y\rangle \in S)$ and $(\langle y, z\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$ and $(\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow \langle x, y\rangle \in R$ and $\langle x, z\rangle \in S$

$\Rightarrow \langle x, z\rangle \in R \cap S$

Therefore $R \cap S$ is transitive.

Hence $R \cap S$ is equivalence.

8. Prove that the relation *"congruence modulo m"* over the set of positive integers is an equivalence relation?

Show also that if $x_1 = y_1$ and $x_2 = y_2$ then $(x_1 + x_2) = (y_1 + y_2)$.

Solution:

Let N be the set of all positive integers we have *"congruence modulo m"* relation on N as $x \equiv y (mod\ m) \Leftrightarrow m |\ x - y$, for $x, y \in N$.

Let $x, y, z \in N$

(i) We have

$x - x = 0 = 0m$

Therefore $x \equiv x \ (mod\ m)$ for $x \in N$.

"Congruence modulo m" is reflexive.

(ii)Let

$x \equiv y (mod\ m)$

$\Rightarrow\ m |\ x - y$

$\Rightarrow x - y = km$, for some integer $k \in Z$

$\Rightarrow y - x = (-k)m$, for some integer $-k \in Z$

$\Rightarrow y \equiv x \ (mod\ m)$

"congruence modulo m" is symmetric on N.

(iii) Let

$x \equiv y(mod\ m)$ and $y \equiv z \ (mod\ m)$

$\Rightarrow x - y = k_1 m$, and $y - x = k_2 m$ for some integer $k_1, k_2 \in Z$

$\Rightarrow (x - y) + (y - z) = (k_1 + k_2)m$

$\Rightarrow x - z = (k_1 + k_2)m$ for some integer $k_1 + k_2$

$\Rightarrow x \equiv z \ (mod\ m)$

"Congruence modulo m" is transitive on N.

Hence *"congruence modulo m"* is an equivalence relation.

Let $x_1 \equiv y_1 (mod\ m)$ and $x_2 \equiv y_2 (mod\ m)$.

Then $m |\ x_1 - y_1$ and $m |\ x_2 - y_2$

i.e., $x_1 - y_1 = k_1 m$ and $x_2 - y_2 = k_2 m$

Now

$(x_1 - y_1) + (x_2 - y_2) = k_1 m + k_2 m$

$(x_1 + x_2) - (y_1 + y_2) = (k_1 + k_2)m$

$\Rightarrow m|(x_1 + x_2) - (y_1 + y_2)$

$(x_1 + x_2) \equiv (y_1 + y_2)(mod\, m)$

9. Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$ be a relation defined on A. Find the transitive closure of R?

Solution:

The matrix of the relation R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = M_R \odot M_R$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M_{R^3} = M_{R^2} \odot M_R$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$
\begin{aligned}
M_{R^4} &= M_{R^3} \odot M_R \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

As $|A| = 4$, we get

$$
\begin{aligned}
M_{R^+} &= M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

Hence

$R^+ = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$

ASSIGNMENT PROBLEMS

Part -A

1. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 2,2\rangle\}$ and $S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 1,3\rangle, \langle 3,3\rangle\}$ be any relations on $X = \{1,2,3\}$. Find $R \cup S, R \cap S, \tilde{R}, R(R), R(\tilde{S}), D(R \cup S), R(R \cap S)$.
2. Give an example for reflexive, symmetric, transitive and irreflexive relations.
3. Give an example of a relation which is neither reflexive nor irreflexive.
4. Give an example of a relation which is neither symmetric not antisymmetric?
5. Find the graph of the relation $R = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

6. Find the relation matrix of
   $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$
7. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$ and
   $= \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$. Find $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$, $R \circ R \circ S$ and $S \circ S \circ S$?
8. Define equivalence relation and equivalence classes?
9. Define Poset?
10. Define reflexive closure?
11. Define transitive closure of the relation R?
12. Let $R = \{\langle 1,2\rangle, \langle 3,5\rangle, \langle 6,1\rangle, \langle 6,3\rangle, \langle 6,4\rangle\}$ be a relation $A = \{1,2,3,4,5,6\}$. Identify the root of the tree of R.
13. Determine whether the relation R is a partial ordered on the set Z, where Z is set of positive integer, and aRb if and only if a=2b.
14. The following relations are on $\{1,3,5\}$. Let R be a relation, xRy if and only if $y = x + 2$, and let S be a relation, xSy if and only if $x \leq y$. Find $R \circ S$ and $S \circ R$?
15. True or False: The relation $<$ on Z^+ is not a partial order since it is not reflexive.

Part B

1. Show that the intersection of equivalence relations is an equivalence relation.
2. Determine whether the relations represented by the following zero-one matrices are equivalence relations.

$$a)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad b)\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. If R and S are symmetric, show that $R \cup S$ and $R \cup S$ are symmetric.
4. Let L be set of all straight lines in the Euclidean plane and R be the relation in L defined by $xRy \Leftrightarrow x$ is perpendicular to y. Is R is Reflexive? Symmetric? Antisymmetric? Transive?
5. Consider the subsets $A = \{1,7,8\}$, $B = \{1,6,9,10\}$ and $C = \{1,9,10\}$ where $E = \{1,2,3 \dots \dots 10\}$ is an universal set. List the non empty minsets generated by A,B and C . Do they form a partition on E?
6. Let $X = \{1,2,3, \dots ..20\}$ and $R = \{\langle x,y\rangle \mid x - y \textit{ is divisible by } 5\}$ be a relation on X. Show that R is an equivalent relation and find the partition of X induced by R.
7. If R is an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes constitute a partition on A.
8. Given the relation matrix M_R and M_S. Explain how to find $M_{R \circ S}$, $M_{S \circ R}$ and M_{R^2}?
9. Let A be s set of books. Let R be a relation on A such that $\langle a,b\rangle \in R$ if ' book a' with cost more and contains fever pages then ' book b'. In general, is R reflexive? Symmetric? Antisymmetric? Transitive?
10. Let R be a binary relation on the set of all positive integers such that $R = \{\langle a,b\rangle \mid a = b^2\}$. Is R reflexive? Symmetric? Antisymmetric? Transitive? An equivalence relation?