

Course Material

COURSE OBJECTIVE: Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

Unit 2: SET THEORY AND GROUP THEORY

Basic concepts of Set theory - Laws of Set theory - Partition of set, Relations - Types of Relations: Equivalence relation, Partial ordering relation - Graphs of relation - Hasse diagram, Functions: Injective, Surjective, Bijective functions, Compositions of functions, Identity and Inverse functions. Groups - Properties of groups - Semi group and Monoid (definition and examples only) - Subgroups.

SET THEORY

The concept of a set is used in various disciplines and particularly in computers.

Basic Definition:

1. "A collection of well defined objects is called a set".

The capitals letters are used to denote sets and small letters are used for denote objects of the set. Any object in the set is called element or member of the set. If x is an element of the set X , then we write $x \in X$, to be read as 'x belongs to X ', and if x is not an element of X , the we write $x \notin X$ to be read as 'x does not belongs to X '.

2. The number of elements in the set A is called *cardinality* of the set A , denoted by $|A|$ or $n(A)$. We note that in any set the elements are distinct. The collection of sets is also a set.

$$S = \{P_1, \{P_2, P_3\}, P_4, P_5\}$$

Here $\{P_2, P_3\}$ itself one set and it is one element of S and $|S|=4$.

3. Let A and B be any two sets. If every element of A is an element of B , then A is called a *subset* of B is denote by ' $A \subseteq B$ '.

We can say that A *contained (included)* in B , (or) B *contains (includes)* A .

Symbolically, $A \subseteq B$ (or) $B \supseteq A$

Logically, $A \subseteq B = (x\forall)\{x \in A \rightarrow x \in B\}$

Let $A = \{1,2,3,4,5\}$, $B = \{1,2,4\}$, $C = \{1,5\}$, $D = \{2\}$, $E = \{1,4,2\}$

Then $B \subseteq A$, $C \subseteq A$, $D \subseteq A$, $D \subseteq B$

$C \not\subseteq B$, since $5 \in C \Rightarrow 5 \notin B$, $E \subseteq B$ and $B \subseteq E$.

Some of the important properties of set inclusion.

For any sets A, B and C

$A \subseteq A$ (Reflexive)

$(A \subseteq B) \wedge (B \subseteq C) \Rightarrow (A \subseteq C)$ (Transitive)

Note that $A \subseteq B$ does not imply $B \subseteq A$ except for the following case.

4. Two sets A and B are said to be *equal* if and only if $A \subseteq B$ and $B \subseteq A$,

i.e., $A = B \Leftrightarrow (A \subseteq B \text{ and } B \subseteq A)$

Example $\{1,2,4\} = \{4,1,2\}$ and $P = \{\{1,2\}, 4\}$, $Q = \{1,2,4\}$ then $P \neq Q$

Since $\{1,2\} \in P$ and $\{1,2\} \notin Q$ eventhough $1,2 \in Q$.

The equality of sets is reflexive, symmetric, and transitive.

5. A set A is said to be a *proper subset* of a set B if $A \subseteq B$ and $A \neq B$.

Symbolically it is written as $A \subset B$. *i.e.*, $A \subset B \Leftrightarrow (A \subseteq B \wedge A \neq B)$

\subset is also called a *proper inclusion*.

6. A set is said to be *universal set* if it includes every set under our discussion. A universal set is denoted by U or E .

In other words, if $p(x)$ is a predicate. $E = \{x | p(x) \vee \neg p(x)\}$

One can observe that universal set contains all the sets.

7. A set is said to be *empty set* or *null set* if it does not contain any element, which is denoted by \emptyset .

In other words, if $p(x)$ is a predicate. $\emptyset = \{x | p(x) \vee \neg p(x)\}$

One can observe that null set is a subset for all sets.

8. For a set A, the set of all subsets of A is called the *power set* of A. The power set of A is denoted by $\rho(A)$ or 2^A i.e., $\rho(A) = \{S | S \subseteq A\}$

Example, Let $A = \{a, b, c\}$

Then $\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$

Then set \emptyset and A are called *improper subsets* of A and the remaining sets are called *proper subsets* of A.

One can easily note that the number of elements of $\rho(A)$ is $2^{|A|}$ i.e., $|\rho(A)| = 2^{|A|}$

SOME OPERATIONS ON SETS

1. Intersection of sets

Definition:

Let A and B be any two sets, the *intersection* of A and B is written as $A \cap B$ is the set of all elements which belong to both A and B.

Symbolically

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

Example $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$ then $A \cap B = \{2, 4, 6\}$. From the definition of intersection it follows that for any sets A, B, C and universal set E.

$$A \cap A = A \qquad A \cap B = B \cap A \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cap E = A \qquad A \cap \emptyset = \emptyset$$

2. Disjoint sets

Definition:

Two set A and B are called *disjoint* if and only if $A \cap B = \emptyset$, that is, A and B have no element in common.

Example $A = \{1,2,3\}$ $B = \{5,7,9\}$ $C = \{3,4\}$

$A \cap B = \emptyset$, $A \cap C = \{3\}$, $B \cap C = \emptyset$

A and B are disjoint and B and C also, but A and C are not disjoint.

3. Mutually disjoint sets

Definition:

A collection of sets is called a *disjoint collection*, if for every pair of sets in the collection, are disjoint. The elements of a disjoint collection are said to be *mutually disjoint*.

Let $A = \{A_i\}_{i \in I}$ be an indexed set, A is mutually disjoint if and only if

$$A_i \cap A_j = \emptyset \text{ for all } i, j \in I, i \neq j.$$

Example

$$A_1 = \{\{1,2\}, \{3\}\}, \quad A_2 = \{\{1\}, \{2,3\}\}, \quad A_3 = \{\{1,2,3\}\}$$

Then $A = \{A_1, A_2, A_3\}$ is a disjoint collection of sets.

$$A_1 \cap A_2 = \emptyset, \quad A_1 \cap A_3 = \emptyset \text{ and} \quad A_2 \cap A_3 = \emptyset$$

4. Unions of sets

Definition:

The *union* of two sets A and B, written as $A \cup B$, is the set of all elements which are elements of A or the elements of B or both.

Symbolically $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Example Let $A = \{1,2,3,4,5,6\}$ $B = \{2,4,6,8\}$ then $A \cup B = \{1,2,3,4,5,6,8\}$

From the union, it is clear that, for any sets A, B,C, and universal set E.

$$A \cup A = A \quad A \cup B = B \cup A \quad A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cup E = E \quad A \cup \emptyset = A$$

5. Relative complement of a set

Definition:

Let A and B are any two sets. The *relative complement* of B in A, written $A - B$, is the set of elements of A which are not elements of B.

Symbolically $A - B = \{x \mid x \in A \text{ or } x \notin B\}$

Note that $A - B = A \cap \bar{B}$.

Example Let $A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$$A - B = \{1,3,5\}$$

$$B - A = \{8\}$$

It is clear from the definition that, for any set A and B.

$$A - B = \emptyset$$

$$A - B \neq B - A$$

$$A - \emptyset = A$$

6. Complement of a set

Definition:

Let A be any set, and E be universal. The relative complement of A in E is called *absolute complement or complement* of A. The complement of A is denoted by \bar{A} (or A^c or $\sim A$)

Symbolically

$$E - A = \bar{A} = \{x \mid x \in E \text{ and } x \notin A\}$$

Example Let $E = \{1,2,3,4, \dots\}$ be universal set and

$A = \{2,4,6,8, \dots\}$ be any set in E.

Then

$$\bar{A} = \{1,3,5,7, \dots\}$$

From the definition, for any sets $A \bar{\bar{A}} = A \quad \bar{\bar{\emptyset}} = E$

$$\bar{\bar{E}} = \emptyset \quad A \cup \bar{A} = E \quad A \cap \bar{A} = \emptyset$$

7. Boolean sum of sets

Definition:

Let A and B are any two sets. The *symmetric difference or Boolean sum* of A and B is the set $A+B$ defined by

$$A + B = (A - B) \cup (B - A) = (A \cap \bar{B}) \cup (B \cap \bar{A})$$

$$\text{(or) } A + B = \{x \mid x \in A \text{ and } x \notin B\} \cup \{x \mid x \in B \text{ and } x \notin A\}$$

Example Let

$$A = \{1,2,3,4,5,6\}$$

$$B = \{2,4,6,8\} \text{ then}$$

$$A + B = \{1,3,5,8\} \text{ From the definition, for any sets A and B.}$$

$$A + A = \emptyset, \quad A + \emptyset = A$$

$$A + E = \bar{A}, \quad A + B = B + A \text{ and}$$

$$A + (B + C) = (A + B) + C$$

8. The principle of duality

If we interchange the symbols \cap, \cup, E and \emptyset, \subseteq and \supseteq, \subset and \supset, \subset , in a set equation or expression. We obtain a new equation or expression is said to be *dual* of the original one (*primal*).

“ If T is any theorem expressed in terms of \cap, \cup and $-$ deducible from the given basic laws, then the dual of T is also a theorem”.

Note that, the theorem T is proved in m steps, then dual of T also proved in m step.

Example The dual of $A \cap \bar{A} = \emptyset$ is given by $A \cup \bar{A} = E$.

Remark: Dual (Dual T) = T.

Identities on sets

$$A \cup A = A$$

Idempotent laws

$$A \cap A = A$$

$$A \cup B = B \cup A$$

Commutative laws

$$A \cap B = B \cap A$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Associative laws

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (A \cap B) = A$$

Absorption laws

$$A \cap (A \cup B) = A$$

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$

De Morgan's laws

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

$$A \cup \emptyset = A$$

$$A \cap \emptyset = \emptyset$$

$$A \cup E = E$$

$$A \cap E = A$$

$$A \cup \bar{A} = E$$

$$A \cap \bar{A} = \emptyset$$

$$\bar{\emptyset} = E$$

$$\bar{E} = \emptyset$$

$$\bar{\bar{A}} = A$$

PROBLEMS

1. $S = \{a, b, p, q\}$, $Q = \{a, p, t\}$. Find $S \cup Q$ and $S \cap Q$?

Solution:

$$S \cup Q = \{a, b, p, q, t\}$$

$$S \cap Q = \{a, p\}$$

2. If $A = \{a, b, c\}$. Find $\rho(A)$?

Solution:

$$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\} \text{ and}$$

$$|A| = 3$$

$$|\rho(A)| = 2^3 = 8$$

3. Write all proper subsets of $A = \{a, b, c\}$.

Solution:

The proper subsets are

$$\rho(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

4. Show that $A \subseteq B \Leftrightarrow A \cap B = A$.

Solution:

If $A \subseteq B$, then $\forall x \in A \Rightarrow x \in B$

Now, let

$$x \in A \Leftrightarrow x \in A \text{ and } x \in B$$

$$\Leftrightarrow x \in A \cap B$$

$$A = A \cap B$$

If $A \cap B = A$, then

$$\text{Let } x \in A, \quad x \in A \cap B \Rightarrow x \in B$$

Therefore $A \subseteq B$.

5. If $A = \{2,5,6,7\}$, $B = \{1,2,3,4\}$, $C = \{1,3,5,7\}$. Find $A - B$, $A - C$, $C - B$ and $B - C$.

Solution:

$$A - B = \{5,6,7\}$$

$$A - C = \{2,6\}$$

$$C - B = \{5,7\}$$

$$B - C = \{2,4\}$$

6. If $A = \{2,3,4\}$, $B = \{1,2\}$, $C = \{4,5,6\}$. Find $A + B$, $B + C$, $A + C$, $A + B + C$ and $(A + B) + (B + C)$.

Solution:

$$A + B = \{1,3,4\}$$

$$B + C = \{1,2,4,5,6\}$$

$$A + C = \{2,3,5,6\}$$

$$A + B + C = \{1,3,5,6\}$$

$$(A + B) + (B + C) = \{2,3,5,6\}$$

Note that

$$A + (B + B) + C = A + (\emptyset) + C = A + C = \{2,3,5,6\}$$

7. Show that $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Solution:

Let

$$x \in A \Rightarrow x \in A \text{ (or) } x \in B$$

$$\Rightarrow x \in A \cup B$$

$$\Rightarrow A \subseteq A \cup B$$

$$\text{Now let } x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

$$\Rightarrow x \in A$$

$$A \cap B \subseteq A$$

Hence $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Remark: $B \subseteq A \cup B$, $A \cap B \subseteq B$ and $A \cap B \subseteq A \cup B$.

8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.

Solution:

$$x \in A - (A \cap B) \Leftrightarrow x \in A \text{ and } x \notin (A \cap B)$$

$$\Leftrightarrow x \in A \text{ and } \{x \notin A \text{ or } x \notin B\}$$

$$\Leftrightarrow \{x \in A \text{ and } x \notin A\} \text{ (or) } \{x \in A \text{ and } x \notin B\}$$

$$\Leftrightarrow \emptyset \text{ (or) } \{x \in A \text{ and } x \notin B\}$$

$$\Leftrightarrow x \in A \text{ and } x \notin B$$

$$A - (A \cap B) \subseteq A - B \text{ and } A - B \subseteq A - (A \cap B)$$

Therefore $A - (A \cap B) = A - B$.

9. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution:

$$x \in A \cup (B \cap C) \Leftrightarrow x \in A \text{ or } x \in B \cap C$$

$$\Leftrightarrow x \in A \text{ or } \{x \in B \text{ and } x \in C\}$$

$$\Leftrightarrow \{x \in A \text{ or } x \in B\} \text{ and } \{x \in A \text{ or } x \in C\}$$

$$\Leftrightarrow \{x \in A \cup B\} \text{ and } \{x \in A \cup C\}$$

$$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$$

Therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

10. Show that $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

Solution:

$$\text{Let } x \in \overline{(A \cup B)} \Leftrightarrow x \notin A \cup B$$

$$\Leftrightarrow x \notin A \text{ and } x \notin B$$

$$\Leftrightarrow x \in \bar{A} \text{ and } x \in \bar{B}$$

$$\Leftrightarrow x \in \bar{A} \cap \bar{B}$$

Therefore $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

11. Show that $(A - B) - C = A - (B \cup C)$.

Solution:

$$(A - B) - C = (A - B) \cap \bar{C} \quad (P - Q = P \cap \bar{Q})$$

$$= (A \cap \bar{B}) \cap \bar{C}$$

$$= A \cap (B \cap \bar{C}) \quad (\text{Associative})$$

$$= A \cap (\overline{B \cup C}) \quad (\text{De Morgan's law})$$

12. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$

Solution:

$$\text{Let } (A \cap B) - (A \cap C)$$

$$= (A \cap B) \cap (\overline{A \cap C})$$

$$= (A \cap B) \cap (\bar{A} \cup \bar{C})$$

$$= (A \cap B \cap \bar{A}) \cup (A \cap B \cap \bar{C})$$

$$= ((A \cap \bar{A}) \cap B) \cup (A \cap B \cap \bar{C})$$

$$= (\emptyset \cap B) \cup (A \cap B \cap \bar{C})$$

$$= \emptyset \cup (A \cap B \cap \bar{C})$$

$$= A \cap (B \cap \bar{C})$$

$$= A \cap (B - C)$$

ASSIGNMENT PROBLEMS

Part -A

1. Define a set
2. Define subset of a set. What is mean by proper subset?

- (i) Find all subset of $A = \{1,2,3\}$
- (ii) Find all proper subsets of A.
3. Define power set.
4. Define disjoint sets with example?
5. If $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8,10\}$. Find $A \cup B, A \cap B, A - B, B - A, A + B$, and $B + A$?
6. Which of the following sets are empty?
7. $\{x \mid x \in R, x + 6 = 6\}$
8. $\{x \mid x \text{ is a real integer such that } x^2 + 1 = 0\}$
9. $\{x \mid x \text{ is a real integer and } x^2 - 4 = 0\}$
10. State duality principle in set theory.
11. Define cardinality of a set.
12. If a set A has n elements, then the number of elements of power set of A is.....
13. Find the intersection of the following sets
- (i) $\{x \mid x^2 - 1 = 0\}, \{x \mid x^2 + 2x + 1 = 0\}$
14. Write the dual of $A \cap \bar{A} = \emptyset$.
15. Let A, B and C sets, such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, can we conclude that $B=C$.
16. State De Morgan's Laws.
17. Whether the union of sets is commutative or not?

PART –B

1. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
2. Verify the De Morgan's laws
 - (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$, (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$
3. Show that the intersection of sets is associative.
4. Show that $A - (B - C) = (A - B) \cup (A \cap C)$.
5. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$
6. Let $A_i = \{1, 2, 3, \dots\}$ for $i = 1, 2, 3, \dots$ find (a) $\bigcup_{i=1}^n A_i$ (b) $\bigcap_{i=1}^n A_i$
7. Prove that $A - (A - B) \subset B$.
8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.
9. Prove that $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$.
10. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove that $B=C$.(cancelation law)
11. Show that $A - (B \cup C) = (A - B) \cap (A - C)$.
12. Show that $A + A = \emptyset$, where + is the symmetric difference of sets.
13. Show that $(R \subseteq S)$ and $(S \subset Q)$ imply $R \subset Q$.
14. Given that $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$. Show that $A \subseteq B$.