FINAL EXAMINATION

INSTRUCTIONS: ATTEMPT ALL THE QUESTIONS

TIME ALLOWED FOR THE EXAM: THREE HOURS

QUESTION 1. (10 Marks)

Suppose $x \in \mathbb{R}$ is $\geq$ all elements of a non-empty subset $S \subset \mathbb{R}$. Explain why this implies $x \geq \sup S$.

QUESTION 2. (10 Marks)

Let $\mathbb{F}$ be an archimedean ordered field, and let $x, y \in \mathbb{F}$ with $x < y$. If $z \in \mathbb{F}$ show that there is a rational multiple qz of z for which $x < qz < y$.

QUESTION 3. (10 Marks)

Suppose S is a non-empty subset of $\mathbb{R}$ and $x \in \mathbb{R}$ is not an upper bound of S. Show that there exists $y > x$ which is also not an upper bound of S.

QUESTION 4. (10 Marks)

Suppose L and U are non-empty subsets of $\mathbb{R}$ such that (i) every element of L is $\leq$ every element of $\mathbb{R}$, and (ii) for any $\varepsilon > 0$ there is an element $l \in L$ and an element $u \in U$ with $u - l < \varepsilon$. Prove that there is a unique real number which is $\leq$ all elements of U and $\geq$ all elements of L.

QUESTION 5. (10 Marks)

Let ε be any positive real number.

(i) Show that there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n^2} < \varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

(ii) Let a be any real number. Show that there is some $n_1 \in \mathbb{N}$ such that

$$2a\frac{1}{n} < \varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq n_1$.

QUESTION 6. (10 Marks)

Suppose a is a positive real number with $a^2 < 7$. Show that

$$(a + \frac{1}{n})^2 < \varepsilon$$

for all $n \in \mathbb{N}$ large enough (i.e. there is some $n' \in \mathbb{N}$ such that the preceding inequality holds for all natural numbers $n \geq n'$).

QUESTION 7. (10 Marks)

Let

$$S = \{x \in \mathbb{R} : x > 0 \,\text{and}\, x^2 < 7\}.$$

(i) Show that $S \neq \emptyset$.

(ii) Show that S is bounded above in $\mathbb{R}$.

(iii) Let $a = \sup S$. Prove that $a^2 \geq 7$. (Hint: Suppose a^2 were less than 7. Then use the result of Problem 3.)

(iv) Prove that a^2 is, in fact, equal to 7. Thus, there is a real number whose square is 7.

QUESTION 8. (10 Marks)

Provide brief explanations/answers for the following:

(i) If U is an open set then $U^0 =$

(ii) For any set A, the interior of the interior of A is the interior of A, i.e. $(A^0)^0 = A^0$.

(iii) The set $[0, \infty)$ is closed as a subset of $\mathbb{R}$.

(iv) If S is closed then its closure $\overline{S}$ is S itself, i.e. $\overline{S} = S$.

QUESTION 9. (20 Marks)

Consider the partition

$$X = \left(\frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N}\right)$$

of $[0,1]$. For the function f on $[0,1]$ given by

$$f(x) = x^2 \qquad \text{for all } x \in [0,1]$$

(i) Show that

$$U(f,X) = \frac{1}{6}\left(1+\frac{1}{N}\right)\left(2+\frac{1}{N}\right)$$

[Hint: Use the sum formula $1^2 + 2^2 + \cdots + k^2 = k(k+1)(2k+1)/6$.]

(ii) Show that

$$L(f,X) = \frac{1}{6}\left(1 - \frac{1}{N}\right)\left(2 - \frac{1}{N}\right)$$

(iii) Assuming that f is integrable, prove that

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

using the definition of the Riemann integral and the results of (i) and (ii).