Limits and Continuity

Limits of Functions

Definition
Suppose that a and ℓ are real numbers and let f be a real-valued function whose domain D includes all points in some open interval about a (except possibly the point a itself). Then ℓ is called the **limit** *of the function f at a if, given any $\epsilon > 0$, there exists a $\delta > 0$ (depending on a and ϵ) such that*

$$|f(x) - \ell| < \epsilon \text{ for all } x \in D \text{ satisfying } 0 < |x - a| < \delta.$$

In this case, we write $\lim_{x \to a} f(x) = \ell$ or $f(x) \to \ell$ as $x \to a$.

Note that the existence of the limit of $f(x)$ as x tends to a does not depend on $f(a)$. Indeed, $f(a)$ may or may not be defined since a is not necessarily in the domain of f. If $f(a)$ and $\lim_{x \to a} f(x)$ both exist, they may or may not be equal. We are only interested in the behaviour of f as x gets *closer to a*. It is implicit in the definition of the limit that a is an accumulation point of the domain D of f.

We can reformulate the above definition in the ϵ-neighbourhood language as follows: $\lim_{x \to a} f(x) = \ell$, if for each ϵ-neighbourhood $N(\ell, \epsilon)$ of ℓ there exists a deleted δ-neighbourhood $N^*(a, \delta)$ of a such that $f(x) \in N(\ell, \epsilon)$ whenever $x \in N^*(a, \delta) \cap D$.

Definition
(1) *Suppose that f is defined for all real numbers $x > k$, where $k \in \mathbb{R}$. Then $\ell \in \mathbb{R}$ is the* **limit of f as x tends to ∞** *if, given $\epsilon > 0$, there exists a real number K such that*

$$|f(x) - \ell| < \epsilon \text{ whenever } x > K.$$

In this case we write $\lim_{x \to \infty} f(x) = \ell$.

(2) *Suppose that f is defined for all real numbers $x < k$, where $k \in \mathbb{R}$. Then $\ell \in \mathbb{R}$ is the* **limit of f as x tends to $-\infty$**, *denoted by $\lim_{x \to -\infty} f(x) = \ell$, if, given $\epsilon > 0$, there exists a real number k such that*

$$|f(x) - \ell| < \epsilon \text{ whenever } x < k.$$

Examples
[1] Show that $\lim_{x \to 2} x^2 = 4$.

Solution: Let $\epsilon > 0$ be given. We need to produce a $\delta > 0$ such that

$$|x^2 - 4| < \epsilon \text{ whenever } 0 < |x - 2| < \delta.$$

Now,

$$|x^2 - 4| = |(x - 2)(x + 2)| = |x - 2||x + 2|.$$

Consider all x which satisfy the inequality $|x-2| < 1$. Then, for all such x, we have $1 < x < 3$. Thus,

$$|x + 2| \le |x| + 2 < 3 + 2 = 5,$$

and so

$$|x^2 - 4| = |x-2||x+2| < 5|x-2|.$$

Choose $\delta = \min\left\{1, \frac{\epsilon}{5}\right\}$. Then, whenever $0 < |x-2| < \delta$, we have that

$$|x^2 - 4| < \epsilon \ . \qquad \square$$

[2] Show that $\lim_{x \to 3}(x^2 + 2x) = 15$.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$|(x^2 + 2x) - 15| < \epsilon \ \text{ for all } x \text{ satisfying } \ 0 < |x-3| < \delta.$$

Note that

$$|(x^2 + 2x) - 15| = |(x+5)(x-3)| = |x+5||x-3|.$$

Since we are interested in the values of x near 3, we may consider those values of x which satisfy the inequality $|x-3| < 1$, i.e., $2 < x < 4$. For all these values we have that $|x+5| < 9$. Therefore, if $|x-3| < 1$, we have that

$$|(x^2 + 2x) - 15| < 9|x-3|.$$

Choose $\delta = \min\left\{1, \frac{\epsilon}{9}\right\}$. Then, working backwards, we have that

$$|(x^2 + 2x) - 15| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x-3| < \delta. \qquad \square$$

[3] Show that $\lim_{x \to -1} \frac{2x+3}{x+2} = 1$.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left|\frac{2x+3}{x+2} - 1\right| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x-(-1)| = |x+1| < \delta.$$

By elementary algebraic manipulation, we have that

$$\left|\frac{2x+3}{x+2} - 1\right| = \left|\frac{(2x+3)-(x+2)}{x+2}\right| = \left|\frac{x+1}{x+2}\right| = \frac{|x+1|}{|x+2|} \ .$$

Since we are interested in the values of x near -1, we may consider those values of x which satisfy the inequality $|x+1| < \frac{1}{2}$, i.e., $\frac{-3}{2} < x < \frac{-1}{2}$. Recognising $|x+2| = |x-(-2)|$ as the distance of x from -2, we have that

$$|x+2| = |x-(-2)| > \left|\frac{-3}{2} - (-2)\right| = \frac{1}{2}.$$

Therefore

$$\left|\frac{2x+3}{x+2} - 1\right| = \frac{|x+1|}{|x+2|} < 2|x+1|.$$

Choose $\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{2}\right\}$. Then whenever $0 < |x+1| < \delta$, we have that

$$\left|\frac{2x+3}{x+2} - 1\right| < \epsilon . \qquad \square$$

[4] Show that $\lim_{x \to 0} f(x)$, where $f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$ does not exist.

Solution: Assume that the limit exists and $\lim_{x \to 0} f(x) = \ell$. Then, with $\epsilon = 1$, there is a $\delta > 0$ such that

$$|f(x) - \ell| < 1 \text{ for all } x \text{ satisfying } 0 < |x| < \delta.$$

Taking $x = \frac{-\delta}{2}$, we have that $|x| = \frac{\delta}{2} < \delta$, and so

$$1 > |f(x) - \ell| = |-1-\ell| = |1+\ell|.$$

Thus, $-2 < \ell < 0$.

On the hand, if $x = \frac{\delta}{2}$, we have that $|x| = \frac{\delta}{2} < \delta$, and so

$$1 > |f(x) - \ell| = |1 - \ell|.$$

Therefore, $0 < \ell < 2$. But there is no real number that can simultaneously satisfy the inequalities $-2 < \ell < 0$ and $0 < \ell < 2$. Therefore $\lim_{x \to 0} f(x)$ does not exist. $\square$

[5] Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left|x \sin \frac{1}{x} - 0\right| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x - 0| < \delta.$$

Now,

$$\left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \leq |x|.$$

Choose $0 < \delta \leq \epsilon$. Then, whenever $0 < |x - 0| = |x| < \delta$, we have that

$$\left|x \sin \frac{1}{x} - 0\right| \leq |x| < \epsilon,$$

which proves that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$. $\square$

[6] Consider the function $f : \mathbb{R} \to \{0, 1\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} . \end{cases}$$

Show that if $a \in \mathbb{R}$, then $\lim_{x \to a} f(x)$ does not exist.

Solution: Assume that there is an $\ell \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = \ell$. Then, with $\epsilon = \frac{1}{4}$, there exists $\delta > 0$ such that

$$|f(x) - \ell| < \frac{1}{4} \text{ for all } x \text{ satisfying } 0 < |x-a| < \delta.$$

If $x \in \mathbb{Q}$, then

$$|1 - \ell| < \frac{1}{4} \text{ whenever } 0 < |x-a| < \delta, \text{ and}$$

if $x \in \mathbb{R} \setminus \mathbb{Q}$, then

$$|\ell| < \frac{1}{4} \text{ whenever } 0 < |x-a| < \delta.$$

Since the set $\{x \in \mathbb{R} : 0 < |x-a| < \delta\}$ contains both rationals and irrationals, we have that

$$1 = |1-0| = |1-\ell+\ell| \leq |1-\ell| + |\ell| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

which is absurd. □

The following theorem highlights the relationship between convergence of sequences and limits of functions.

Theorem
Let f be a function which is defined in some open interval I containing $a \in \mathbb{R}$, except possibly at a. Then $\lim_{x\to a} f(x) = \ell$ if and only if for every sequence $(a_n) \subset I \setminus \{a\}$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = \ell$.

$\prec$Assume that $\lim_{x\to a} f(x) = \ell$ and let $(a_n) \subset I \setminus \{a\}$ be a sequence such that $\lim_{n\to\infty} a_n = a$. Then, given $\epsilon > 0$, there is a $\delta > 0$ and an $N \in \mathbb{N}$ such that

$$|f(x) - \ell| < \epsilon \text{ for all } x \in I \text{ satisfying } 0 < |x-a| < \delta \text{ and } |a_n - a| < \delta \text{ for all } n \geq N.$$

Now, $0 < |a_n - a| < \delta$ since $a_n \neq a$ for all $n \geq N$. Therefore

$$|f(a_n) - \ell| < \epsilon \text{ for all } n \geq N.$$

That is, $\lim_{n\to\infty} f(a_n) = \ell$.

For the converse, assume that for every sequence $(a_n) \subset I \setminus \{a\}$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = \ell$.

Claim: $\lim_{x\to a} f(x) = \ell$. If the claim were false, then there would exist an $\epsilon_0 > 0$ such that for every $\delta > 0$ with $0 < |x-a| < \delta$, we have

$$|f(x) - \ell| \geq \epsilon_0.$$

Let $n \in \mathbb{N}$ and take $\delta = \frac{1}{n}$. Then we can find $a_n \in I \setminus \{a\}$ such that $0 < |a_n - a| < \frac{1}{n}$ and

$$|f(a_n) - \ell| \geq \epsilon_0.$$

Clearly, (a_n) is a sequence in $I \setminus \{a\}$ with the property that $\lim_{n\to\infty} a_n = a$ and

$$|f(a_n) - \ell| \geq \epsilon_0 \text{ for all } n \in \mathbb{N}.$$

That is, $\lim_{n\to\infty} f(a_n) \neq \ell$, a contradiction. ■

The condition that $a_n \neq a$ for all $n \in \mathbb{N}$ in Theorem 5.1.4 is essential. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{for } x \neq 0 \\ \frac{1}{2} & \text{for } x = 0. \end{cases}$$

Let (a_n) be the sequence where $a_n = 0$ for all $n \in \mathbb{N}$. Then $a_n \in \mathbb{R}(=$ domain of $f)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = 0$. Since $f(a_n) = f(0) = \dfrac{1}{2}$ for all $n \in \mathbb{N}$, and $\lim_{x\to 0} f(x) = 1$, it follows that

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 1 = \lim_{x\to 0} f(x).$$

Theorem
(Uniqueness of Limits). *Let f be a function which is defined on some open interval I containing a, except possibly at a. If $\lim_{x\to a} f(x) = \ell_1$ and $\lim_{x\to a} f(x) = \ell_2$, then $\ell_1 = \ell_2$.*

$\prec$If $\ell_1 \neq \ell_2$, let $\epsilon = \dfrac{|\ell_1 - \ell_2|}{3}$. Then, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - \ell_1| < \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_1, \text{ and}$$

$$|f(x) - \ell_2| < \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x-a| < \delta$, we have

$$0 < |\ell_1 - \ell_2| \leq |f(x) - \ell_1| + |f(x) - \ell_2| < \frac{|\ell_1 - \ell_2|}{3},$$

which is impossible. ■

Algebra of Limits

Theorem
Let $\ell_1, \ell_2, a \in \mathbb{R}$. Suppose that f and g are real-valued functions defined on some open interval I containing a, except possibly at a itself, and that $\lim_{x\to a} f(x) = \ell_1$ and $\lim_{x\to a} g(x) = \ell_2$. Then,

(1) $\lim_{x\to a}[f(x) \pm g(x)] = \ell_1 \pm \ell_2$.

(2) $\lim_{x\to a}[f(x)g(x)] = \ell_1\ell_2$.

(3) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell_1}{\ell_2}$ *provided $g(x) \neq 0$ for all $x \in I$ and $\ell_2 \neq 0$.*

$\prec$

(1) Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - \ell_1| < \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_1, \text{ and}$$

$$|g(x) - \ell_2| < \frac{\epsilon}{2} \text{ whenever } x \in I \text{ and } 0 < |x-a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $x \in I$ and $0 < |x - a| < \delta$, we have

$$|\,[f(x)+g(x)]-[\ell_1+\ell_2]\,| = |\,[f(x)-\ell_1]+[g(x)-\ell_2]\,| \le |f(x)-\ell_1|+|g(x)-\ell_2| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

That is, $f(x) + g(x) \to \ell_1 + \ell_2$ as $x \to a$.

A similar argument shows that $f(x) - g(x) \to \ell_1 - \ell_2$ as $x \to a$.

(2) With $\epsilon = 1$, there exists a $\delta_1 > 0$ such that

$$|f(x) - \ell_1| < 1 \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1.$$

This implies that

$$|f(x)| \le |f(x) - \ell_1| + |\ell_1| < 1 + |\ell_1| \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1.$$

Now, for all $x \in I$ with $0 < |x - a| < \delta_1$, we have

$$\begin{aligned} |f(x)g(x) - \ell_1\ell_2| &= |f(x)g(x) - f(x)\ell_2 + f(x)\ell_2 - \ell_1\ell_2| \\ &\le |f(x)||g(x) - \ell_2| + |\ell_2||f(x) - \ell_1| \\ &< (1 + |\ell_1|)|g(x) - \ell_2| + |\ell_2||f(x) - \ell_1|\,. \end{aligned}$$

Given $\epsilon > 0$, there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$\begin{aligned} |f(x) - \ell_1| &< \frac{\epsilon}{2(1 + |\ell_2|)} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_2, \text{ and} \\ |g(x) - \ell_2| &< \frac{\epsilon}{2(1 + |\ell_1|)} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_3. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, whenever $x \in I$ and $0 < |x - a| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - \ell_1\ell_2| &< (1 + |\ell_1|)\left[\frac{\epsilon}{2(1 + |\ell_1|)}\right] + |\ell_2|\left[\frac{\epsilon}{2(1 + |\ell_2|)}\right] \\ &= \frac{\epsilon}{2} + \frac{\epsilon|\ell_2|}{2(1 + |\ell_2|)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

That is, $f(x)g(x) \to \ell_1\ell_2$ as $x \to a$.

(3) It is enough to show that $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{\ell_2}$ provided $g(x) \neq 0$ for all $x \in I$ and $\ell_2 \neq 0$. Since $\ell_2 \neq 0$, $\epsilon = \frac{|\ell_2|}{2} > 0$. Therefore there exists a $\delta_1 > 0$ such that

$$|g(x) - \ell_2| < \frac{|\ell_2|}{2} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1.$$

Now, for all $x \in I$ satisfying $0 < |x - a| < \delta_1$, we have

$$|\ell_2| \le |\ell_2 - g(x)| + |g(x)| < \frac{|\ell_2|}{2} + |g(x)|.$$

That is, $\frac{|\ell_2|}{2} < |g(x)|$ for all $x \in I$ satisfying $0 < |x - a| < \delta_1$. It now follows that for all $x \in I$ satisfying $0 < |x - a| < \delta_1$,

$$\left|\frac{1}{g(x)} - \frac{1}{\ell_2}\right| = \left|\frac{\ell_2 - g(x)}{g(x)\ell_2}\right| = \frac{|\ell_2 - g(x)|}{|g(x)\ell_2|} < \frac{2|\ell_2 - g(x)|}{|\ell_2||\ell_2|} = \frac{2|\ell_2 - g(x)|}{\ell_2^2}.$$

Given $\epsilon > 0$ there exist $\delta_2 > 0$ such that

$$|g(x) - \ell_2| < \frac{\epsilon \ell_2^2}{2} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $x \in I$ and $0 < |x - a| < \delta$, we have

$$\left| \frac{1}{g(x)} - \frac{1}{\ell_2} \right| < \frac{2|\ell_2 - g(x)|}{\ell_2^2} < \frac{2}{\ell_2^2} \cdot \frac{\epsilon \ell_2^2}{2} = \epsilon.$$

That is, $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{\ell_2}$ provided $\ell_2 \neq 0$. ■

Theorem
Let $\ell_1, \ell_2, a \in \mathbb{R}$. Suppose that f and g are real-valued functions defined on some open interval I containing a, except possibly at a itself, and that $f(x) \leq g(x)$ for all $x \in I$. If $\lim_{x \to a} f(x) = \ell_1$ and $\lim_{x \to a} g(x) = \ell_2$, then $\ell_1 \leq \ell_2$.

$\prec$If $\ell_2 < \ell_1$, let $\epsilon = \frac{\ell_1 - \ell_2}{2}$. Now, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} |f(x) - \ell_1| &< \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1, \text{ and} \\ |g(x) - \ell_2| &< \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_2. \end{aligned}$$

That is,

$$\begin{aligned} \ell_1 - \epsilon < f(x) &< \ell_1 + \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1, \text{ and} \\ \ell_2 - \epsilon < g(x) &< \ell_2 + \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_2. \end{aligned}$$

That is,

$$\begin{aligned} \frac{\ell_1 + \ell_2}{2} &< f(x) \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_1, \text{ and} \\ g(x) &< \frac{\ell_1 + \ell_2}{2} \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta_2. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $x \in I$ and $0 < |x - a| < \delta$, we have

$$g(x) < \frac{\ell_1 + \ell_2}{2} < f(x),$$

and so $g(x) < f(x)$, a contradiction. ■

Theorem
(Squeeze Theorem). *Suppose that f, g and h are real-valued functions defined on some open interval I containing a, except possibly at a itself, and that $f(x) \leq g(x) \leq h(x)$ for all $x \in I$. If $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} h(x) = \ell$, then $\lim_{x \to a} g(x) = \ell$.*

$\prec$Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\begin{aligned} |f(x) - \ell| &< \epsilon \text{ whenever } 0 < |x - a| < \delta_1, \text{ and} \\ |h(x) - \ell| &< \epsilon \text{ whenever } 0 < |x - a| < \delta_2. \end{aligned}$$

That is,

$$\begin{aligned} \ell - \epsilon < f(x) &< \ell + \epsilon \text{ whenever } 0 < |x - a| < \delta_1, \text{ and} \\ \ell - \epsilon < h(x) &< \ell + \epsilon \text{ whenever } 0 < |x - a| < \delta_2. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x - a| < \delta$, we have

$$\ell - \epsilon < f(x) \leq g(x) \leq h(x) < \ell + \epsilon.$$

Thus,

$$|g(x) - \ell| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x - a| < \delta.$$

That is, $\lim_{x \to a} g(x) = \ell$. ■

Exercise

[1] Show that $f(x) \to 0$ as $x \to a$ if and only if $|f(x)| \to 0$ as $x \to a$.

[2] Let $a, \ell \in \mathbb{R},\ D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Show that if $f(x) \to \ell$ as $x \to a$, then $|f(x)| \to |\ell|$ as $x \to a$. Does the converse hold? Justify your answer.

[3] Let $a, \ell \in \mathbb{R},\ D \subset \mathbb{R},\ f : D \to \mathbb{R}$ and $\ell > 0$. Show that if $\lim_{x \to a} f(x) = \ell$, then there exists a deleted ϵ-neighbourhood $N^*(a, \epsilon)$ of a such that $f(x) > 0$ for all $x \in N^*(a, \epsilon) \cap D$.

[4] Let $a, \ell \in \mathbb{R},\ D \subset \mathbb{R},\ f : D \to \mathbb{R}$ and $\lim_{x \to a} f(x) = \ell$. Show that there exists a deleted ϵ-neighbourhood $N^*(a, \epsilon)$ and a positive real number M such that $|f(x)| \leq M$ for all $x \in N^*(a, \epsilon) \cap D$.

Continuous Functions

When discussing the limit $\lim_{x \to a} f(x)$, we made no reference to $f(a)$, the value of the function f at a. In fact, we emphasized that $f(a)$ was unimportant in the analysis of $\lim_{x \to a} f(x)$. In this section we want to bring $f(a)$ into the picture; we want to relate the limit $\lim_{x \to a} f(x)$ to the value of f at a.

Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is said to be **continuous at** *$a \in D$ if, given any $\epsilon > 0$, there exists a $\delta > 0$ (which generally depends on ϵ and a) such that*

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in D \text{ and } |x - a| < \delta.$$

The function f is **continuous on** *D if it is continuous at each point of D. If f is not continuous at a, we say that f is* **discontinuous** *there.*

Let us reformulate this definition in the language of neighbourhoods:

Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is said to be **continuous at** *$a \in D$ if, for each ϵ-neighbourhood $N(f(a), \epsilon)$ of $f(a)$, there is a δ-neighbourhood $N(a, \delta)$ of a such that*

$$f(x) \in N(f(a), \epsilon) \text{ whenever } x \in N(a, \delta) \cap D.$$

Note that, in contrast to the *deleted* δ-neighbourhoods used in the definition of $\lim_{x \to a} f(x) = \ell$, for continuity we use δ-neighbourhoods.

Examples

[1] Show that the function $f(x) = x^2$ is continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$ be given and $a \in \mathbb{R}$. We need to produce a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta.$$

Now,

$$|f(x) - f(a)| = |x^2 - a^2| = |(x - a)(x + a)|.$$

Since we are interested in the behaviour of f near a, we may restrict our attention to those real numbers x that satisfy the inequality $|x - a| < 1$. These real numbers satisfy the inequalities $a - 1 < x < a + 1$. Therefore, for all these real numbers, we have

$$|x + a| \leq |x| + |a| < |a + 1| + |a| \leq 1 + 2|a|.$$

Now, take $\delta = \min\left\{1, \dfrac{\epsilon}{1 + 2|a|}\right\}$. Then, $|x - a| < \delta$ implies that

$$|f(x) - f(a)| = |x^2 - a^2| < \epsilon.$$

That is, f is continuous at a. Since a was arbitrarily chosen in $\mathbb{R}$, it follows that f is continuous on $\mathbb{R}$. □

[2] Show that the function $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ is continuous at 0.

Solution: Let $\epsilon > 0$ be given. We need to produce a $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon \text{ whenever } |x - 0| < \delta.$$

Now

$$|f(x) - f(0)| = \left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \leq |x|.$$

Choose $0 < \delta \leq \epsilon$. Then, $|x - 0| < \delta$ implies that

$$|f(x) - f(0)| = \left|x \sin \frac{1}{x} - 0\right| \leq |x| < \delta \leq \epsilon.$$

That is, f is continuous at 0. □

[3] Show that the function $f : \mathbb{R} \to \{-1,\ 1\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is discontinuous at every real number.

Solution: Assume that f is continuous at some $a \in \mathbb{R}$. Then, given $\epsilon = 1$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta.$$

Since rationals and irrationals are dense in $\mathbb{R}$, the interval $|x-a| < \delta$ contains both rationals and irrationals. If $x \in \mathbb{Q}$ and $|x - a| < \delta$, then

$$|1 - f(a)| < 1, \text{ whence } 0 < f(a) < 2.$$

On the other hand, if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $|x - a| < \delta$, then

$$|-1 - f(a)| < 1, \text{ whence } -2 < f(a) < 0.$$

But there is no real number that can simultaneously satisfy both the inequalities $0 < f(a) < 2$ and $-2 < f(a) < 0$. Therefore f is discontinuous at every $a \in \mathbb{R}$. □

Show that the function $f(x) = \dfrac{1}{x}$ is continuous at 1.

Solution: Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$|f(x) - f(1)| < \epsilon \text{ whenever } |x - 1| < \delta.$$

Since we are interested in the values of x near 1, we may consider those x for which $|x - 1| < \dfrac{1}{2}$. These x satisfy the inequalities

$$\frac{1}{2} < x < \frac{3}{2}.$$

Now, for all the x which satisfy $|x - 1| < \dfrac{1}{2}$, we have

$$|f(x) - f(1)| = \left|\frac{1}{x} - 1\right| = \frac{|x-1|}{x} < 2|x - 1|.$$

Choose $\delta = \min\left\{\dfrac{1}{2}, \dfrac{\epsilon}{2}\right\}$. Then, whenever $|x - 1| < \delta$, we have that

$$\left|\frac{1}{x} - 1\right| < \epsilon.$$

That is, f is continuous at $x = 1$. □

Show that if a is an isolated point in the domain D of f, then f is continuous at a.

Solution: Since $a \in D$, f is defined at a. Let $\epsilon > 0$ be given. Since a is an isolated point of D, there is a δ-neighbourhood $N(a, \delta)$ of a such that $N(a, \delta) \cap D = \{a\}$. Assume that $x \in D$ and $|x - a| < \delta$; i.e., $x \in N(a, \delta)$. Then $x = a$ since $N(a, \delta) \cap D = \{a\}$. Hence

$$|f(x) - f(a)| = |f(a) - f(a)| = 0 < \epsilon.$$

Therefore f is continuous at a. □

We can deduce from Example 5 that if $f : \mathbb{Z} \to \mathbb{R}$, then f is continuous at every point of $\mathbb{Z}$. □

The following theorem gives a criterion for continuity at a point in terms of sequences. On some occasions it is easier to apply this formulation than the $\epsilon - \delta$ definition. In particular, this formulation comes handy when one wants to use contradiction to prove discontinuity of a function.

Theorem
Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous at $a \in D$ if and only if for every sequence $(a_n) \subset D$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = f(a)$.

$\prec$Suppose that f is continuous at $a \in D$ and that (a_n) is a sequence in D such that $\lim_{n\to\infty} a_n = a$. Given $\epsilon > 0$, there is a $\delta > 0$ and an $N \in \mathbb{N}$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta \text{ and } |a_n - a| < \delta \text{ for all } n \geq N.$$

Therefore

$$|f(a_n) - f(a)| < \epsilon \text{ for all } n \geq N.$$

That is, $\lim_{n\to\infty} f(a_n) = f(a)$.

For the converse, assume that for every sequence $(a_n) \subset D$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = f(a)$ and that f is *not* continuous at a. Then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ with $0 < |x - a| < \delta$, we have

$$|f(x) - f(a)| \geq \epsilon_0.$$

For $n \in \mathbb{N}$, let $\delta = \dfrac{1}{n}$. Then we can find $a_n \in D$ such that $0 < |a_n - a| < \dfrac{1}{n}$ and

$$|f(a_n) - f(a)| \geq \epsilon_0.$$

Clearly, (a_n) is a sequence in D with the property that $\lim_{n\to\infty} a_n = a$ and

$$|f(a_n) - f(a)| \geq \epsilon_0 \text{ for all } n \in \mathbb{N}.$$

That is, $\lim_{n\to\infty} f(a_n) \neq f(a)$, a contradiction. ■

Examples

[1] Find the limit of the sequence $\left\{\ell\text{n}\left(\dfrac{n+1}{n}\right)\right\}$, if it exists.

Solution: Since

$$\lim_{n\to\infty} \frac{n+1}{n} = 1$$

and the function $f(x) = \ell\text{n}\ x$ is continuous on $(0, \infty)$, it follows from Theorem 5.2.4 that

$$\lim_{n\to\infty} \ell\text{n}\left(\frac{n+1}{n}\right) = \ell\text{n}\ 1 = 0.$$

That is, the sequence $\left\{\ell\text{n}\left(\dfrac{n+1}{n}\right)\right\}$ converges to 0. □

[2] Show that the function $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is continuous only at $x = 0$.

Solution: Let us first show that f is continuous at $x = 0$. Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon \text{ whenever } |x - 0| < \delta.$$

Now, since $0 \in \mathbb{Q}$, we have that

$$|f(x) - f(0)| = |f(x) - 0| = |f(x)|.$$

If $x \in \mathbb{Q}$, then $|f(x)| = |x|$, and if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $|f(x)| = |-x| = |x|$. Choose $\delta = \epsilon$. Then whenever $|x| < \delta$, we have that $|f(x) - f(0)| < \epsilon$, i.e., f is continuous at $x = 0$.

Next, we show that f is discontinuous on $a \in \mathbb{R} \setminus \{0\}$. Assume that f is continuous at some $a \in \mathbb{R} \setminus \{0\}$. If $a \in \mathbb{Q}$, then for each $n \in \mathbb{N}$ there is an $a_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$|a_n - a| < \frac{1}{n}.$$

That is, the sequence (a_n) converges to a. Since $a_n \in \mathbb{R} \setminus \mathbb{Q}$ for each $n \in \mathbb{N}$, $f(a_n) = -a_n$, and since $a \in \mathbb{Q}$, $f(a) = a$. Therefore

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} (-a_n) = -\lim_{n \to \infty} a_n = -a \neq a = f(a).$$

Similarly, if $a \in \mathbb{R} \setminus \mathbb{Q}$, then for each $n \in \mathbb{N}$ there is an $a_n \in \mathbb{Q}$ such that

$$|a_n - a| < \frac{1}{n}.$$

Again, the sequence (a_n) converges to a. Since $a_n \in \mathbb{Q}$ for each $n \in \mathbb{N}$, $f(a_n) = a_n$, and since $a \in \mathbb{R} \setminus \mathbb{Q}$, $f(a) = -a$. Therefore

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = a \neq -a = f(a).$$

Thus f is discontinuous at a. □

Exercise

[1] Show that the function $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is continuous only at $x = 0$.

[2] Show that the function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is discontinuous at every point of $\mathbb{R}$.

The following theorem asserts that continuity is preserved by the standard algebraic operations on functions.

Theorem

Let f and g be functions with common domain $D \subset \mathbb{R}$, and let $a \in D$. If f and g are continuous at a, then so are the functions

(i) $f \pm g$,

(ii) cf *for each* $c \in \mathbb{R}$,

(iii) $|f|$,

(iv) fg,

(v) $\dfrac{f}{g}$, *provided* $g(a) \neq 0$.

$\prec$Exercise. ■

Theorem
Let f be a function which is continuous at $a \in \mathbb{R}$. Suppose that g is a function which is continuous at the point $f(a)$. Then the composite function $g \circ f$ is continuous at a.

$\prec$Let $\epsilon > 0$ be given. Then there exist $\eta > 0$ and $\delta > 0$ such that

$$\begin{aligned} |g(y) - g(f(a))| &< \epsilon \text{ whenever } |y - f(a)| < \eta, \text{ and} \\ |f(x) - f(a)| &< \eta \text{ whenever } |x - a| < \delta. \end{aligned}$$

(Now δ depends on η, which in turn, depends on ϵ. Therefore δ depends on ϵ.) Hence, for all $x \in \mathbb{R}$ with $|x - a| < \delta$, we have that

$$|g(f(x)) - g(f(a))| < \epsilon.$$

That is, $g \circ f$ is continuous at a. ■

The next theorem, called the Intermediate Value Theorem, asserts that if the domain of a continuous function is an interval, then so is its range.

Theorem
(Intermediate Value Theorem). *If f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for each number k between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = k$.*

$\prec$For definiteness, assume that $f(a) < f(b)$. Let $S = \{x \in [a, b] \mid f(x) \leq k\}$. Then $S \neq \emptyset$ since $a \in S$. Thus, $c = \sup S$ exists as a real number in $[a, b]$. By Theorem 4.1.18, there exists a sequence (x_n) in S such that $\lim\limits_{n \to \infty} x_n = c$. Since $a \leq x_n \leq b$ for each $n \in \mathbb{N}$, we have that $a \leq c \leq b$, and so f is continuous at c. This then implies that $\lim\limits_{n \to \infty} f(x_n) = f(c)$. As $f(x_n) \leq k$ for each $n \in \mathbb{N}$, we deduce that $f(c) \leq k$, and so $c \in S$. It now remains to show that $f(c) \geq k$. To this end, we first observe that since $c \in S$ and $c = \sup S$, $c + \frac{1}{n} \notin S$ for each $n \in \mathbb{N}$. Also, since $k < f(b)$, we have that $c < b$. Therefore, by Corollary 3.1.17, there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < b - c$. Hence, for each $n \geq N$, we have that

$$\frac{1}{n} < b - c, \quad \text{i.e.,} \quad c + \frac{1}{n} < b.$$

This implies that for all $n \geq N$, $c + \frac{1}{n} \in [a, b]$ and $c + \frac{1}{n} \notin S$. Thus, $f(c + \dfrac{1}{n}) > k$ for all $n \geq N$. By continuity of f, we have that $f(c) \geq k$, whence $f(c) = k$. ■

One of the many interesting consequences of the Intermediate Value Theorem is the following *fixed-point theorem.*

Theorem
(Fixed-point Theorem). *If f is continuous on a closed interval $[a, b]$ and $f(x) \in [a, b]$ for each $x \in [a, b]$, then f has a fixed point; i.e., there exists a point $c \in [a, b]$ such that $f(c) = c$.*

$\prec$If $f(a) = a$ or $f(b) = b$, then we are done. We therefore assume that $a < f(a)$ and $f(b) < b$. Let $g(x) = f(x) - x$ for every $x \in [a, b]$. Clearly, g is a continuous function on $[a, b]$, $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. That is, 0 is an intermediate value for g on $[a, b]$. Hence, by the Intermediate Value Theorem (Theorem 5.2.9), there exists a $c \in [a, b]$ such that $g(c) = 0$. This, of course, implies that $f(c) = c$. ■

Uniform Continuity

Before giving the formal definition of uniform continuity we need to look closely at the definition of continuity given earlier. We said that *a function f, with domain $D \subset \mathbb{R}$, is continuous at $a \in D$ if, given any $\epsilon > 0$ there exists a $\delta > 0$ (which depends on ϵ and a) such that*

$$|f(x) - f(a)| < \epsilon \text{ whenever } x \in D \text{ and } |x - a| < \delta.$$

For continuity at another point $b \in D$, for the same ϵ, a $\delta' > 0$ would exist such that

$$|f(x) - f(b)| < \epsilon \text{ whenever } x \in D \text{ and } |x - b| < \delta'.$$

The δ and δ' may not be the same. Therefore, δ *depends* on ϵ as well as the point a. For this reason, continuity is a local concept – it describes what happens to a function in a neighbourhood of a point.

We now define uniform continuity.

Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is said to be **uniformly continuous on** *D if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in D \text{ and } |x - y| < \delta.$$

The most important point to note here is that δ does not depend on any particular point of the domain D – the same δ works for all points of D. Therefore uniform continuity is a global concept.

Examples

[1] Show that the function $f(x) = x$ is uniformly continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$ be given. We must produce a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in \mathbb{R} \text{ and } |x - y| < \delta.$$

Since $|f(x) - f(y)| = |x - y|$, we may choose $0 < \delta \leq \epsilon$. Then, for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have that

$$|f(x) - f(y)| = |x - y| < \delta \leq \epsilon.$$

That is, f is uniformly continuous on $\mathbb{R}$. □

[2] Show that the function $f(x) = x^2$ is *not* uniformly continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$ be given. We must show that for every $\delta > 0$ there exist $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and

$$|f(x) - f(y)| = |x^2 - y^2| \geq \epsilon.$$

Choose $x, y \in \mathbb{R}$ with $x - y = \dfrac{\delta}{2}$ and $x + y = \dfrac{2\epsilon}{\delta}$. Then $|x - y| < \delta$ and

$$|x^2 - y^2| = |x + y||x - y| \geq \frac{2\epsilon}{\delta} \cdot \frac{\delta}{2} = \epsilon.$$

Thus, f is not uniformly continuous on $\mathbb{R}$. □

[3] Show that the function $f(x) = x^2$ is uniformly continuous on $[-1, 1]$.

Solution: Let $\epsilon > 0$ be given. Then for all $x, y \in [-1, 1]$ we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \le 2|x - y|.$$

Choose $\delta = \frac{\epsilon}{2}$. Then, for all $x, y \in [-1, 1]$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \le 2|x - y| < 2\frac{\epsilon}{2} = \epsilon.$$

Hence f is uniformly continuous on $[-1, 1]$. □

[4] Show that the function $f(x) = \frac{1}{x}$ is *not* uniformly continuous on $(0, 1]$.

Solution: Let $\epsilon = \frac{1}{2}$ and $\delta > 0$. Then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Take $x = \frac{1}{n}$, and $y = \frac{1}{n+1}$. Then

$$|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta \quad \text{and} \quad |f(x) - f(y)| = |n - (n+1)| = 1 > \frac{1}{2}.$$

Hence f is not uniformly continuous on $(0, 1]$. □

[5] Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$, where $a > 0$.

Solution: Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in [a, \infty) \text{ and } |x - y| < \delta.$$

Now, for all $x, y \in [a, \infty)$,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \frac{|x - y|}{xy} \le \frac{|x - y|}{a^2}.$$

Take $\delta = a^2\epsilon$. Then, for all $x, y \in [a, \infty)$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| \le \frac{|x - y|}{a^2} < \epsilon.$$

That is, f is uniformly continuous on $[a, \infty)$. □

Theorem

If $f : D \to \mathbb{R}$ is uniformly continuous on D, then it is continuous there.

≺Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in D \text{ and } |x - y| < \delta.$$

Let $y = a \in D$. Then

$$|f(x) - f(a)| < \epsilon \text{ for all } x, y \in D \text{ such that } |x - y| < \delta.$$

Thus, f is continuous at $a \in D$. Since $a \in D$ was arbitrarily chosen, f is continuous on D. ■

Definition
A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a **Lipschitz condition** *on an interval $I \subset \mathbb{R}$ if there is a positive real number M such that*

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } \ x, y \in I.$$

If $M < 1$, then f is called a **contraction map**.

Examples

[1] Show that the function $f(x) = x^2$ satisfies a Lipschitz condition on $[0, 2]$.

Solution: For all $x, y \in [0, 2]$, we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 4|x - y|. \qquad \square$$

[2] Show that the function $f(x) = |x|$ satisfies a Lipschitz condition on $\mathbb{R}$.

Solution: For all $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y|. \qquad \square$$

[3] Show that the function $f(x) = \sin x$ satisfies a Lipschitz condition on $\mathbb{R}$.

Solution: Let $x, y \in \mathbb{R}$. Then

$$\begin{aligned} |f(x) - f(y)| = |\sin x - \sin y| &= \left|\sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right)\right| \\ &= \left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-y}{2}\right)\right| \\ &\leq |x - y|, \end{aligned}$$

where we have used the two facts:

$$\left|\cos\left(\frac{x+y}{2}\right)\right| \leq 1 \quad \text{and} \quad \left|\sin\left(\frac{x-y}{2}\right)\right| \leq \frac{|x-y|}{2}. \qquad \square$$

Theorem
If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition on an interval $I \subset \mathbb{R}$, then f is uniformly continuous there.

$\prec$Since f satisfies a Lipschitz condition on I, there exists a positive real number M such that

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } \ x, y \in I.$$

Let $\epsilon > 0$ be given and take $\delta = \dfrac{\epsilon}{M}$. Then, whenever $x, y \in I$ and $|x - y| < \delta$, we have that

$$|f(x) - f(y)| \leq M|x - y| < M\frac{\epsilon}{M} = \epsilon.$$

That is, f is uniformly continuous on I. ■

Theorem
If f is contraction map on a closed interval $[a, b]$ such that $f(x) \in [a, b]$ for each $x \in [a, b]$, then f has a unique fixed point; i.e., there exists exactly one point $c \in [a, b]$ such that $f(c) = c$.

$\prec$Since f is a contraction map, it is (uniformly) continuous on $[a, b]$. Furthermore, f satisfies the hypotheses of the fixed-point theorem. Therefore there exists a point $c \in [a, b]$ such that $f(c) = c$.

To prove uniqueness, assume that there is a $d \in [a, b]$ such that $f(d) = d$. Since f is a contraction map, there exists an $M \in \mathbb{R}$ such that $0 < M < 1$ and

$$|c - d| = |f(c) - f(d)| \leq M|c - d| < |c - d|,$$

which is impossible. Thus $c = d$. ■

Continuous Functions and Compact Sets

Theorem
A continuous image of a compact set is compact, i.e., if K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous on K, then the set

$$f(K) := \{y \in \mathbb{R} \mid f(x) = y \text{ for some } x \in K\}$$

is compact.

$\prec$Let (y_n) be a sequence in $f(K)$. Then, for each $n \in \mathbb{N}$, there is an $x_n \in K$ such that $y_n = f(x_n)$. Since K is compact, the sequence (x_n) has a subsequence $\left(x_{n_k}\right)$ which converges to some $x \in K$. Using continuity of f, we have, by Theorem 5.2.4, that $f\left(x_{n_k}\right) \overset{k\to\infty}{\longrightarrow} f(x) \in f(K)$. Hence, the subsequence $\left(y_{n_k}\right) = \left(f\left(x_{n_k}\right)\right)$ of (y_n) converges to $y = f(x) \in f(K)$. ■

Definition
A real-valued function f with domain D is said to be **bounded** *on D if there exists a positive real number M such that*

$$|f(x)| \leq M \text{ for all } x \in D.$$

A continuous function may not be bounded even when its domain is a bounded set. One such example is the function $f(x) = \dfrac{1}{x}$ defined on $(0, 1)$. As x approaches 0 from the right, f grows without bound.

The next theorem asserts that a *continuous* real-valued function defined on a compact set is always bounded there.

Corollary
If K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous on K, then f is bounded on K. That is,

$$M = \sup\{f(x) \mid x \in K\} \text{ and } m = \inf\{f(x) \mid x \in K\}$$

are finite. Moreover, there are points x_1 and x_2 in K such that $f(x_1) = M$ and $f(x_2) = m$.

$\prec$Since K is compact and f is continuous, it follows from Theorem 5.2.18 that $f(K)$ is a compact. By the Heine-Borel Theorem (Theorem 3.3.7), we have that $f(K)$ is closed and bounded. Therefore M and m are finite. Since $f(K)$ is closed, M and m belong to $f(K)$. Therefore there are points x_1 and x_2 such that $M = f(x_1)$ and $m = f(x_2)$. ■

While the Intermediate Value Theorem (Theorem 5.2.9) assures us that a continuous function takes an interval into an interval, the following theorem tells us that, in fact, a continuous function takes a closed and bounded interval into a closed and bounded one! Its proof is contained in Corollary 5.2.20.

Theorem
(Extreme Value Theorem). *If f is continuous on a closed interval $[a, b]$, then there exist points u and v in $[a, b]$ such that*

$$f(u) \leq f(x) \leq f(v) \ \text{ for all } \ x \in [a, b], \ \text{ i.e., } \ x \in [a, b] \ \Rightarrow \ f(x) \in [f(u), f(v)].$$

The following result asserts that a continuous function on a compact set is uniformly continuous on K.

Theorem
If K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous on K, then f is uniformly continuous.

$\prec$Assume that f is continuous on K but *not* uniformly continuous there. Then, there is an $\epsilon > 0$ such that, for each $\delta > 0$, there are points $x, y \in K$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. In particular, for each $n \in \mathbb{N}$, there are points $x_n, \ y_n \in K$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$. Since K is compact, the sequence (x_n) has a subsequence $\left(x_{n_k}\right)$ which converges to some $x \in K$. Similarly, the sequence (y_n) has a subsequence $\left(y_{n_k}\right)$ which converges to some $y \in K$. Since

$$0 \leq |x - y| \leq \left|x - x_{n_k}\right| + \left|x_{n_k} - y_{n_k}\right| + \left|y_{n_k} - y\right| \ \to \ 0 \ \text{ as } \ k \to \infty,$$

it follows that $x = y$. Since f is continuous on K, we have that $f\left(x_{n_k}\right) \overset{k\to\infty}{\longrightarrow} f(x)$ and $f\left(y_{n_k}\right) \overset{k\to\infty}{\longrightarrow} f(y)$. Hence, there are natural numbers N_1 and N_2 such that

$$\left|f(x_{n_k}) - f(x)\right| \ < \ \frac{\epsilon}{2} \ \text{ for all } \ k \geq N_1, \ \text{ and}$$

$$\left|f(y_{n_k}) - f(x)\right| \ < \ \frac{\epsilon}{2} \ \text{ for all } \ k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then for all $k \geq N$ we have

$$0 < \epsilon \leq \left|f(x_{n_k}) - f(y_{n_k})\right| \leq \left|f(x_{n_k}) - f(x)\right| + \left|f(y_{n_k}) - f(x)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is absurd. ■

Exercise

[1] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that if $f(q) = 0$ for all $q \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$. More generally, show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that vanishes on a dense set, then f is identically zero.

[2] Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **linear** if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that the function $f(x) = cx$, where $c \in \mathbb{R}$, is a continuous linear function. Show that, in fact, every continuous linear function f is of this form.

[3] Let $S \subset \mathbb{R}$. The **inverse image** of S under f, denoted by $f^{-1}(S)$, is the set

$$f^{-1}(S) = \{x \in \mathbb{R} : f(x) \in S\}.$$

Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if the inverse image $f^{-1}(V)$ of every open set V is open.

[4] Show that the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.

[5] Show that the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \sin \dfrac{1}{x}$ is continuous but not uniformly continuous on $\mathbb{R}^+$.

[6] Let $S \subset \mathbb{R}$ and, for $x \in \mathbb{R}$, define

$$f(x) = \inf\{|x - s| : s \in S\}.$$

Show that if $x \notin \overline{S}$, then $f(x) > 0$. Also, show that

$$|f(x) - f(y)| \leq |x - y| \quad \text{for all } \; x, y \in \mathbb{R}.$$

This says that f satisfies a Lipschitz condition on $\mathbb{R}$.