

# Sequences of Real Numbers

## Introduction

### Definition

A **sequence** is a function whose domain is the set  $\mathbb{N}$  of natural numbers. If  $f$  is such a sequence, let  $f(n) = x_n$  denote the value of the function  $f$  at  $n \in \mathbb{N}$ . In this case, we denote the sequence  $f$  by  $(x_n)_{n=1}^{\infty}$  (or simply by  $(x_n)$ ).

### Examples

[1]  $\left(\frac{n}{n+1}\right)$  is the sequence  $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$ .

[2]  $((-1)^n)$  is the sequence  $(-1, 1, -1, \dots)$ .

[3]  $(2^n)$  is the sequence  $(2, 4, 8, \dots)$ .

Sequences are also frequently specified by giving a recursion formula. For example, if  $x_{n+2} = \frac{x_n + x_{n+1}}{3}$ , where  $x_1 = 0$  and  $x_2 = 1$ , then the terms of the sequence  $(x_n)$  are:  $(0, 1, \frac{1}{3}, \frac{4}{9}, \frac{7}{27}, \dots)$ .

### Remarks

[1] The order of the terms of a sequence is an important part in the definition of a sequence. For example, the sequence  $(1, 5, 7, \dots)$  is *not* the same as the sequence  $(1, 7, 5, \dots)$ .

[2] There is a distinction between the *terms* of a sequence and the *values* of a sequence. A sequence has infinitely many terms while its values may or may not be finite.

[3] It is not necessary for the terms of a sequence to be different. For example,  $(1, 2, 2, 2, \dots)$  is a perfectly good sequence.

### Exercise

Write down the first five terms of each of the following sequences.

(i)  $\left(\frac{n^2 + 2n}{3n}\right)$  (ii)  $\left(\frac{\cos n\pi}{n^2}\right)$  (iii)  $\left(\frac{1}{\sqrt{n}}\right)$  (iv)  $\left(\sin \frac{n\pi}{2}\right)$ .

### Definition

A sequence  $(x_n)$  is said to be

[1] **bounded above** if there is a real number  $K$  such that  $x_n \leq K$  for all  $n \in \mathbb{N}$ ;

[2] **bounded below** if there is a real number  $k$  such that  $k \leq x_n$  for all  $n \in \mathbb{N}$ ;

[3] **bounded** if it is bounded above and bounded below; otherwise it is **unbounded**.

It is easy to see that a sequence  $(x_n)_n$  is bounded if and only if there is a positive real number  $M$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

### Examples

[1] The sequence  $\left(\frac{1}{n}\right)$  is bounded since  $0 < \frac{1}{n} \leq 1$  for all  $n \in \mathbb{N}$ .

[2] The sequence  $\left(n + \frac{1}{n}\right)$  is bounded below by 2 but is not bounded above.

[3] The sequence  $((-1)^n n)$  is not bounded above and it is not bounded below.

**Definition**

[1] A sequence  $(x_n)$  is said to **converge** to a real number  $\ell$  if, given  $\epsilon > 0$ , there exists a natural number  $N$  (which depends on  $\epsilon$ ) such that

$$|x_n - \ell| < \epsilon \text{ for all } n \geq N.$$

Symbolically,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N) \Rightarrow |x_n - \ell| < \epsilon].$$

If  $(x_n)$  converges to  $\ell$ , then we say that  $\ell$  is the **limit** of the sequence  $(x_n)$  as  $n$  increases without bound, and we write

$$\lim_{n \rightarrow \infty} x_n = \ell \text{ or } x_n \rightarrow \ell \text{ as } n \rightarrow \infty.$$

[2] If the sequence  $(x_n)$  does not converge to a real number, we say that it **diverges**.

[3] A sequence  $(x_n)$  is said to **diverge to**  $\infty$ , denoted by  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ , if for any positive real number  $M$ , there is an  $N \in \mathbb{N}$  such that

$$x_n > M \text{ for all } n \geq N.$$

Similarly,  $(x_n)$  **diverges to**  $-\infty$ , denoted by  $x_n \rightarrow -\infty$  as  $n \rightarrow \infty$ , if for any negative real number  $K$ , there is an  $N \in \mathbb{N}$  such that

$$x_n < K \text{ for all } n \geq N.$$

It is clear from the definition that convergence or divergence of a sequence is about the behaviour of the ‘tail-end’ of a sequence. Therefore, altering a finite number of terms of a sequence does not affect its convergence or divergence.

**Examples**

[1] Show that a sequence  $(x_n)$  converges to zero if and only if the sequence  $(|x_n|)$  converges to zero.

Solution: Assume that the sequence  $(x_n)$  converges to zero. Then, given  $\epsilon > 0$ , there exists a natural number  $N$  (which depends on  $\epsilon$ ) such that

$$|x_n - 0| = |x_n| < \epsilon \text{ for all } n \geq N.$$

Now, for all  $n \geq N$ , we have

$$||x_n| - 0| = |x_n| < \epsilon.$$

That is, the sequence  $(|x_n|)$  converges to zero.

For the converse, assume that the sequence  $(|x_n|)$  converges to zero. That is, given  $\epsilon > 0$ , there exists a natural number  $N$  (which depends on  $\epsilon$ ) such that

$$||x_n| - 0| = |x_n| < \epsilon \text{ for all } n \geq N.$$

It now follows that the sequence  $(x_n)$  converges to zero.

[2] Show that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

Solution: Let  $\epsilon > 0$  be given. We must find an  $N \in \mathbb{N}$  such that

$$\left| \frac{1}{n} - 0 \right| < \epsilon \text{ for all } n \geq N.$$

By the Archimedean Property, there is an  $N \in \mathbb{N}$  such that  $0 < \frac{1}{N} < \epsilon$ . Thus, if  $n \geq N$ , then we have that

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

That is,  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

[3] Show that  $\lim_{n \rightarrow \infty} \left( 1 - \frac{1}{2^n} \right) = 1$ .

Solution: Let  $\epsilon > 0$  be given. We need to find an  $N \in \mathbb{N}$  such that

$$\left| \left( 1 - \frac{1}{2^n} \right) - 1 \right| < \epsilon \text{ for all } n \geq N.$$

Noting that

$$\left| \left( 1 - \frac{1}{2^n} \right) - 1 \right| = \frac{1}{2^n} = \frac{1}{(1+1)^n}, \text{ and}$$

$$(1+1)^n = \sum_{k=0}^n \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \geq \binom{n}{0} + \binom{n}{1} = 1 + n,$$

we have that

$$\frac{1}{2^n} = \frac{1}{(1+1)^n} \leq \frac{1}{n+1} < \frac{1}{n}.$$

Now, by the Archimedean Property, there is an  $N \in \mathbb{N}$  such that  $0 < \frac{1}{N} < \epsilon$ . Therefore, for all  $n \geq N$  we have

$$\left| \left( 1 - \frac{1}{2^n} \right) - 1 \right| = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Thus,  $\lim_{n \rightarrow \infty} \left( 1 - \frac{1}{2^n} \right) = 1$ .

[4] Find  $\lim_{n \rightarrow \infty} \frac{n}{n^2 - 2}$ .

Solution: By trying a few values of  $n$ , we conjecture that  $\lim_{n \rightarrow \infty} \frac{n}{n^2 - 2} = 0$ . Let us prove this conjecture. Let  $\epsilon > 0$  be given. We need to find an  $N \in \mathbb{N}$  such that

$$\left| \frac{n}{n^2 - 2} - 0 \right| < \epsilon \text{ for all } n \geq N.$$

We first note that

$$\left| \frac{n}{n^2 - 2} \right| = \frac{n}{|n^2 - 2|}.$$

If we take  $N \geq 2$ , then for all  $n \geq N$ , we have that

$$\frac{n}{|n^2 - 2|} = \frac{n}{n^2 - 2} = \frac{1}{n - \frac{2}{n}} < \frac{1}{n - 1} \leq \frac{1}{N - 1}.$$

If we can choose  $N$  so that  $N \geq 2$  and  $\frac{1}{N-1} < \epsilon$ , then we are done. That is, we need to choose  $N$  so that  $N \geq 2$  and  $N > 1 + \frac{1}{\epsilon}$ . Choose  $N > \max\left(2, 1 + \frac{1}{\epsilon}\right)$ . Then, by working backwards, we have that

$$\left| \frac{n}{n^2 - 2} - 0 \right| < \epsilon \text{ for all } n \geq N.$$

[5] Show that the sequence  $((-1)^n)$  diverges.

Solution: Assume that this sequence converges to some real number  $\ell$ . Then, with  $\epsilon = \frac{1}{2}$ , there is an  $N \in \mathbb{N}$  such that

$$|(-1)^n - \ell| < \frac{1}{2} \text{ for all } n \geq N.$$

In particular,

$$|(-1)^{n+1} - \ell| < \frac{1}{2}.$$

Therefore, for all  $n \geq N$ ,

$$2 = |(-1)^n - (-1)^{n+1}| \leq |(-1)^n - \ell| + |\ell - (-1)^{n+1}| < \frac{1}{2} + \frac{1}{2} = 1,$$

which is absurd.

[6] Show that the sequence  $(1 + (-1)^n)$  diverges.

Solution: Assume that this sequence converges to some real number  $\ell$ . Then, with  $\epsilon = 1$ , there exists a number  $N \in \mathbb{N}$  such that

$$|(1 + (-1)^n) - \ell| < 1 \text{ for all } n \geq N.$$

Now, if  $n \geq N$  is odd, then we have

$$|(1 + (-1)^n) - \ell| = |\ell| < 1, \text{ whence } -1 < \ell < 1,$$

and if  $n \geq N$  is even, we have that

$$|(1 + (-1)^n) - \ell| = |2 - \ell| < 1, \text{ whence } 1 < \ell < 3.$$

But this is impossible.

[7] Show that if  $x \in \mathbb{R}$  and  $|x| < 1$ , then  $\lim_{n \rightarrow \infty} x^n = 0$ .

Solution: If  $x = 0$ , then there is nothing to prove. Assume that  $x \neq 0$ . Since  $|x| < 1$ , we have that  $\frac{1}{|x|} > 1$ . Thus, there is a positive real number  $a$  such that

$$\frac{1}{|x|} = 1 + a.$$

Let  $\epsilon > 0$  be given. We want to find an  $N \in \mathbb{N}$  such that

$$|x^n - 0| < \epsilon \text{ for all } n \geq N.$$

Now,

$$\frac{1}{|x|} = 1 + a \Rightarrow |x^n| = |x|^n = \frac{1}{(1+a)^n}.$$

Using the binomial theorem, we have that

$$(1+a)^n = \sum_{k=0}^n \binom{n}{k} a^k = 1 + na + \frac{n(n-1)a^2}{2} + \dots + a^n > na,$$

and consequently,

$$|x^n| = \frac{1}{(1+a)^n} < \frac{1}{na}.$$

If we can find an  $N \in \mathbb{N}$  such that  $\frac{1}{na} < \epsilon$  for all  $n \geq N$ , then we are done. Since  $a\epsilon > 0$ , we have, by the Archimedean Property, that there is an  $N \in \mathbb{N}$  such that  $\frac{1}{N} < a\epsilon$ . Hence, for all  $n \geq N$ , we have that

$$\frac{1}{n} \leq \frac{1}{N} \Rightarrow \frac{1}{na} \leq \frac{1}{Na} < \epsilon,$$

and so  $|x^n| < \epsilon$ .

[8] Suppose that  $(x_n)$  is a sequence such that  $x_n > 0$  for all  $n \in \mathbb{N}$ . Show that  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$  if and only if  $\lim_{n \rightarrow \infty} \frac{1}{x_n} = 0$ .

Solution: By definition,  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ , if and only if for any  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$x_n > \frac{1}{\epsilon} \text{ for all } n \geq N.$$

This is equivalent to the statement that

$$\frac{1}{x_n} < \epsilon \text{ for all } n \geq N, \text{ which, in turn, is equivalent to } \lim_{n \rightarrow \infty} \frac{1}{x_n} = 0.$$

**Theorem**

Let  $(s_n)$  and  $(t_n)$  be sequences of real numbers and let  $s \in \mathbb{R}$ . If for some positive real number  $k$  and some  $N_1 \in \mathbb{N}$ , we have

$$|s_n - s| \leq k|t_n| \text{ for all } n \geq N_1$$

and if  $\lim_{n \rightarrow \infty} t_n = 0$ , then  $\lim_{n \rightarrow \infty} s_n = s$ .

<Let  $\epsilon > 0$  be given. Since  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ , there exists an  $N_2 \in \mathbb{N}$  such that

$$|t_n| < \frac{\epsilon}{k} \text{ for all } n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$  we have

$$|s_n - s| \leq k|t_n| < \frac{\epsilon}{k} \cdot k = \epsilon.$$

That is,  $\lim_{n \rightarrow \infty} s_n = s$ . ■

**Example**

Show that  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .

Solution: Since  $\sqrt[n]{n} \geq 1$  for each  $n \in \mathbb{N}$ , there is a nonnegative real number  $a_n$  such that

$$\sqrt[n]{n} = 1 + a_n.$$

Thus, by the binomial theorem, we have

$$n = (1 + a_n)^n = \sum_{k=0}^n \binom{n}{k} a_n^k = 1 + na_n + \frac{n(n-1)a_n^2}{2} + \dots + a_n^n \geq 1 + \frac{n(n-1)a_n^2}{2}.$$

Therefore,

$$n - 1 \geq \frac{n(n-1)a_n^2}{2}, \text{ whence } a_n^2 \leq \frac{2}{n}, \text{ or } a_n \leq \sqrt{\frac{2}{n}} \text{ for all } n \geq 2.$$

Now, since

$$|\sqrt[n]{n} - 1| = |a_n| = a_n \leq \sqrt{\frac{2}{n}}$$

and  $\lim_{n \rightarrow \infty} \sqrt{\frac{2}{n}} = 0$ , we have, by Theorem 4.1.9, that  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ . □

The next theorem says that a convergent sequence has only one limit. It is therefore unambiguous to talk of *the* limit of a convergent sequence.

**Theorem**

Let  $(s_n)$  be a sequence of real numbers. If  $\lim_{n \rightarrow \infty} s_n = \ell_1$  and  $\lim_{n \rightarrow \infty} s_n = \ell_2$ , then  $\ell_1 = \ell_2$ .

<Let  $\epsilon > 0$  be given. Then there exist natural numbers  $N_1$  and  $N_2$  such that

$$|s_n - \ell_1| < \frac{\epsilon}{2} \text{ for all } n \geq N_1, \text{ and}$$

$$|s_n - \ell_2| < \frac{\epsilon}{2} \text{ for all } n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$  we have that

$$|\ell_1 - \ell_2| = |(s_n - \ell_2) + (\ell_1 - s_n)| \leq |s_n - \ell_2| + |s_n - \ell_1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since  $\epsilon > 0$  is arbitrary, we have that  $\ell_1 = \ell_2$ . ■

**Proposition**

A sequence  $(x_n)$  converges to  $\ell \in \mathbb{R}$  if and only if for each  $\epsilon > 0$ , the set  $\{n \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\}$  is finite.

<Assume that the sequence  $(x_n)$  converges to  $\ell$ . Then, given  $\epsilon > 0$ , there is a natural number  $N$  such that, for all  $n \geq N$ ,

$$|x_n - \ell| < \epsilon \iff \{x_n \mid n \geq N\} \subset (\ell - \epsilon, \ell + \epsilon).$$

It now follows that  $\{n \in \mathbb{N} \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\} \subset \{1, 2, \dots, N - 1\}$ , a finite set.

Conversely, let  $\epsilon > 0$  be given and assume that the set  $\{n \in \mathbb{N} \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\}$  is finite. Let  $N = \max\{n \in \mathbb{N} \mid x_n \notin (\ell - \epsilon, \ell + \epsilon)\} + 1$ . If  $n \geq N$ , then

$$x_n \in (\ell - \epsilon, \ell + \epsilon) \iff |x_n - \ell| < \epsilon,$$

and so  $x_n \rightarrow \ell$  as  $n \rightarrow \infty$ . ■

**Theorem**

Every convergent sequence of real numbers is bounded.

<Let  $(s_n)$  be a sequence of real numbers which converges to  $s$ , say. Then, with  $\epsilon = 1$ , there exists an  $N \in \mathbb{N}$  such that

$$|s_n - s| < 1 \text{ for all } n \geq N.$$

By the triangle inequality, we have that

$$|s_n| \leq |s_n - s| + |s| < 1 + |s| \text{ for all } n \geq N.$$

Let  $M = \max\{|s_1|, |s_2|, \dots, |s_N|, |s| + 1\}$ . Then  $|s_n| \leq M$  for all  $n \in \mathbb{N}$ . That is, the sequence  $(s_n)$  is bounded. ■

The converse of Theorem 4.1.13 is not necessarily true. That is, there are sequences which are bounded but do not converge. One such example is the sequence  $((-1)^n)$ . We shall however see later that every bounded sequence which is *monotone* will always converge.

**Exercise**

- [1] Show that if  $s_n \rightarrow s$ , then  $|s_n| \rightarrow |s|$ . Does the converse hold?
- [2] Show that if  $(s_n)$  and  $(t_n)$  are sequences with  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$  and if  $s_n \leq t_n$  for all  $n \in \mathbb{N}$ , then  $s \leq t$ .
- [3] Let  $(s_n)$  and  $(t_n)$  be sequences such that  $(s_n)$  is bounded and  $\lim_{n \rightarrow \infty} t_n = 0$ . Show that  $\lim_{n \rightarrow \infty} s_n t_n = 0$ .
- [4] Which of the following sequences are bounded?

- (i)  $\left(\frac{n}{n+4}\right)$     (ii)  $\left(\frac{n^2+n-4}{n+5}\right)$     (iii)  $\left(1 - \frac{1}{2^n}\right)$     (iv)  $(1 + (-1)^n)$
- (v)  $(\sqrt[n]{n})$     (vi)  $\left(\frac{1 + (-1)^n}{n}\right)$     (vii)  $\left(\frac{1 + 2^n}{2^n}\right)$ .

**Theorem**

(Squeeze Theorem). Suppose that  $(s_n)$ ,  $(t_n)$  and  $(u_n)$  are sequences such that  $s_n \leq t_n \leq u_n$  for all  $n \in \mathbb{N}$ . If  $\lim_{n \rightarrow \infty} s_n = \ell = \lim_{n \rightarrow \infty} u_n$ , then  $\lim_{n \rightarrow \infty} t_n = \ell$ .

<Let  $\epsilon > 0$  be given. Then there exist  $N_1$  and  $N_2$  in  $\mathbb{N}$  such that

$$\begin{aligned} |s_n - \ell| &< \epsilon \text{ for all } n \geq N_1 \quad \text{and} \\ |u_n - \ell| &< \epsilon \text{ for all } n \geq N_2. \end{aligned}$$

That is,

$$\begin{aligned} \ell - \epsilon &< s_n < \ell + \epsilon \text{ for all } n \geq N_1 \quad \text{and} \\ \ell - \epsilon &< u_n < \ell + \epsilon \text{ for all } n \geq N_2. \end{aligned}$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ , we have

$$\ell - \epsilon < s_n \leq t_n \leq u_n < \ell + \epsilon,$$

and consequently,

$$|t_n - \ell| < \epsilon \text{ for all } n \geq N.$$

That is,  $\lim_{n \rightarrow \infty} t_n = \ell$ . ■

**Examples**

[1] Show that  $\lim_{n \rightarrow \infty} \frac{\cos \frac{n\pi}{2}}{n^2} = 0$ .

Solution: Since

$$0 \leq \left| \frac{\cos \frac{n\pi}{2}}{n^2} - 0 \right| = \left| \frac{\cos \frac{n\pi}{2}}{n^2} \right| \leq 1 \cdot \frac{1}{n^2}, \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0,$$

it follows that  $\lim_{n \rightarrow \infty} \frac{\cos \frac{n\pi}{2}}{n^2} = 0$ . □

[2] Show that for any  $x$ , with  $|x| < 1$ ,  $\lim_{n \rightarrow \infty} nx^n = 0$ .

Solution: Without loss of generality, we assume that  $x \neq 0$  and  $n > 1$ . Since  $|x| < 1$ , there is a positive real number  $a$  such that

$$\frac{1}{|x|} = 1 + a.$$

Then

$$\frac{1}{|x^n|} = (1 + a)^n = \sum_{r=0}^n \binom{n}{r} a^r \geq \frac{n(n-1)}{2} a^2$$

for some  $a > 0$ . Thus,

$$\begin{aligned} |x^n| &\leq \frac{2}{n(n-1)a^2} \\ \Rightarrow |nx^n| &\leq \frac{2}{(n-1)a^2} \\ \Rightarrow \frac{-2}{(n-1)a^2} &\leq nx^n \leq \frac{2}{(n-1)a^2}. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \frac{-2}{(n-1)a^2} = 0 = \lim_{n \rightarrow \infty} \frac{2}{(n-1)a^2},$$

we have, by the Squeeze Theorem, that

$$\lim_{n \rightarrow \infty} nx^n = 0.$$

[3] Show that for any  $x \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ .

Solution: Let  $N$  be the first integer greater than  $|x|$ . Then, if  $n \geq N$ ,

$$\left| \frac{x^n}{n!} \right| = \frac{|x|^n}{n!} = \frac{|x|^{N-1} |x|^{n-N+1}}{(N-1)! N(N+1) \cdots n} = \frac{|x|^{N-1}}{(N-1)!} \cdot \frac{|x|^{n-N+1}}{N(N+1) \cdots n}.$$

Let  $K = \frac{|x|^{N-1}}{(N-1)!}$ . Then  $K$  is a constant which is independent of  $n$ . Thus,

$$\left| \frac{x^n}{n!} \right| = K \cdot \frac{|x|^{n-N+1}}{N(N+1) \cdots n} < K \cdot \frac{|x|^{n-N+1}}{N \cdot N \cdot N \cdots N} = K \cdot \left( \frac{|x|}{N} \right)^{n-N+1},$$

whence

$$-K \left( \frac{|x|}{N} \right)^{n-N+1} < \frac{x^n}{n!} < K \left( \frac{|x|}{N} \right)^{n-N+1}.$$

Since  $\frac{|x|}{N} < 1$  and  $n - N + 1 \rightarrow \infty$  as  $n \rightarrow \infty$ , it follows that

$$\lim_{n \rightarrow \infty} \left( \frac{|x|}{N} \right)^{n-N+1} = 0,$$

and consequently,

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0. \quad \square$$

**Exercise**

[1] Show that  $\lim_{n \rightarrow \infty} \frac{n}{2^n} = 0$ .

[2] Show that  $\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$ .

**Theorem**

Let  $S$  be a subset of  $\mathbb{R}$  which is bounded above. Then there exists a sequence  $(s_n)$  in  $S$  such that

$$\lim_{n \rightarrow \infty} s_n = \sup S.$$

<Let  $c = \sup S$ . By the characterisation of supremum (Theorem 3.1.9), for each  $n \in \mathbb{N}$  there exists  $s_n \in S$  such that

$$c - \frac{1}{n} < s_n \leq c.$$

Since  $\lim_{n \rightarrow \infty} \left( c - \frac{1}{n} \right) = c = \lim_{n \rightarrow \infty} c$ , we have, by the Squeeze Theorem, that

$$\lim_{n \rightarrow \infty} s_n = c = \sup S. \quad \blacksquare$$

## Algebra of Limits

The following lemma asserts that if the sequence  $(t_n)$  converges to  $t \neq 0$ , then the sequence  $(t_n)$  is “bounded away from zero”.

**Lemma**

If the sequence  $(t_n)$  converges to  $t \neq 0$ , then there is an  $N \in \mathbb{N}$  such that  $|t_n| > \frac{|t|}{2}$  for all  $n \geq N$ .

<Since  $t \neq 0$ ,  $|t| > 0$ . Let  $\epsilon = \frac{|t|}{2}$ . Then there exists an  $N \in \mathbb{N}$  such that

$$|t_n - t| < \epsilon \text{ for all } n \geq N.$$

Thus,

$$|t_n| = |t - (t - t_n)| \geq |t| - |t - t_n| > |t| - \frac{|t|}{2} = \frac{|t|}{2} \text{ for all } n \geq N.$$

**Theorem**

Let  $(s_n)$  and  $(t_n)$  be sequences of real numbers which converge to  $s$  and  $t$  respectively. Then

- (i)  $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t.$   
 (ii)  $\lim_{n \rightarrow \infty} s_n t_n = st.$   
 (iii)  $\lim_{n \rightarrow \infty} \frac{s_n}{t_n} = \frac{s}{t}$  if  $t_n \neq 0$  for all  $n \in \mathbb{N}$  and  $t \neq 0.$

<

- (i) Let  $\epsilon > 0$  be given. Then there exist  $N_1$  and  $N_2$  in  $\mathbb{N}$  such that

$$|s_n - s| < \frac{\epsilon}{2} \text{ for all } n \geq N_1 \text{ and}$$

$$|t_n - t| < \frac{\epsilon}{2} \text{ for all } n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ , we have

$$\begin{aligned} |(s_n + t_n) - (s + t)| &= |(s_n - s) + (t_n - t)| \\ &\leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence,  $\lim_{n \rightarrow \infty} (s_n + t_n) = s + t.$

- (ii) Let  $\epsilon > 0$  be given. Now,

$$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s t_n + s t_n - st| = |(s_n - s)t_n + (t_n - t)s| \\ &\leq |s_n - s||t_n| + |t_n - t||s|. \end{aligned}$$

Since  $(t_n)$  is convergent, it is bounded. Therefore there is a positive real number  $K$  such that

$$|t_n| \leq K \text{ for all } n \in \mathbb{N}.$$

Thus,

$$|s_n t_n - st| \leq |s_n - s||t_n| + |t_n - t||s| \leq |s_n - s|K + |t_n - t||s|.$$

Let  $M = \max\{K, |s|\}$ . Then

$$|s_n t_n - st| \leq M(|s_n - s| + |t_n - t|).$$

Since  $s_n \rightarrow s$  and  $t_n \rightarrow t$  as  $n \rightarrow \infty$ , there exist  $N_1$  and  $N_2$  in  $\mathbb{N}$  such that

$$|s_n - s| < \frac{\epsilon}{2(M+1)} \text{ for all } n \geq N_1 \text{ and}$$

$$|t_n - t| < \frac{\epsilon}{2(M+1)} \text{ for all } n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ , we have

$$|s_n t_n - st| \leq M(|s_n - s| + |t_n - t|) < (M+1) \left( \frac{\epsilon}{2(M+1)} + \frac{\epsilon}{2(M+1)} \right) = \epsilon.$$

Hence,  $\lim_{n \rightarrow \infty} s_n t_n = st.$

- (iii) It suffices to show that  $\lim_{n \rightarrow \infty} \frac{1}{t_n} = \frac{1}{t}$  if  $t_n \neq 0$  for all  $n \in \mathbb{N}$  and  $t \neq 0$ . Once this has been shown we can then apply (ii). By Lemma 4.2.1, there is an  $N_1 \in \mathbb{N}$  such that

$$|t_n| > \frac{|t|}{2} \text{ for all } n \geq N_1.$$

Again, there exists an  $N_2 \in \mathbb{N}$  such that

$$|t_n - t| < \frac{\epsilon|t|^2}{2} \text{ for all } n \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ , we have

$$\left| \frac{1}{t_n} - \frac{1}{t} \right| = \left| \frac{t - t_n}{t t_n} \right| = \frac{|t - t_n|}{|t| |t_n|} < \frac{2|t - t_n|}{|t|^2} < \frac{2}{|t|^2} \cdot \frac{\epsilon|t|^2}{2} = \epsilon.$$

Hence,  $\lim_{n \rightarrow \infty} \frac{1}{t_n} = \frac{1}{t}$ . ■

### Exercise

[1] Let  $(s_n)$  and  $(t_n)$  be sequences of real numbers. Prove or disprove the following statements.

- (i) If  $(s_n)$  converges and  $(t_n)$  diverges, then the sequence  $(s_n + t_n)$  diverges.
- (ii) If both  $(s_n)$  and  $(t_n)$  diverge, then the sequence  $(s_n + t_n)$  also diverges.
- (iii) If both  $(s_n)$  and  $(t_n)$  diverge, then the sequence  $(s_n t_n)$  also diverges.
- (iv) If both  $(s_n)$  and  $(t_n)$  diverge and  $t_n \neq 0$  for all  $n \in \mathbb{N}$ , then the sequence  $\left(\frac{s_n}{t_n}\right)$  diverges.
- (v) If both  $(s_n)$  and  $(s_n t_n)$  converge, then the sequence  $(t_n)$  converges.
- (vi) If both  $(s_n)$  and  $(s_n t_n)$  diverge, then the sequence  $(t_n)$  diverges.

[2] (a) Show that if the sequence  $(s_n)$  converges to  $s$ , then the sequence  $\{s_n^2\}$  converges to  $s^2$ .

- (b) Use (a) and the fact that for all  $x, y \in \mathbb{R}$ ,  $xy = \frac{1}{4}[(x + y)^2 - (x - y)^2]$  to give an alternative proof of Theorem 4.2.2(ii).

## Monotone Sequences

### Definition

Let  $(s_n)$  be a sequence of real numbers. We say that  $(s_n)$  is

- (a) **increasing** if for each  $n \in \mathbb{N}$ ,  $s_n \leq s_{n+1}$ .
- (b) **strictly increasing** if for each  $n \in \mathbb{N}$ ,  $s_n < s_{n+1}$ .
- (c) **decreasing** if for each  $n \in \mathbb{N}$ ,  $s_{n+1} \leq s_n$ .
- (d) **strictly decreasing** if for each  $n \in \mathbb{N}$ ,  $s_{n+1} < s_n$ .
- (e) **monotone** if  $(s_n)$  is increasing or decreasing.
- (f) **strictly monotone** if  $(s_n)$  is strictly increasing or strictly decreasing.

**Remark**

An increasing sequence  $(s_n)$  is bounded below by  $s_1$ ; a decreasing sequence  $(t_n)$  is bounded above by  $t_1$ . It therefore follows that an increasing sequence is bounded if and only if it is bounded above. Similarly, a decreasing sequence is bounded if and only if it is bounded below.

**Examples**

- [1] The sequence  $(1, 1, 2, 3, 5, \dots)$  is increasing.
- [2] The sequence  $(3, 1, 0, 0, -3, -7, \dots)$  is decreasing.
- [3] The sequence  $(n^2)$  is strictly increasing.
- [4] The sequence  $(-n)$  is strictly decreasing.

The regular behaviour of monotone sequences makes it easier to determine its convergence or divergence.

**Theorem**

(Monotone Convergence Theorem). *A monotone sequence converges if and only if it is bounded.*

<We have already proved in Theorem 4.1.13 that if a sequence converges then it is bounded.

To prove the converse, let  $(s_n)$  be a bounded increasing sequence and let  $S = \{s_n \mid n \in \mathbb{N}\}$ . Since  $S$  is bounded above, it has a supremum,  $\sup S = s$ , say. We claim that  $\lim_{n \rightarrow \infty} s_n = s$ . Let  $\epsilon > 0$  be given. By the characterisation of supremum (Theorem 3.1.9), there exists  $s_N \in S$  such that

$$s - \epsilon < s_N \leq s_n \leq s < s + \epsilon \text{ for all } n \geq N.$$

Thus,  $|s_n - s| < \epsilon$  for all  $n \geq N$ .

The proof for the case when the sequence  $(s_n)$  is decreasing is similar. ■

**Examples**

- [1] Show that  $\left(\frac{n+1}{n}\right)$  is a convergent sequence.

Solution: We show that the sequence  $\left(\frac{n+1}{n}\right)$  is (1) monotone, and (2) bounded. Its convergence will then follow from the Monotone Convergence Theorem (Theorem 4.2.7).

Monotonicity: Let  $s_n = \frac{n+1}{n}$ . Then

$$\begin{aligned} \frac{s_{n+1}}{s_n} &= \frac{n+2}{n+1} \div \frac{n+1}{n} = \frac{n+2}{n+1} \times \frac{n}{n+1} = \frac{n^2+2n}{(n+1)^2} \\ &< \frac{n^2+2n+1}{(n+1)^2} = \frac{(n+1)^2}{(n+1)^2} = 1. \end{aligned}$$

Thus,

$$s_n = \frac{n+1}{n} > \frac{n+2}{n+1} = s_{n+1} \text{ for all } n \in \mathbb{N}.$$

Therefore, the sequence  $\left(\frac{n+1}{n}\right)$  is monotone decreasing.

Another proof of monotonicity: Consider  $f(x) = \frac{x+1}{x}$  for all  $x \in [1, \infty)$ . Then,

$$f'(x) = \frac{x - (x+1)}{x^2} = \frac{-1}{x^2} < 0 \text{ for all } x \in [1, \infty).$$

Thus,  $f$  is decreasing on  $[1, \infty)$ . Therefore

$$f(n) > f(n+1) \text{ i.e., } \frac{n+1}{n} > \frac{n+2}{n+1} \text{ for all } n \in \mathbb{N}.$$

Boundedness:

$$\frac{n+1}{n} = 1 + \frac{1}{n} > 1 \text{ for all } n \in \mathbb{N}.$$

Thus, the sequence  $\left(\frac{n+1}{n}\right)$  is bounded below by 1. □

[2] Show that  $\left(\left(1 + \frac{1}{n}\right)^n\right)$  is a convergent sequence.

Solution: We need an easy preliminary result:

$$r! \geq 2^{r-1} \text{ for } r \geq 2.$$

This follows from the fact that

$$r! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots r \geq 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 2^{r-1} \text{ for } r \geq 2.$$

We establish the existence of the limit by showing that

(a)  $\left(1 + \frac{1}{n}\right)^n < 3$  for each  $n \in \mathbb{N}$  (boundedness),

(b)  $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$  for each  $n > 2$  (monotonicity).

Now, by the binomial theorem,

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= \sum_{r=0}^n \binom{n}{r} \frac{1}{n^r} = \sum_{r=0}^n \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \frac{1}{n^r} \\ &= 1 + \frac{n}{n} + \sum_{r=2}^n \frac{1}{r!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{r-1}{n}\right); \end{aligned} \quad (*)$$

the first two terms correspond to  $r = 0$  and  $r = 1$ , respectively. Then we obtain (a) from (\*) by noting that

$$\left(1 + \frac{1}{n}\right)^n < 2 + \sum_{r=2}^n \frac{1}{r!} < 2 + \sum_{r=2}^n \frac{1}{2^{r-1}}$$

and so

$$\left(1 + \frac{1}{n}\right)^n < 2 + \frac{\frac{1}{2}(1 - \frac{1}{2^{n-1}})}{1 - \frac{1}{2}} < 3.$$

We obtain (b) from (\*) by noting that if we replace  $n$  by  $n + 1$ , each of the brackets in (\*) becomes bigger, so that each of the terms under the  $\sum$  sign becomes bigger; and there is also one more positive term in the series. That's all we want! □

(The limit that the sequence  $\left(\left(1 + \frac{1}{n}\right)^n\right)$  converges to is denoted by the letter  $e$ .)

**Theorem**

(Nested Intervals Theorem). For each  $n \in \mathbb{N}$ , let  $I_n = [a_n, b_n]$ , where  $-\infty < a_n < b_n < \infty$ . If  $I_{n+1} \subset I_n$  for each  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$ , then  $\bigcap_{n=1}^{\infty} I_n$  consists of exactly one point.

<Since  $I_{n+1} \subset I_n$ , we have that  $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$  for all  $n \in \mathbb{N}$ . Thus,  $(a_n)$  is an increasing sequence of real numbers. Let  $A$  be the set of all endpoints  $a_n$ , i.e.,  $A = \{a_n : n \in \mathbb{N}\}$ .

**Claim:**  $a_k \leq b_\ell$  for all  $k, \ell \in \mathbb{N}$ . Indeed, if  $k \leq \ell$ , then  $a_k \leq a_\ell \leq b_\ell$ . On the other hand, if  $\ell \leq k$ , then  $a_k \leq b_k \leq b_\ell$ .

It now follows that  $A$  is a nonempty set which is bounded above (by every  $b_n$ ). By the Completeness Axiom,  $A$  has a supremum,  $a = \sup A$  (say). Clearly,  $a_n \leq a \leq b_n$  for all  $n \in \mathbb{N}$ . Hence  $a \in I_n$  for each  $n \in \mathbb{N}$ , and consequently,  $a \in \bigcap_{n=1}^{\infty} I_n$ . We showed in the Monotone Convergence Theorem (Theorem 4.2.7) that  $\lim_{n \rightarrow \infty} a_n = a$ . Assume that  $b \in \bigcap_{n=1}^{\infty} I_n$ . Then  $a_n \leq b \leq b_n$  for all  $n \in \mathbb{N}$ , and so  $0 \leq b - a_n \leq b_n - a_n$  for all  $n \in \mathbb{N}$ . Since  $\lim_{n \rightarrow \infty} 0 = 0 = \lim_{n \rightarrow \infty} (b_n - a_n)$ , we have, by the Squeeze Theorem, that  $\lim_{n \rightarrow \infty} (b - a_n) = 0$ , whence  $\lim_{n \rightarrow \infty} a_n = b$ . Thus,  $a = b$ . ■

**Remark**

The Nested Intervals Theorem (Theorem 4.2.9) may fail for a decreasing sequence of open or half-open intervals. For example, if  $I_n = \left(0, \frac{1}{n+1}\right]$  or  $I_n = [n, \infty)$  for each  $n \in \mathbb{N}$ , then  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .

### Subsequences

**Definition**

Let  $(s_n)$  be a sequence of real numbers and let  $(n_k)_{k \in \mathbb{N}}$  be a sequence of natural numbers such that  $n_1 < n_2 < n_3 < \dots$ . Then the sequence  $(s_{n_k})$  is called a **subsequence** of  $(s_n)$ . That is, a subsequence  $(s_{n_k})$  of the sequence  $(s_n)$  is a strictly increasing function  $\phi : k \mapsto s_{n_k}$ .

**Example**

Let  $(s_n)$  be the sequence  $\left(1, 2, \frac{1}{2}, 3, \frac{1}{3}, \dots\right)$ . Then  $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$  and  $\left(1, 2, \frac{1}{3}, \dots\right)$  are subsequences of  $(s_n)$ .

**Theorem**

Let  $(s_n)$  be a sequence which converges to  $s$ . Then any subsequence of  $(s_n)$  converges to  $s$ .

<Let  $(s_{n_k})$  be a subsequence of  $(s_n)$  and let  $\epsilon > 0$  be given. Then there exists an  $N \in \mathbb{N}$  such that

$$|s_n - s| < \epsilon \text{ for each } n \geq N.$$

Thus, when  $k \geq N$  we have that  $n_k \geq k \geq N$  and so

$$|s_{n_k} - s| < \epsilon \text{ for all } k \geq N.$$

That is,  $\lim_{k \rightarrow \infty} s_{n_k} = s$ . ■

Now we state and prove the version of the Bolzano-Weierstrass Theorem that applies to sequences. Compare this with Theorem 3.3.8 (Bolzano-Weierstrass Theorem for sets).

**Theorem**

(Bolzano-Weierstrass Theorem for sequences). *Every bounded infinite sequence  $(s_n)$  of real numbers has a convergent subsequence.*

<Since  $(s_n)$  is bounded, there exists  $M > 0$  such that  $|s_n| < M$  for all  $n \in \mathbb{N}$ , i.e.,  $-M < s_n < M$  for all  $n \in \mathbb{N}$ . If  $(s_n)$  has a finite range  $S = \{s_n : n \in \mathbb{N}\}$ , then there is (at least) one term of the sequence  $(s_n)$  which occurs infinitely many times in  $(s_n)$ . Call this term  $x$ , and let

$$s_{n_1} = s_{n_2} = s_{n_3} = \dots = x.$$

Therefore, there exists a strictly increasing sequence  $(n_k)_k$  of natural numbers such that  $s_{n_k} = x$  for all  $k \in \mathbb{N}$ . Thus,  $\{s_{n_k}\}$  is a subsequence of  $(s_n)$  which converges to  $x$ .

Now, suppose that  $S$  is infinite. Then either  $[-M, 0]$  or  $[0, M]$  contains  $s_n$  for infinitely many  $n \in \mathbb{N}$ . Call such an interval  $I_1 = [a_1, b_1]$ . Note that  $|I_1| = |b_1 - a_1| = M$ . Now, bisect  $I_1$ . One of the two subintervals of  $I_1$  contains  $s_n$  for infinitely many  $n \in \mathbb{N}$ . Call that subinterval  $I_2 = [a_2, b_2]$ . Clearly,  $I_1 \supset I_2$  and  $|I_2| = |b_2 - a_2| = \frac{M}{2}$ . Bisect  $I_2$ . One of the two subintervals of  $I_2$  contains  $s_n$  for infinitely many  $n \in \mathbb{N}$ . Call that subinterval  $I_3 = [a_3, b_3]$ . Clearly,  $I_1 \supset I_2 \supset I_3$  and  $|I_3| = |b_3 - a_3| = \frac{M}{2^2}$ . Continue in this manner to obtain a sequence of intervals  $I_1, I_2, I_3, \dots$  with  $I_1 \supset I_2 \supset I_3 \supset \dots$  and

$$|I_n| = |b_n - a_n| = \frac{M}{2^{n-1}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By the Nested Intervals Theorem (Theorem 4.2.9), we have that  $\bigcap_{n=1}^{\infty} I_n$  consists of exactly one point,  $\ell$  say. We obtain a convergent subsequence as follows: choose  $s_{n_1} \in I_1$ . Next, choose  $s_{n_2} \in I_2$  with  $n_2 > n_1$ . Next, choose  $s_{n_3} \in I_3$  with  $n_3 > n_2$ . Continue in this manner. (Such a selection is possible since  $I_n$  contains infinitely many terms of the sequence  $(s_n)$ .) Then  $(s_{n_k})$  is a subsequence of  $(s_n)$  with  $s_{n_k} \in I_k$  for all  $k \in \mathbb{N}$ . Since  $\ell$  is also in  $I_k$ , we have that

$$|s_{n_k} - \ell| < \frac{M}{2^{k-1}} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

That is,  $\lim_{k \rightarrow \infty} s_{n_k} = \ell$ . ■

### Cauchy Sequences

**Definition**

A sequence  $(s_n)$  is called a **Cauchy sequence** if, given any  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that

$$|s_n - s_m| < \epsilon \text{ for all } n, m \geq N.$$

Symbolically,

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n, m \in \mathbb{N})[(n \geq N) \wedge (m \geq N) \Rightarrow (|s_n - s_m| < \epsilon)].$$

Equivalently,  $(s_n)$  is a Cauchy sequence if  $\lim_{n, m \rightarrow \infty} |s_n - s_m| = 0$ .

**Examples**

[1] Show that the sequence  $(s_n)$ , where  $s_n = \frac{n+1}{n}$ , is a Cauchy sequence.

Solution: For all  $n, m \in \mathbb{N}$ ,

$$|s_n - s_m| = \left| \left( \frac{n+1}{n} \right) - \left( \frac{m+1}{m} \right) \right| = \left| \frac{mn + m - nm - n}{nm} \right| = \left| \frac{m-n}{nm} \right| \leq \frac{m+n}{nm}.$$

Therefore, if  $m \geq n$ , then

$$|s_n - s_m| \leq \frac{m+n}{nm} \leq \frac{2m}{nm} = \frac{2}{n}.$$

Let  $\epsilon > 0$  be given. Then there is an  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \frac{\epsilon}{2}$ . Thus, for all  $n \geq N$ , we have

$$|s_n - s_m| = \left| \binom{n+1}{n} - \binom{m+1}{m} \right| < \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

Hence  $(s_n)$  is a Cauchy sequence. □

[2] Show that the sequence  $(s_n)$ , where  $s_n = 1 - \frac{1}{2!} + \dots + \frac{(-1)^{n+1}}{n!}$ , is a Cauchy sequence.

Solution: For  $n, m \in \mathbb{N}$  with  $m \geq n$ , we have that

$$\begin{aligned} |s_n - s_m| &= \left| \left( 1 - \frac{1}{2!} + \dots + \frac{(-1)^{n+1}}{n!} \right) - \left( 1 - \frac{1}{2!} + \dots + \frac{(-1)^{m+1}}{m!} \right) \right| \\ &= \left| \frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \dots + \frac{(-1)^{m+1}}{m!} \right| \\ &\leq \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{m!} \\ &\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{m-1}} = \frac{1}{2^n} \left[ 1 + \frac{1}{2} + \dots + \frac{1}{2^{m-n-1}} \right] \\ &= \frac{2}{2^n} \left[ 1 - \left( \frac{1}{2} \right)^{m-n} \right] < \frac{2}{2^n} = \frac{1}{2^{n-1}}. \end{aligned}$$

Since  $\frac{1}{2^{n-1}} \rightarrow 0$  as  $n \rightarrow \infty$ , given any  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$\frac{1}{2^{n-1}} = \left| \frac{1}{2^{n-1}} - 0 \right| < \epsilon \text{ for all } n \geq N.$$

Thus,

$$|s_n - s_m| = \left| \left( 1 - \frac{1}{2!} + \dots + \frac{(-1)^{n+1}}{n!} \right) - \left( 1 - \frac{1}{2!} + \dots + \frac{(-1)^{m+1}}{m!} \right) \right| < \frac{1}{2^{n-1}} < \epsilon$$

for all  $m \geq n \geq N$ . That is,  $(s_n)$  is a Cauchy sequence. □

**Theorem**

Every convergent sequence  $(s_n)$  is a Cauchy sequence.

<Assume that  $(s_n)$  converges to  $s$ . Then, given any  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that

$$|s_n - s| < \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Now, for all  $n, m \geq N$ , we have that

$$|s_n - s_m| = |(s_n - s) + (s - s_m)| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

That is,  $(s_n)$  is a Cauchy sequence. ■

**Theorem**

Every Cauchy sequence  $(s_n)$  is bounded.

<Let  $\epsilon = 1$ . Then there exists an  $N \in \mathbb{N}$  such that

$$|s_n - s_m| < 1 \text{ for all } n, m \geq N.$$

Choose a  $k \geq N$  and observe that

$$|s_n| = |s_n - s_k + s_k| \leq |s_n - s_k| + |s_k| < 1 + |s_k| \text{ for all } n \geq N.$$

Let  $M = \max\{|s_1|, |s_2|, \dots, |s_N|, |s_k| + 1\}$ . Then  $|s_n| \leq M$  for all  $n \in \mathbb{N}$ , and therefore  $(s_n)$  is bounded. ■

**Theorem**

Every Cauchy sequence  $(s_n)$  of real numbers converges.

<By Theorem 4.2.18,  $(s_n)$  is bounded, and therefore, by the Bolzano-Weierstrass Theorem (Theorem 4.2.14),  $(s_n)$  has a subsequence  $\{s_{n_k}\}$  which converges to some real number  $\ell$ . We claim that the sequence  $(s_n)$  converges to  $\ell$ . Let  $\epsilon > 0$  be given. Then there exist natural numbers  $N_1$  and  $N_2$  such that

$$|s_n - s_m| < \frac{\epsilon}{2} \text{ for all } n, m \geq N_1 \text{ and}$$

$$|s_{n_k} - \ell| < \frac{\epsilon}{2} \text{ for all } k \geq N_2.$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ , we have

$$|s_n - \ell| \leq |s_n - s_{n_k}| + |s_{n_k} - \ell| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore,  $\lim_{n \rightarrow \infty} s_n = \ell$ . ■

Combining Theorem 4.2.17 and Theorem 4.2.19, we get:

**Cauchy's Convergence Criterion for sequence:** A sequence  $(s_n)$  of real numbers converges if and only if it is a Cauchy sequence.

**Examples**

[1] Use Cauchy's Criterion to show that the sequence  $\left(\frac{(-1)^n}{n}\right)$  converges.

Solution: We must show that the sequence  $\left(\frac{(-1)^n}{n}\right)$  is Cauchy. To that end, let  $\epsilon > 0$  and  $s_n = \frac{(-1)^n}{n}$ . Then, for all  $n, m \in \mathbb{N}$  with  $m \geq n$ ,

$$|s_n - s_m| = \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.$$

Now, there is an  $N \in \mathbb{N}$  such that  $\frac{2}{N} < \epsilon$ . Thus, for all  $n \geq N$ , we have

$$|s_n - s_m| = \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| \leq \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

Thus,  $\left(\frac{(-1)^n}{n}\right)$  is a Cauchy sequence. □

[2] Show that the sequence  $(s_n)$ , where  $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ , diverges.

Solution: It suffices to show that  $(s_n)$  is *not* a Cauchy sequence. Now, for  $n, m \in \mathbb{N}$  with  $n > m$ , we have

$$\begin{aligned} |s_n - s_m| &= \left| \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) - \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} \right) \right| \\ &= \left| \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \right| = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \\ &> \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{n-m \text{ terms}} = \frac{n-m}{n}. \end{aligned}$$

In particular, if we take  $n = 2m$ , we get

$$|s_n - s_m| = \left| \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) - \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} \right) \right| > \frac{n-m}{n} = \frac{1}{2}.$$

Thus,  $(s_n)$  is not Cauchy. □

**Exercise**

- [1] Show that if a  $(x_n)$  contains two subsequences that converge to different limits, then  $(x_n)$  diverges.
- [2] Show that every subsequence of a bounded sequence is bounded.
- [3] Show that if  $(x_n)$  is a Cauchy sequence, then so is  $\{|x_n|\}$ .

**Limit Superior and Limit Inferior**

Let  $(x_n)$  be a bounded sequence of real numbers and for each  $n \in \mathbb{N}$ , let

$$E_n = \{x_n, x_{n+1}, \dots\} = \{x_k \mid k \geq n\}.$$

Then  $E_n$  is a bounded subset of  $\mathbb{R}$ . Set

$$s_n = \inf E_n \text{ and } S_n = \sup E_n.$$

Clearly,  $E_j \subset E_i$  for all  $i, j \in \mathbb{N}$  such that  $i < j$ . In particular,  $E_{k+1} \subset E_k$  for each  $k \in \mathbb{N}$ . Therefore

$$s_k \leq s_{k+1} \leq S_{k+1} \leq S_k.$$

These inequalities show that  $(s_n)$  is a monotone increasing sequence of real numbers and  $(S_n)$  is a monotone decreasing sequence of real numbers. Note further that, for each  $k \in \mathbb{N}$ ,

$$s_1 \leq s_k \leq s_{k+1} \leq S_{k+1} \leq S_k \leq S_1.$$

Therefore, the increasing sequence  $(s_n)$  is bounded above by  $S_1$  and the decreasing sequence  $(S_n)$  is bounded below by  $s_1$ . In fact, for each  $n \in \mathbb{N}$ ,  $S_n$  is an upper bound for the sequence  $(s_n)$  and for each  $n \in \mathbb{N}$ ,  $s_n$  is a lower bound for the sequence  $(S_n)$ . Therefore, by the Monotone Convergence Theorem

(Theorem 4.2.7), the sequence  $(s_n)$  converges to the supremum  $s = \sup_n s_n$  and the sequence  $(S_n)$  converges to the infimum  $S = \inf_n S_n$ . That is,

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k = \sup_n s_n \text{ and } S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sup_{k \geq n} x_k = \inf_n S_n.$$

The number  $s$  is called the **limit inferior** of the sequence  $(x_n)$  and the number  $S$  is called the **limit superior** of the sequence  $(x_n)$ . We write

$$s = \liminf_{n \rightarrow \infty} x_n \text{ and } S = \limsup_{n \rightarrow \infty} x_n.$$

We also use the notation  $\underline{\lim}_{n \rightarrow \infty} x_n$  for the limit inferior and  $\overline{\lim}_{n \rightarrow \infty} x_n$  for the limit superior.

**Definition**

Let  $(x_n)$  be a bounded sequence of real numbers. The **limit inferior** of the sequence  $(x_n)$ , denoted by  $\liminf_{n \rightarrow \infty} x_n$ , is defined by

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right) = \sup_{n \geq 1} \left( \inf_{k \geq n} x_k \right).$$

Similarly, we define the **limit superior** of the sequence  $(x_n)$ , denoted by  $\limsup_{n \rightarrow \infty} x_n$ , as

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right) = \inf_{n \geq 1} \left( \sup_{k \geq n} x_k \right).$$

**Examples**

[1] Consider the sequence  $(x_n)$ , where, for each  $n \in \mathbb{N}$ ,  $x_n = (-1)^n$ . Then

$$\inf_{k \geq n} x_k = \inf_{k \geq n} (-1)^k = -1 \text{ and } \sup_{k \geq n} x_k = \sup_{k \geq n} (-1)^k = 1.$$

Therefore

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right) = -1 \text{ and}$$

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right) = 1.$$

[2] Consider the sequence  $(x_n)$ , where, for each  $n \in \mathbb{N}$ ,  $x_n = (-1)^n + \frac{1}{n}$ . Then

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right) = -1 \text{ and}$$

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right) = 1.$$

[3] Consider the sequence  $(x_n)$ , where, for each  $n \in \mathbb{N}$ ,  $x_n = \frac{1}{n}$ . Then

$$\begin{aligned} E_1 &= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \\ E_2 &= \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \\ E_3 &= \{\frac{1}{3}, \frac{1}{4}, \dots\} \\ &\vdots \\ E_n &= \{\frac{1}{n}, \frac{1}{n+1}, \dots\} \\ &\vdots \end{aligned}$$

Therefore

$$\sup_{k \geq n} x_k = \sup E_n = \frac{1}{n} \text{ and } \inf_{k \geq n} x_k = \inf E_n = 0.$$

It now follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0 \text{ and} \\ \liminf_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right) = \lim_{n \rightarrow \infty} (0) = 0. \end{aligned}$$

**Lemma**

Let  $(x_n)$  be a bounded sequence of real numbers,  $\alpha = \liminf_{n \rightarrow \infty} x_n$ , and  $\beta = \limsup_{n \rightarrow \infty} x_n$ . Then there is a subsequence of  $(x_n)$  which converges to  $\alpha$  and a subsequence of  $(x_n)$  which converges to  $\beta$ .

<Let  $E_k = \{x_k, x_{k+1}, \dots\} = \{x_n \mid n \geq k\}$ ,  $s_k = \inf E_k$  and  $S_k = \sup E_k$ . Then

$$\limsup_{n \rightarrow \infty} x_n = \beta = \lim_{k \rightarrow \infty} S_k. \tag{4.1}$$

Since  $S_k$  is the supremum of the set  $E_k$ , given  $\epsilon > 0$ , there is an index  $n_k \geq k$  such that

$$S_k - \frac{\epsilon}{2} < x_{n_k} \leq S_k < S_k + \frac{\epsilon}{2} \iff |S_k - x_{n_k}| < \frac{\epsilon}{2}. \tag{4.2}$$

Also, since  $\lim_{k \rightarrow \infty} S_k = \beta$ , there is a natural number  $N$  such that, for all  $k \geq N$ ,

$$|S_k - \beta| < \frac{\epsilon}{2}. \tag{4.3}$$

From (4.2) and (4.3), we have that for all  $n_k \geq k \geq N$ ,

$$|x_{n_k} - \beta| \leq |x_{n_k} - S_k| + |S_k - \beta| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus,  $(x_{n_k})$  is a subsequence of  $(x_n)$  which converges to  $\beta$ .

The proof of the second part is similar. ■

**Theorem**

Let  $(x_n)$  be a bounded sequence of real numbers. Then

(i)  $\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$ .

(ii)  $\limsup_{n \rightarrow \infty} x_n = -\liminf_{n \rightarrow \infty} (-x_n)$ .

(iii) The sequence  $(x_n)$  converges if and only if  $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$ . In this case,

$$\lim_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n.$$

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(i) For each  $n \in \mathbb{N}$ ,  $\inf_{k \geq n} x_k \leq \sup_{k \geq n} x_k$ . Therefore

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right) \leq \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right) = \limsup_{n \rightarrow \infty} x_n.$$

(ii) For each  $n \in \mathbb{N}$ ,  $\inf_{k \geq n} (-x_k) = -\sup_{k \geq n} x_k$ . Therefore

$$\liminf_{n \rightarrow \infty} (-x_n) = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} (-x_k) \right) = \lim_{n \rightarrow \infty} \left( -\sup_{k \geq n} x_k \right) = -\limsup_{n \rightarrow \infty} x_n.$$

(iii) Assume that the sequence  $(x_n)$  converges to  $x$ . By Lemma 4.2.24, there are subsequences  $(x_{n_k})$  and  $(y_{n_\ell})$  of  $(x_n)$  which converge to  $\liminf_{n \rightarrow \infty} x_n$  and  $\limsup_{n \rightarrow \infty} x_n$  respectively. Since every subsequence of a convergent sequence converges to the same limit as the sequence itself, it follows that

$$\liminf_{n \rightarrow \infty} x_n = \lim_{k \rightarrow \infty} x_{n_k} = x = \lim_{\ell \rightarrow \infty} y_{n_\ell} = \limsup_{n \rightarrow \infty} x_n.$$

Conversely, assume that  $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$ . Then for each  $n \in \mathbb{N}$ ,

$$s_n = \inf\{x_n, x_{n+1}, \dots\} \leq x_n \leq S_n = \sup\{x_n, x_{n+1}, \dots\}.$$

Since

$$\lim_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} S_n,$$

it follows, by the Squeeze Theorem, that the sequence  $(x_n)$  converges and  $\lim_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$ . ■

**Exercise**

[1] For each of the following, find the limit superior, limit inferior and the limit of the sequence.

(a)  $(x_n)$ , where  $x_n = \frac{(-1)^n}{n+1}$ .

(b)  $(x_n)$ , where  $x_n = (-1)^n + (-1)^{n+2}$ .

(c)  $(x_n)$ , where  $x_n = 2(-1)^n + \frac{n}{n+1}$ .

## Sequential Characterization of Closed Sets

### Theorem

Let  $K$  be a nonempty subset of  $\mathbb{R}$  and  $x \in \mathbb{R}$ . Then

- (a)  $x \in \overline{K}$  if and only if there is a sequence  $(x_n) \subset K$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .
- (b)  $K$  is closed if and only if  $K$  contains the limit of every convergent sequence in  $K$ .

<

- (a) Assume that  $x \in \overline{K}$ . Then either  $x \in K$  or  $x \in K'$ . If  $x \in K$ , then the constant sequence  $(x, x, x, \dots)$  in  $K$  converges to  $x$ . If  $x \in K'$ , then, for each  $n \in \mathbb{N}$ , the interval  $(x - \frac{1}{n}, x + \frac{1}{n})$  contains a point  $x_n \in K$  distinct from  $x$ . It now follows that  $|x_n - x| < \frac{1}{n}$ . Clearly,  $(x_n) \subset K$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

Conversely, assume that there is a sequence  $(x_n) \subset K$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . Then, either  $x \in K$  or every  $\epsilon$ -neighbourhood of  $x$  contains a point  $x_n \neq x$ , in which case  $x \in K'$ . Thus  $x \in \overline{K}$ .

- (b) By Corollary 3.2.18,  $K$  is closed if and only if  $K = \overline{K}$ . Hence, (b) follows from (a). ■

## Sequential Compactness

### Definition

A subset  $K$  of  $\mathbb{R}$  is said to be **sequentially compact** if every sequence in  $K$  has a subsequence that converges to a point in  $K$ .

The following theorem asserts that for subsets of  $\mathbb{R}$  compactness and sequential compactness are equivalent. In fact, this is true in any “metric space”. We shall discuss metric spaces later.

### Theorem

A subset  $K$  of  $\mathbb{R}$  is compact if and only if it is sequentially compact.

<Assume that  $K$  is compact and let  $(x_n)$  be a sequence in  $K$ . Then, by the Heine-Borel Theorem (Theorem 3.3.7),  $K$  is closed and bounded. Therefore the sequence  $(x_n)$  is bounded. By Bolzano-Weierstrass Theorem (Theorem 4.2.14)  $(x_n)$  has a subsequence  $(x_{n_k})$  which converges to some  $x \in \mathbb{R}$ . Since  $K$  is closed, we have by Theorem 7.3.6, that  $x \in K$ . Hence,  $K$  is sequentially compact.

Conversely, assume that  $K$  is *not* compact. Then, by the Heine-Borel Theorem (Theorem 3.3.7), either  $K$  is not closed or  $K$  is not bounded. If  $K$  is not closed, then there is a sequence  $(x_n)$  in  $K$  that converges to a point outside of  $K$ . But then every convergent subsequence of  $(x_n)$  will converge to a point outside of  $K$ . Therefore  $K$  is not sequentially compact. If  $K$  is not bounded, then there is a sequence  $(x_n)$  in  $K$  such that  $|x_n| > n$  for each  $n \in \mathbb{N}$ . Thus, every subsequence of  $(x_n)$  is unbounded, and so, by Theorem 4.1.13, no subsequence of  $(x_n)$  converges (to a point in  $K$ ). Hence  $K$  is not sequentially compact. ■

### Exercise

- [1] Let a sequence  $(x_n)$  of real numbers be defined recursively by

$$x_1 = 0, \quad x_{n+1} = \frac{3x_n + 1}{x_n + 3} \quad \text{for all } n \geq 1.$$

- (a) Show, by induction, that  $0 \leq x_n \leq 1$  for all  $n \in \mathbb{N}$ .
- (b) Show that the sequence  $(x_n)$  is monotonically increasing.
- (c) Does the sequence  $(x_n)$  converge? If so, find its limit.
- (d) Does  $\sup\{x_n \mid n \in \mathbb{N}\}$  exist? If so, find it.