

Real Numbers and their Properties

Real Numbers as a Complete Ordered Field

FIELD AXIOMS

Definition

A **field** is a set \mathbb{F} together with two binary operations $+$: $\mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ (called **addition**) and \times : $\mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ (called **multiplication**) such that for all $x, y, z \in \mathbb{F}$,

A1. $x + y = y + x$;

A2. $(x + y) + z = x + (y + z)$;

A3. There is an element $0 \in \mathbb{F}$, called the **additive identity**, such that $x + 0 = x$ for each $x \in \mathbb{F}$;

A4. For each $x \in \mathbb{F}$, there is an element $-x \in \mathbb{F}$, called the **additive inverse** of x , such that $x + (-x) = 0$;

M1. $x \times y = y \times x$;

M2. $(x \times y) \times z = x \times (y \times z)$;

M3. There is an element $1 \in \mathbb{F}$, called the **multiplicative identity**, such that $x \times 1 = x$;

M4. For each $x \in \mathbb{F} \setminus \{0\}$, there is an element $x^{-1} \in \mathbb{F}$, called the **multiplicative inverse** of x , such that $x \times x^{-1} = 1$;

D1. $x \times (y + z) = x \times y + x \times z$.

Note that a field is a triple $(\varphi, +, \times)$, where φ is a set, $+$ and \times are binary operations satisfying the above properties. We shall abuse notation by simply writing φ for a field. To simplify notation, we shall write xy instead of $x \times y$ and $\frac{x}{y}$ for $x \times y^{-1}$.

Exercise

[1] Let \mathbb{F} be a field.

(a) Show that the additive and multiplicative identities are unique.

(b) Let $x \in \mathbb{F}$ and $y \in \mathbb{F} \setminus \{0\}$. Show that $-x$ and y^{-1} are unique.

ORDER AXIOMS

Definition

An **ordered field** is a field \mathbb{F} on which an order relation $<$ is defined such that

(i) (trichotomy) for every $x, y \in \varphi$, exactly one of the following holds:

$$x < y, \quad x = y, \quad y < x;$$

(ii) (transitivity) for all x, y and $z, x < y \wedge y < z \Rightarrow x < z$;

(iii) For all $x, y,$ and z in $\varphi, x < y \Rightarrow x + z < y + z$. Furthermore, if $z > 0$, then $xz < yz$.

The sets \mathbb{R} and \mathbb{Q} are examples of ordered fields (under the usual $<$ relation).

Exercise

Show that $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ is a field and that \mathbb{Z}_7 is not an ordered field under the usual $<$ relation.

COMPLETENESS AXIOM

Definition

Let B be a subset of an ordered field \mathbb{F} .

- (a) An element $u \in \varphi$ is an **upper bound** for B if $x \leq u$ for all $x \in B$.
- (b) An element $\ell \in \varphi$ is a **lower bound** for B if $\ell \leq x$ for all $x \in B$.
- (c) B is said to be **bounded** if it has both an upper and a lower bound.
- (d) An element $M \in \varphi$ is the **least upper bound** for B if
 - (i) M is an upper bound for B and,
 - (ii) for all upper bounds α for B , we have $M \leq \alpha$.

The least upper bound for B is also called the **supremum** for B and is usually abbreviated as $\text{lub}(B)$ or $\text{sup } B$.

- (e) An element $m \in \varphi$ is the **greatest lower bound** for B if
 - (i) m is a lower bound for B and,
 - (ii) for all lower bounds β for B , we have $\beta \leq m$.

The greatest lower bound for B is also called the **infimum** for B and is usually abbreviated as $\text{glb}(B)$ or $\text{inf } B$.

Proposition

A nonempty subset S of an ordered field φ can have at most one least upper bound.

Proof. Assume that λ and ν are both least upper bounds for S . Then, by definition of the least upper bound, $\lambda \leq \nu \leq \lambda$. Hence, $\lambda = \nu$.

Definition

An ordered field φ is said to be **complete** if every nonempty subset S of φ which is bounded above has the least upper bound.

Exercise

Show that in a complete field φ , every nonempty subset S of φ which is bounded below has the greatest lower bound.

Theorem

(Characterization of supremum) Let S be a nonempty subset of an ordered field φ , and $M \in \varphi$. Then $M = \text{sup } S$ if and only if

- (i) M is an upper bound for S , and
- (ii) for any $\epsilon \in \varphi$ with $\epsilon > 0$, there is an element $s \in S$ such that $M - \epsilon < s$

Proof. Assume that M is the supremum for S , i.e., $M = \text{sup } S$. Then, by definition, M is an upper bound for S . If there is an $\epsilon' \in \varphi$ with $\epsilon' > 0$ for which $M - \epsilon' \geq s$ for all $s \in S$, then $M - \epsilon'$ is an upper bound for S which is smaller than M , a contradiction.

For the converse, assume that (i) and (ii) hold. Since S is bounded above, it has a supremum, A (say). Since M is an upper bound for S , we must have that $A \leq M$. If $A < M$, then with $\epsilon = M - A$, there is an element s in S such that

$$M - (M - A) < s \leq A, \quad \text{i.e., } A < A,$$

which is absurd. Therefore $A = M$, i.e., M is the supremum of S . ■

Definition

Let $(\varphi, <)$ and $(\mathbb{G}, <)$ be ordered fields. An **order isomorphism** between φ and \mathbb{G} is a bijection $\phi : \varphi \rightarrow \mathbb{G}$ such that for all $x, y \in \varphi$,

(i) $\phi(x + y) = \phi(x) + \phi(y)$;

(ii) $\phi(xy) = \phi(x)\phi(y)$;

(iii) if $x < y$, then $\phi(x) < \phi(y)$.

- ◇ A complete ordered field exists. We denote it by \mathbb{R} and call it the field of real numbers.
- ◇ There is an order isomorphism between any two complete ordered field.
- ◇ It follows from the above two statements that there is essentially one complete ordered field, viz. \mathbb{R} . “Essentially” here means that there is an order isomorphism between any complete ordered field and the field \mathbb{R} of real numbers.

Theorem

Let A and B be nonempty subsets of \mathbb{R} which are bounded above. Then the set

$$S = \{a + b : a \in A, b \in B\}$$

is bounded above and $\sup S = \sup A + \sup B$.

Proof. Let $c \in S$. Then $c = a + b$ for some $a \in A$ and $b \in B$. Thus, $c = a + b \leq \sup A + \sup B$. Therefore $\sup A + \sup B$ is an upper bound for S . Since $\sup S$ is the *least* upper bound for S , we have that $\sup S \leq \sup A + \sup B$.

It now remains to show that $\sup A + \sup B \leq \sup S$. To that end, let $\epsilon > 0$ be given. By Theorem 3.1.9, there exist elements $x_\epsilon \in A$ and $y_\epsilon \in B$ such that

$$\sup A - \frac{\epsilon}{2} < x_\epsilon, \quad \text{and}$$

$$\sup B - \frac{\epsilon}{2} < y_\epsilon.$$

Thus, $\sup A + \sup B - \epsilon < x_\epsilon + y_\epsilon \leq \sup S$. Since this is true for any $\epsilon > 0$, we have that $\sup A + \sup B \leq \sup S$, whence $\sup A + \sup B = \sup S$. ■

Exercise

- [1] Let $S \subset \mathbb{R}$ be bounded above and let $x \in \mathbb{R}$. Show that if $x < \sup S$, then there exists an $s \in S$ such that $x < s$.

[2] Let S be a subset of \mathbb{R} which is bounded below. Show that the set

$$L = \{\ell \in \mathbb{R} : \ell \text{ is a lower bound for } S\}$$

is bounded above and $\sup L = \inf S$.

[3] Show that if a set S of real numbers is bounded below, then $\inf S$ exists.

[4] Let $S \subset T \subset \mathbb{R}$, where $S \neq \emptyset$.

(i) Show that if T is bounded above, then so is S and $\sup S \leq \sup T$.

(ii) Show that if T is bounded below, then so is S and $\inf T \leq \inf S$.

[5] For a subset S of \mathbb{R} , let $-S = \{-s : s \in S\}$. Show that if S is bounded below, then $-S$ is bounded above and $\sup(-S) = -\inf S$.

[6] Formulate and prove the characterization of infimum analogous to Theorem 3.1.9.

[7] Let A and B be non-empty bounded subsets of \mathbb{R} .

(i) Show that the set $S = \{a+b : a \in A, b \in B\}$ is bounded below and $\inf S = \inf A + \inf B$.

(ii) Show that the set $D = \{a-b : a \in A, b \in B\}$ is bounded above and $\sup D = \sup A - \inf B$.

(iii) Show that the set $D = \{a-b : a \in A, b \in B\}$ is bounded below and $\inf D = \inf A - \sup B$.

(iv) Show that the set $A \cup B$ is bounded above and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

[8] Let A and B be non-empty bounded subsets of \mathbb{R}^+ , the set of positive real numbers. Show that the set $P = \{ab : a \in A, b \in B\}$ is bounded and $\sup P = \sup A \cdot \sup B$, $\inf P = \inf A \cdot \inf B$.

The Archimedean Property of the Real Numbers

The following property of real numbers is one of the major consequences of the Completeness Axiom.

Theorem

(Archimedean Property). *The set \mathbb{N} of natural numbers is not bounded above.*

Proof. Assume that \mathbb{N} is bounded above. By the Completeness Axiom, $\sup \mathbb{N}$ exists. Let $m = \sup \mathbb{N}$. Then, by Theorem 3.1.9, with $\epsilon = 1$, there is an element $k \in \mathbb{N}$ such that $m - 1 < k$. This implies that $m < k + 1 \leq m$, which is absurd. ■

There are several equivalent formulations of the Archimedean Property. We shall mention just a few of them as corollaries of Theorem 3.1.13.

Corollary

For every real number b there exists an integer m such that $m < b$.

Proof. For every real number b there is a natural number n such that $n > -b$. Hence, $m = -n < b$. ■

Corollary

Given any real number x , there exists an integer k such that $x - 1 \leq k < x$.

<Let $x \in \mathbb{R}$. By Corollary 3.1.14, there is an integer m such that $m < x$. By the Archimedean Property, there is a natural number n such that $x < n$. Hence, $m < x < n$. Choose the largest integer k from the finite collection $m, m + 1, \dots, n$ such that $k < x$. Then $k + 1 \geq x$, and consequently, $x - 1 \leq k < x$. ■

Corollary

If x and y are two positive real numbers, then there exists a natural number n such that $nx > y$.

<Assume that $nx \leq y$ for all $n \in \mathbb{N}$. Then $n \leq \frac{y}{x}$ for all $n \in \mathbb{N}$. This means that \mathbb{N} is bounded above (by $\frac{y}{x}$), contradicting Theorem 3.1.13. ■

Corollary

If $\epsilon > 0$ then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. Take $x = \epsilon$ and $y = 1$ in Corollary 3.1.16. ■

Since \mathbb{R} is a field in which order and completeness axioms and the Archimedean Property hold, \mathbb{R} is called a **complete ordered Archimedean field**.

The following theorem asserts that we can approximate any real number as closely as we wish by a rational number. A similar statement also holds for irrational numbers.

Theorem

(Density of Rationals in Reals). If x and y are real numbers such that $x < y$, then there exists a rational number r such that $x < r < y$. That is, between any two distinct real numbers there is a rational number.

Proof. By Corollary 3.1.17, there is a natural number n such that $\frac{1}{n} < y - x$. That is,

$$x < y - \frac{1}{n}. \tag{3.1}$$

Also, by Corollary 3.1.15, there is an integer k such that

$$ny - 1 \leq k < ny, \text{ i.e. } y - \frac{1}{n} \leq \frac{k}{n} < y. \tag{3.2}$$

Combining (3.1) and (3.2), we have that

$$x < y - \frac{1}{n} \leq \frac{k}{n} < y. \tag{3.3}$$

Corollary

(Density of Irrationals in Reals). If x and y are real numbers such that $x < y$, then there exists an irrational number z such that $x < z < y$. That is, between any two distinct real numbers there is an irrational number.

Proof. By Theorem 3.1.18, there are rational numbers r_1 and r_2 such that

$$x < r_1 < r_2 < y.$$

Then $z = r_1 + \frac{r_2 - r_1}{\sqrt{2}}$ is an irrational number such that $x < z < y$. ■

Corollary

Let b be any real number and let $S = \{q \in \mathbb{Q} : q < b\}$. Then $b = \sup S$. That is, every real number is a supremum of a set of rational numbers.

Proof. Of course, b is an upper bound for the set S . By the Completeness Axiom, S has a supremum, $c = \sup S$ say. By definition of supremum, $c \leq b$. If $c < b$, then by Theorem 3.1.18, there exists an $q \in \mathbb{Q}$ such that $c < q < b$. But then $q < b$ implies that $q \in S$, and $c < q$ contradicts the fact that c is the supremum of S . Therefore $c = b$. ■

It is now easy to see why the completeness axiom fails in \mathbb{Q} . Indeed, if η is an irrational number and $S = \{q \in \mathbb{Q} : 0 \leq q < \eta\}$, then S is bounded, with $\sup S = \eta$. However η does not belong to the set \mathbb{Q} .

Topology of the Real Numbers

In this section we briefly discuss some elementary topological properties of the set \mathbb{R} of real numbers.

The Absolute Value Function

Definition

Let $x \in \mathbb{R}$. The **absolute value** of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

It is clear from the definition that the absolute value of any real number is always nonnegative.

Theorem

(Properties of the Absolute Value Function). Let $x, y \in \mathbb{R}$. Then,

$$[1] \quad |x| \geq x \text{ and } |x| \geq -x.$$

$$[2] \quad |x| = |-x|.$$

$$[3] \quad |xy| = |x||y|.$$

$$[4] \quad \left| \frac{x}{y} \right| = \frac{|x|}{|y|} \text{ for } y \neq 0.$$

$$[5] \quad |x + y| \leq |x| + |y|. \text{ (Triangle Inequality.)}$$

<Exercise.

If we think of the real numbers as points on the real line, then $|x - y|$ is just the distance between the real numbers x and y .

Exercise

Let $x, y, z \in \mathbb{R}$. Show that

$$[1] \quad |x - y| + |y - z| \geq |x - z|.$$

$$[2] \quad |x| + |y| \geq |x - y|.$$

$$[3] \quad |x| - |y| \leq |x - y|.$$

$$[4] \quad |x| = \max\{-x, x\}.$$

$$[5] \quad |x - y| < \epsilon \text{ for all } \epsilon > 0 \text{ if and only if } x = y \text{ if and only if } |x - y| = 0.$$

Open Sets and Closed Sets

Definition

Let $a \in \mathbb{R}$ and $\epsilon > 0$.

[1] An ϵ -**neighbourhood** of a is the set

$$N(a, \epsilon) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

[2] A **deleted ϵ -neighbourhood** of a is the set

$$N^*(a, \epsilon) = \{x \in \mathbb{R} : 0 < |x - a| < \epsilon\}.$$

It is clear that

$$N(a, \epsilon) = (a - \epsilon, a + \epsilon) \text{ and } N^*(a, \epsilon) = (a - \epsilon, a) \cup (a, a + \epsilon).$$

Definition

A subset U of \mathbb{R} is said to be **open** if for each $s \in U$ there is an $\epsilon > 0$ such that $(s - \epsilon, s + \epsilon) \subset U$.

Examples

[1] For any $a, b \in \mathbb{R}$ with $a < b$, the set (a, b) is open. Indeed, if $s \in (a, b)$, then $a < s < b$. Take $\epsilon = \min\{s - a, b - s\}$. We claim that $(s - \epsilon, s + \epsilon) \subset (a, b)$. To prove the claim, let $t \in (s - \epsilon, s + \epsilon)$. Then,

$$s - \epsilon < t < s + \epsilon \Rightarrow s - (s - a) \leq s - \epsilon < t < s + \epsilon \leq s + (b - s),$$

whence $a < t < b$, which proves the claim.

[2] The sets $(-1, 0) \cup (3, 7)$ and $(-\infty, 4) \cup (6, 9) \cup (12, 20)$ are open.

[3] The sets \mathbb{R}, \emptyset are open.

[4] The set \mathbb{Q} of rational numbers is not open in \mathbb{R} . Indeed, if $r \in \mathbb{Q}$, then for any $\epsilon > 0$ the interval $(r - \epsilon, r + \epsilon)$ contains an irrational number. Thus, \mathbb{Q} is not open.

[5] The set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not open. The reasoning is the same as in the previous example.

Theorem

(1) A union of an arbitrary collection of open sets in \mathbb{R} is an open set.

(2) An intersection of a finite collection of open sets in \mathbb{R} is an open set.

<(1) Let $\{U_i \mid i \in I\}$ be a collection of open sets in \mathbb{R} . We want to show that the set $\bigcup_{i \in I} U_i$ is open.

Let $x \in \bigcup_{i \in I} U_i$. Then $x \in U_k$ for some $k \in I$. Since U_k is open, we can find an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U_k$. Since $U_k \subset \bigcup_{i \in I} U_i$, we have that $(x - \epsilon, x + \epsilon) \subset \bigcup_{i \in I} U_i$, which proves the desired result.

(2) Let U_1, U_2, \dots, U_n be open sets in \mathbb{R} . We want to show that the set $\bigcap_{k=1}^n U_k$ is open. Let $x \in \bigcap_{k=1}^n U_k$. Then $x \in U_k$ for all $k = 1, 2, \dots, n$. Since U_k is open for each $k = 1, 2, \dots, n$, we can find an $\epsilon_k > 0$ such that $(x - \epsilon_k, x + \epsilon_k) \subset U_k$ for each $k = 1, 2, \dots, n$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$. Then $(x - \epsilon, x + \epsilon) \subset U_k$ for each $k = 1, 2, \dots, n$, and consequently, $(x - \epsilon, x + \epsilon) \subset \bigcap_{k=1}^n U_k$. That is, the set

$\bigcap_{k=1}^n U_k$ is open. ■

An arbitrary intersection of open sets need not be open. For example, if $I_k = \left(\frac{-1}{k}, \frac{1}{k}\right)$, then $\bigcap_{k=1}^{\infty} I_k = \{0\}$, which is not open.

The following theorem completely characterises open subsets of \mathbb{R} .

Theorem

A subset of \mathbb{R} is open if and only if it is a countable union of disjoint open intervals in \mathbb{R} .

<Let G be an open subset of \mathbb{R} and $x \in G$. Denote by I_x the union of all open intervals in G that contain x . Note that since G is open, there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset G$. There is therefore at least one open interval that contains x and is contained in G . Clearly $I_x \subseteq G$. Let

$$\alpha_x = \inf\{a \in \mathbb{R} \mid (a, x] \subset G\} \text{ and } \beta_x = \sup\{b \in \mathbb{R} \mid [x, b) \subset G\}.$$

Claim 1: $I_x = (\alpha_x, \beta_x)$.

$I_x \subseteq (\alpha_x, \beta_x)$: Let $y \in I_x$. Then there is an open interval $(a, b) \subset G$ with $x \in (a, b)$ such that $y \in (a, b)$. Since $(a, b) \subset G$ and $x \in (a, b)$, it follows that $(a, x] \subset G$ and $[x, b) \subset G$. Thus, $\alpha_x \leq a$ and $b \leq \beta_x$. Therefore

$$\alpha_x \leq a < y < b \leq \beta_x$$

and consequently $y \in (\alpha_x, \beta_x)$.

$(\alpha_x, \beta_x) \subseteq I_x$: It is clear that (α_x, β_x) is an open interval and $x \in (\alpha_x, \beta_x)$. It remains to show that $(\alpha_x, \beta_x) \subset G$. To that end, let $y \in (\alpha_x, \beta_x)$. The either $y \leq x$ or $x \leq y$. Without loss of generality, we assume that $y \leq x$. By characterization of α_x (as the infimum), given any $\epsilon > 0$, there is an $a_\epsilon \in \mathbb{R}$ such that $(a_\epsilon, x] \subset G$ and $a_\epsilon < \alpha_x + \epsilon$. In particular, taking $\epsilon = \frac{y - \alpha_x}{2}$, we have that

$$a_\epsilon < \alpha_x + \frac{y - \alpha_x}{2} = \frac{\alpha_x + y}{2}.$$

Since $\alpha_x < y$, it follows that

$$a_\epsilon < \frac{\alpha_x + y}{2} < \frac{y + y}{2} = y \leq x.$$

Therefore $y \in (a_\epsilon, x] \subset G$, and so $y \in G$.

If $x \leq y$, then we can similarly show that $y \in G$.

Claim 2: If $x, y \in G$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Assume that $z \in I_x \cap I_y$. Then $z \in I_x$ and $z \in I_y$. Therefore, I_x is an open interval in G which contains z , and so $I_x \subseteq I_z$. Since $x \in I_x \subseteq I_z$, I_z is an open interval in G which contains x . Hence $I_z \subseteq I_x$. It now follows that $I_x = I_z$. Similarly, $I_z = I_y$, whence $I_x = I_y$. This shows that $\{I_x \mid x \in G\}$ is a collection of disjoint open intervals.

Claim 3: $G = \bigcup_{x \in G} I_x$. Since $I_x \subseteq G$ for each $x \in G$, it follows that $\bigcup_{x \in G} I_x \subseteq G$. On the other hand, if $x \in G$, then, since G is open, there is an open interval $(a, b) \subset G$ such that $x \in (a, b)$. Since $(a, b) \subseteq I_x$, it follows that $x \in I_x$, whence $G \subseteq \bigcup_{x \in G} I_x$.

Claim 4: We can replace the intervals I_x by a countable collection of disjoint intervals. For each $x \in G$, $\alpha_x < x < \beta_x$. Since rationals are dense in reals, there are rational numbers r_x and s_x such that $\alpha_x < r_x < x < s_x < \beta_x$ for each $x \in G$. Thus,

$$x \in (r_x, s_x) \subseteq (\alpha_x, \beta_x) = I_x \text{ for each } x \in G.$$

It now follows that

$$G \subseteq \bigcup_{x \in G} (r_x, s_x) \subseteq \bigcup_{x \in G} I_x = G.$$

That is, $G = \bigcup_{x \in G} (r_x, s_x)$. The intervals $\{(r_x, s_x) \mid x \in G\}$ are clearly disjoint.

Conversely, assume that G is a countable union of disjoint open intervals. Since an open interval is an open set and an arbitrary union of open sets is open, it follows that G is also open. ■

Definition

Let S be a subset of \mathbb{R} , and $x \in \mathbb{R}$. Then

- (a) $x \in S$ is called an **interior point** of S if there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. The set of all interior points of a set S is denoted by S° or $\text{int}(S)$.
- (b) x is called a **boundary point** of S if for every $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains points of S as well as points of $\mathbb{R} \setminus S$. The set of boundary points of S is denoted by ∂S or $\text{bd}(S)$.
- (c) $x \in S$ is called an **isolated point** of S if there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap S = \{x\}$.

It is clear from the definition that each point of an open set S is an interior point of S . Also, every isolated point of a set S is a boundary point of S .

Examples

- [1] Let $S = \{x \in \mathbb{R} : 0 \leq x < 1\}$. Then $S^\circ = \{x \in \mathbb{R} : 0 < x < 1\}$, $\partial S = \{0, 1\}$. S does not have isolated points.
- [2] Let $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Then each point of S is an isolated point of S . Therefore $S \subset \partial S$.
- [3] Let $S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots, 0 \right\}$. Then $\partial S = S$.
- [4] The set \mathbb{N} of natural numbers consists of isolated points only. Therefore, every point of \mathbb{N} is a boundary point. Clearly, $\mathbb{N}^\circ = \emptyset$.
- [5] $\mathbb{Z}^\circ = \emptyset$, and $\mathbb{R}^\circ = \mathbb{R}$.
- [6] Each set with only finitely many elements consists entirely of isolated points.

The following Theorem asserts that elements of a set $S \subset \mathbb{R}$ can be divided into two groups: those that are interior to the set and those that are on the boundary of the set S .

Theorem

Let $S \subset \mathbb{R}$. Then each point of S is either an interior point of S or is a boundary point of S .

<Let $s \in S$. If s is not an interior point of S , then for each $\epsilon > 0$ the interval $(s - \epsilon, s + \epsilon)$ contains a point in $\mathbb{R} \setminus S$. Since $(s - \epsilon, s + \epsilon)$ already contains a point s of the set S , we have that this interval contains a point in S as well as a point in $\mathbb{R} \setminus S$. Hence s is a boundary point of S . ■

Definition

A subset S of \mathbb{R} is said to be **closed** if its complement $\mathbb{R} \setminus S$ is open.

Examples

- [1] The interval $[a, b]$ is closed since its complement $(-\infty, a) \cup (b, \infty)$ is open.
- [2] The sets \mathbb{R} and \emptyset are closed.
- [3] The set \mathbb{Q} of rational numbers is not closed.
- [4] The set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not closed.
- [5] The set $\{1, 3, 4, 7\}$ is closed since its complement $(-\infty, 1) \cup (1, 3) \cup (3, 4) \cup (4, 7) \cup (7, \infty)$ is open.

[6] The set $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$ is closed since its complement

$$(-\infty, 0) \cup \left[\bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k} \right) \right] \cup (1, \infty)$$

is open.

Definition

Let S be a subset of \mathbb{R} .

(1) A point $x \in \mathbb{R}$ is an **accumulation point** of S if for every $\epsilon > 0$ there exists an element $s \in S$ such that $0 < |x - s| < \epsilon$; i.e., $[(x - \epsilon, x) \cup (x, x + \epsilon)] \cap S \neq \emptyset$. In other words, x is an accumulation point of S if every deleted ϵ -neighbourhood $N^*(x, \epsilon)$ of x contains a point of S .

The set of all accumulation points of S is called the **derived set** of S and is denoted by S' .

(2) S is said to be **dense in itself** if $S \subset S'$.

(3) S is called **perfect** if $S = S'$.

(4) The **closure** of S is the set $\overline{S} = S \cup S'$.

Remarks

(1) An accumulation point of a set S need not be an element of S .

(2) A real number x is an accumulation point of a set $S \subset \mathbb{R}$ if for each $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains infinitely many elements of S . Indeed, if x is an accumulation point of S then, for any $\epsilon > 0$, there exists an element $s_1 \in S$ with $s_1 \neq x$, such that $0 < |x - s_1| < \epsilon$. Taking $\epsilon_1 = |x - s_1|$, there exists an element $s_2 \in S$ with $s_2 \neq x$, such that $0 < |x - s_2| < \epsilon_1 < \epsilon$. Taking $\epsilon_2 = |x - s_2|$, there exists $s_3 \in S$ with $s_3 \neq x$ such that $0 < |x - s_3| < \epsilon_2 < \epsilon$. Continuing in this way we obtain a sequence (s_n) with the property that $s_n \neq x$ and $|s_n - x| < \epsilon$ for all n .

[1] Elements of a set $S \subset \mathbb{R}$ can be divided into two groups: **isolated points** = those points that can be separated from the rest of the set with an open interval, and **accumulation points** = those points that cannot be separated from the rest of the set with an open interval.

Examples

[1] Let $S = \{x \in \mathbb{R} : 0 < x \leq 1\}$. Then $S' = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Therefore $\overline{S} = S \cup S' = S'$.

[2] If $S = \{x \in \mathbb{R} : a \leq x \leq b\}$, then $S' = S$. Therefore $\overline{S} = S$.

[3] If $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$, then $S' = \{0\}$. Indeed, if $\epsilon > 0$, then there is an $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < \epsilon$. Therefore $-\epsilon < \frac{1}{m} < \epsilon$; i.e., $\frac{1}{m} \in (-\epsilon, \epsilon)$. Since $\frac{1}{m} \in S$, it follows that for each $\epsilon > 0$, $N^*(0, \epsilon) \cap S \neq \emptyset$. Also $\overline{S} = S \cup S' = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$. Note that $0 \notin S$.

[4] Every real number is an accumulation point of the set \mathbb{Q} of rational numbers; that is, $\mathbb{Q}' = \mathbb{R}$. Indeed, if $x \in \mathbb{R}$ and $\epsilon > 0$, then the interval $(x - \epsilon, x + \epsilon)$ contains infinitely many rational numbers. It now follows that $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

[5] If \mathbb{Z} is the set of integers, then $\mathbb{Z}' = \emptyset$. Indeed, for any $x \in \mathbb{R}$ we can find an $\epsilon > 0$ small enough such that $(x - \epsilon, x + \epsilon)$ contains no integer, except possibly when x is itself an integer. It thus follows that $\overline{\mathbb{Z}} = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z} \cup \emptyset = \mathbb{Z}$.

[6] A finite set has no accumulation points. Indeed, if $S = \{s_1, s_2, \dots, s_n\}$, and $x \in \mathbb{R}$, then taking $\epsilon = \min\{|s_j - x|, j = 1, 2, \dots, n\}$, we have that $\epsilon > 0$ and

$$[(x - \epsilon, x) \cup (x, x + \epsilon)] \cap S = \emptyset.$$

Thus $S' = \emptyset$, and so $\overline{S} = S$.

Another way of seeing that a finite set S has no accumulation points is simply that no ϵ -neighbourhood can contain infinitely many points of S since S is finite!

Theorem

Let $S \subset \mathbb{R}$. Then S is closed if and only if S contains all its accumulation points.

<Suppose that S is closed and let $x \in S'$. We want to show that $x \in S$. If $x \notin S$, then $x \in \mathbb{R} \setminus S$. Since S is closed, $\mathbb{R} \setminus S$ is open. Therefore there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus S$. This then implies that $(x - \epsilon, x + \epsilon) \cap S = \emptyset$. But this contradicts the fact that $x \in S'$. Thus $S' \subset S$.

To prove the converse, assume that $S' \subset S$. We want to show that S is closed, or equivalently, that $\mathbb{R} \setminus S$ is open. To this end, let $x \in \mathbb{R} \setminus S$. Then $x \notin S'$, and so there is an $\epsilon > 0$ such that

$$[(x - \epsilon, x) \cup (x, x + \epsilon)] \cap S = \emptyset.$$

Since $x \notin S$, we have that $(x - \epsilon, x + \epsilon) \cap S = \emptyset$. Thus $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus S$, whence $\mathbb{R} \setminus S$ is open. ■

Corollary

Let $S \subset \mathbb{R}$. Then S is closed if and only if $\overline{S} = S$.

<Assume that S is closed. Then, by Theorem 3.2.17, $S' \subset S$. Therefore $\overline{S} = S \cup S' \subset S \cup S = S$. But $S \subset S \cup S' = \overline{S}$. Thus $S = \overline{S}$.

Conversely, assume that $S = \overline{S}$. Then $S' \subset S \cup S' = \overline{S} = S$. Thus S contains all its accumulation points and, consequently, S is closed. ■

It follows from Theorem 3.2.17 that the sets $[a, b]$, \mathbb{Z} , $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0\right\}$ are all closed, as is any finite set.

Theorem

If $S \subset \mathbb{R}$ is closed and bounded, then $\sup S$ and $\inf S$ belong to S .

<Let $s = \sup S$. ($\sup S$ exists because S is bounded above.) Then, for any $\epsilon > 0$, there is an $x_\epsilon \in S$ such that $s - \epsilon < x_\epsilon \leq s < s + \epsilon$. If $x_\epsilon = s$, then $s \in S$ and we are done. If $x_\epsilon < s$, then $0 < |s - x_\epsilon| < \epsilon$. That is, for every $\epsilon > 0$, there is an $x_\epsilon \in S$ such that $0 < |s - x_\epsilon| < \epsilon$. Thus $s \in S'$. Since S is closed, $s \in S$.

A similar argument shows that $\inf S \in S$. ■

Exercise

[1] Let $S = \left\{n + \frac{1}{m}, n, m \in \mathbb{Z}, m > 0\right\}$. Find S' .

[2] Let S and T be subsets of \mathbb{R} .

(a) Show that if $S \subset T$, then $S' \subset T'$.

(b) Show that if $S \subset T$, then $\overline{S} \subset \overline{T}$.

(c) Show that \overline{S} is a closed subset of \mathbb{R} .

(d) Show that if F is a closed subset of \mathbb{R} and $S \subset F$, then $\overline{S} \subset F$.

(e) Show that $\overline{S} = \bigcap \{F \subset \mathbb{R} \mid F \text{ is closed and } S \subset F\}$. Deduce that \overline{S} is the smallest closed set containing S .

- (f) Show that $\overline{\overline{S}} = \overline{S}$.
- (g) Show that $(S \cup T)' = S' \cup T'$. Deduce that $\overline{S \cup T} = \overline{S} \cup \overline{T}$.
- (h) Show that $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$.
- (i) Is it true that $\overline{S \cap T} = \overline{S} \cap \overline{T}$?
- [3] (a) Show that a union of a finite collection of closed sets in \mathbb{R} is a closed set.
- (b) Show that an intersection of an arbitrary collection of closed sets in \mathbb{R} is a set.
- [4] Let S be a subset of \mathbb{R} .
- (a) Show that S' is closed.
- (b) Show that if S is bounded, then so is S' .
- (c) Show that if S is bounded, then so is \overline{S} .
- [5] Let S and T be subsets of \mathbb{R} .
- (a) Show that if $S \subset T$, then $S^\circ \subset T^\circ$.
- (b) Show that T° is an open subset of \mathbb{R} .
- (c) Show that T is open if and only if $T = T^\circ$.
- (d) Show that if G is an open subset of \mathbb{R} and $G \subset T$, then $G \subset T^\circ$.
- (e) Show that $T^\circ = \bigcup \{G \subset \mathbb{R} \mid G \text{ is open and } G \subset T\}$. Deduce that T° is the largest open set contained in T .
- (f) Show that $T^{\circ\circ} = T^\circ$.
- (g) Show that $(S \cap T)^\circ = S^\circ \cap T^\circ$.
- (h) Show that $S^\circ \cup T^\circ \subseteq (S \cup T)^\circ$.
- (i) Is it true that $(S \cup T)^\circ = S^\circ \cup T^\circ$?

Compactness

Definition

An **open cover** of a set $S \subset \mathbb{R}$ is a collection $\mathcal{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ of open sets such that

$$S \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha.$$

If $\mathcal{G}' \subseteq \mathcal{G}$ and \mathcal{G}' is also an open cover for S , then \mathcal{G}' is called a **subcover** for S . If, in addition, \mathcal{G}' has a finite number of elements, then \mathcal{G}' is called a **finite subcover** of S .

Examples

- [1] Let $S = [0, \infty)$ and, for each $n \in \mathbb{N}$, let $G_n = (-1, n)$. Then $\mathcal{G} = \{G_n \mid n \in \mathbb{N}\}$ is an open cover for S .
- [2] Let $S = [0, 1]$ and for each $n \in \mathbb{N}$, let $A_n = (-\frac{1}{n}, 1 + \frac{1}{n})$. Then $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$ is an open cover for S .
- [3] Let $S = (0, 1)$ and $\mathcal{U} = \{(\frac{1}{n}, 2) \mid n \in \mathbb{N}\}$. Then \mathcal{U} is an open cover for S . Indeed, let $x \in (0, 1)$. Then, by the Archimedean Property, there is a natural number m such that $0 < \frac{1}{m} < x$. Therefore $x \in (\frac{1}{m}, 2)$, whence, $(0, 1) \subset \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 2)$.

Definition

A subset S of \mathbb{R} is said to be **compact** if every open cover for S has a finite subcover.

Examples

[1] A finite subset of \mathbb{R} is compact. Indeed, let $S = \{x_1, x_2, \dots, x_n\}$ and let $\mathcal{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ be an open cover for S . Then each x_i belongs to some G_{α_i} in \mathcal{G} . The set $\mathcal{G}' = \{G_{\alpha_i} \mid 1 \leq i \leq n\}$ is a finite subcover for S .

[2] The set \mathbb{R} of real numbers is not compact since the open cover $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{N}\}$ of \mathbb{R} does not have a finite subcover.

Theorem

Let S be a compact subset of \mathbb{R} . If F is a closed subset of S , then F is compact.

<Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover for F . Then $\mathcal{G} = \mathcal{U} \cup \{F^c\}$ is an open cover for S . Since S is compact, the cover \mathcal{G} is reducible to a finite subcover. That is, there are indices $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$S \subset \bigcup_{i=1}^n U_{\alpha_i} \cup F^c.$$

Since $F \subset S$ and $F \cap F^c = \emptyset$, it follows that $F \subset \bigcup_{i=1}^n U_{\alpha_i}$. Hence F is compact. ■

Theorem

Let a and b be real numbers such that $-\infty < a < b < \infty$. Then the interval $[a, b]$ is compact.

<Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover for the interval $[a, b]$ and let

$$A = \{x \in [a, b] \mid [a, x] \text{ has a finite subcover in } \mathcal{U}\}.$$

Clearly $A \neq \emptyset$ since $a \in A$. Also, A is bounded above as $x \leq b$ for all $x \in A$. By the Completeness Axiom, $\sup A$ exists. Let $c = \sup A$. Then $a \leq c \leq b$.

Claim 1: The element c belongs to A .

Proof of Claim 1: Since \mathcal{U} is an open cover for the interval $[a, b]$, there is an index i_0 such that $c \in U_{i_0}$. Then, since U_{i_0} is open, there is an $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subseteq U_{i_0}$. In particular, $(c - \epsilon, c) \subseteq U_{i_0}$. By characterization of the supremum, there is an element $x_\epsilon \in A$ such that $c - \epsilon < x_\epsilon \leq c$. Therefore $a \leq x_\epsilon \leq c$ and so the interval $[a, c] = [a, x_\epsilon] \cup [x_\epsilon, c]$ has a finite subcover in \mathcal{U} . [The interval $[a, x_\epsilon]$ is finitely covered in \mathcal{U} since $x_\epsilon \in A$. The interval $[x_\epsilon, c]$ is covered by the set U_{i_0} .] It now follows that $c \in A$.

Claim 2: $c = b$.

Proof of Claim 2: If $c < b$, then, since \mathcal{U} is an open cover for $[a, b]$, there is a U_{i_0} in \mathcal{U} such that $c \in U_{i_0}$. Since U_{i_0} is open, there is a $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq U_{i_0}$. Choose $\delta > \delta' > 0$ small enough such that $[c, c + \delta'] \subseteq U_{i_0}$ and $c + \delta' < b$. Then $c + \delta' \in [a, b]$ and $[a, c + \delta'] = [a, c] \cup [c, c + \delta']$ has a finite subcover in \mathcal{U} . [The interval $[a, c]$ is finitely covered in \mathcal{U} since $c \in A$ and the interval $[c, c + \delta']$ is covered by the open set $U_{i_0} \in \mathcal{U}$.] This shows that $c + \delta'$ is in A . But this contradicts the fact that c is the supremum of A . Hence, $c = b$.

By definition of the set A , we conclude that $[a, b]$ can be covered by finitely many elements of \mathcal{U} . That is, $[a, b]$ is compact. ■

Theorem

(Heine-Borel Theorem). A subset K of \mathbb{R} is compact if and only if K is closed and bounded.

Assume that K is compact. We show that K is closed and bounded.

Closedness of K : It suffices to show that the complement, $\mathbb{R} \setminus K$, of K is open. To that end, let $x_0 \in \mathbb{R} \setminus K$ and for each $k \in \mathbb{N}$, let

$$U_k = \{x \in \mathbb{R} \mid |x - x_0| > \frac{1}{k}\} = (-\infty, x_0 - \frac{1}{k}) \cup (x_0 + \frac{1}{k}, \infty).$$

Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R} \setminus \{x_0\}$ and $\mathcal{U} = \{U_k \mid k \in \mathbb{N}\}$ is an open cover for K . Since K is compact, this cover of

K is reducible to a finite subcover. That is, there are indices k_1, k_2, \dots, k_n such that $K \subset \bigcup_{j=1}^n U_{k_j}$. Let $k_{\max} = \max\{k_1, k_2, \dots, k_n\}$. Then

$$K \subset \bigcup_{j=1}^n U_{k_j} = (-\infty, x_0 - \frac{1}{k_{\max}}) \cup (x_0 + \frac{1}{k_{\max}}, \infty) = \{x \in \mathbb{R} \mid |x - x_0| > \frac{1}{k_{\max}}\}.$$

Hence,

$$\{x \in \mathbb{R} \mid |x - x_0| < \frac{1}{k_{\max}}\} \subset \{x \in \mathbb{R} \mid |x - x_0| \leq \frac{1}{k_{\max}}\} \subset \mathbb{R} \setminus K,$$

whence $\mathbb{R} \setminus K$ is open and so K is closed.

Boundedness of K : Let $\mathcal{U} = \{(-k, k) \mid k \in \mathbb{N}\}$. Then \mathcal{U} is an open cover for K . Indeed,

$$K \subset \mathbb{R} = \bigcup_{k \in \mathbb{N}} (-k, k).$$

Since K is compact, there are natural numbers k_1, k_2, \dots, k_n such that $K \subset \bigcup_{j=1}^n (-k_j, k_j)$. Let $k_{\max} = \max\{k_1, k_2, \dots, k_n\}$. Then

$$K \subset \bigcup_{j=1}^n (-k_j, k_j) = (-k_{\max}, k_{\max}).$$

It now follows that K is bounded since it is contained in the bounded interval $(-k_{\max}, k_{\max})$.

Conversely, assume that K is a closed and bounded subset of \mathbb{R} . Then there are real numbers a and b such that $K \subset [a, b]$. It now follows from Theorem 3.3.6 and Theorem 3.3.5 that K is compact. ■

We now apply the Heine-Borel Theorem to prove another important result: the Bolzano-Weierstrass Theorem (for sets).

Theorem

(Bolzano-Weierstrass Theorem for sets). *Every bounded infinite set of real numbers has at least one accumulation point.*

<Let S be a bounded infinite set of real numbers. Suppose that S has no accumulation points. That is, the derived set of S , S' , is empty. Therefore $\overline{S} = S \cup S' = S$, and so S is closed. Thus S is a closed and bounded subset of \mathbb{R} . By the Heine-Borel Theorem (Theorem 3.3.7), S is compact. Since S has no accumulation points, given any $x \in S$, there is an $\epsilon > 0$ such that $S \cap N(x, \epsilon) = \{x\}$. Therefore, the collection $\{N(x, \epsilon) \mid x \in S, \epsilon > 0\}$ is an open cover for S . Since S is compact, there exist x_1, x_2, \dots, x_n in S and positive numbers $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ such that $S \subset \bigcup_{k=1}^n N(x_k, \epsilon_k)$. But then

$$S = S \cap \bigcup_{k=1}^n N(x_k, \epsilon_k) = \bigcup_{k=1}^n (S \cap N(x_k, \epsilon_k)) = \bigcup_{k=1}^n \{x_k\} = \{x_1, x_2, \dots, x_n\}.$$

That is, $S = \{x_1, x_2, \dots, x_n\}$, a finite set. This is a contradiction since S has infinitely many points. ■

Exercise

[1] We showed in Theorem 3.3.5 that a closed subset F of a compact set K is compact. Supply another proof to this statement by using the Heine-Borel Theorem.

- [2] Show that the interval $(0, 1)$ is not compact by
- (a) showing that $\mathcal{U} = \{(\frac{1}{n}, 2) \mid n \in \mathbb{N}\}$ is an open cover for $(0, 1)$ with no finite subcover.
 - (b) using the Heine-Borel Theorem.
- [3] Show that the interval $[0, \infty)$ is not compact by
- (a) showing that $\mathcal{G} = \{(-1, n) \mid n \in \mathbb{N}\}$ is an open cover for $[0, \infty)$ with no finite subcover.
 - (b) using the Heine-Borel Theorem.
- [4] Show that the set \mathbb{N} is not compact by
- (a) finding an open cover for \mathbb{N} that has no finite subcover.
 - (b) using the Heine-Borel Theorem.
- [5] Show that if F is closed and K compact, then $F \cap K$ is compact by
- (a) using the definition of compactness.
 - (b) applying the Heine-Borel Theorem.
- [6] Show that an arbitrary intersection of compact subsets of \mathbb{R} is compact.
- [7] Show that if A and B are compact subsets of \mathbb{R} then so is $A \cup B$ by
- (a) using the definition of compactness.
 - (b) applying the Heine-Borel Theorem.
- [8] Find an infinite collection $\{K_n \mid n \in \mathbb{N}\}$ of compact sets in \mathbb{R} such that $\bigcup_{n \in \mathbb{N}} K_n$ is **not** compact.
 This shows that an arbitrary union of compact sets is not compact.
- [9] Show that a subset K of \mathbb{R} is compact if and only if every infinite subset of K has an accumulation point in K .