Cardinality: the size of a set

Definition

Two sets A and B are said to **have the same cardinality**, *denoted by $|A| = |B|$, if there is a one-to-one function from A onto B. Sets that have the same cardinality are also said to be* **equipotent** *or* **equinumerous**.

Examples

[1] $\mathbb{N}_0$ has the same cardinality as $\mathbb{N}$.

Proof. Define $f : \mathbb{N}_0 \to \mathbb{N}$ by $f(n) = n + 1$ for each $n \in \mathbb{N}_0$.

Claim: f is one-to-one. Let $n,\ m \in \mathbb{N}_0$ such that $f(n) = f(m)$ Then $n + 1 = m + 1$, and consequently $n = m$.

Claim: f is onto. Let $m \in \mathbb{N}$. Then $m - 1 \in \mathbb{N}_0$ and $f(m - 1) = m - 1 + 1 = m$.

[2] Let $\mathbb{E} = \{2n \ : \ n \in \mathbb{N}\} -$ the set of even natural numbers. Then $\mathbb{N}$ and $\mathbb{E}$ have the same cardinality.

Proof. Define $f : \mathbb{N} \to \mathbb{E}$ by $f(n) = 2n$ for each $n \in \mathbb{N}$.

Claim: f is one-to-one. Let n_1 and n_2 be elements of $\mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $2n_1 = 2n_2$ and consequently $n_1 = n_2$.

Claim: f is onto. Let $m \in \mathbb{E}$. Then $m = 2k$ for some $k \in \mathbb{N}$. Hence, $f(k) = 2k = m$.

[3] $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

Proof. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} -\dfrac{n}{2} & \text{if} \quad n \text{ is even} \\[2ex] \dfrac{n-1}{2} & \text{if} \quad n \text{ is odd.} \end{cases}$$

In tabular form

n	1	2	3	4	5	6	7	$\cdots$
$f(n)$	0	-1	1	-2	2	-3	3	$\cdots$

It is clear that $\text{dom}(f) = \mathbb{N}$ and $\text{ran}(f) \subseteq \mathbb{Z}$.

Claim: f is one-to-one. For each $n \in \mathbb{N}$, $f(n) < 0$ if n is even and $f(n) \geq 0$ if n is odd. Let $m_1,\ m_2 \in \mathbb{N}$ such that $f(m_1) = f(m_2)$. We must show that $m_1 = m_2$. If m_1 and m_2 are both even, then

$$\begin{aligned} f(m_1) = f(m_2) \quad &\iff \quad -\frac{m_1}{2} = -\frac{m_2}{2} \\ &\iff \quad m_1 = m_2. \end{aligned}$$

If m_1 and m_2 are both odd, then

$$f(m_1) = f(m_2) \iff \frac{m_1 - 1}{2} = \frac{m_2 - 1}{2} \iff m_1 = m_2.$$

Hence, f is one-to-one.

Claim: f is onto. If $m \in \mathbb{Z}$ and is negative, then $-2m$ is in $\mathbb{N}$ and is even. Therefore

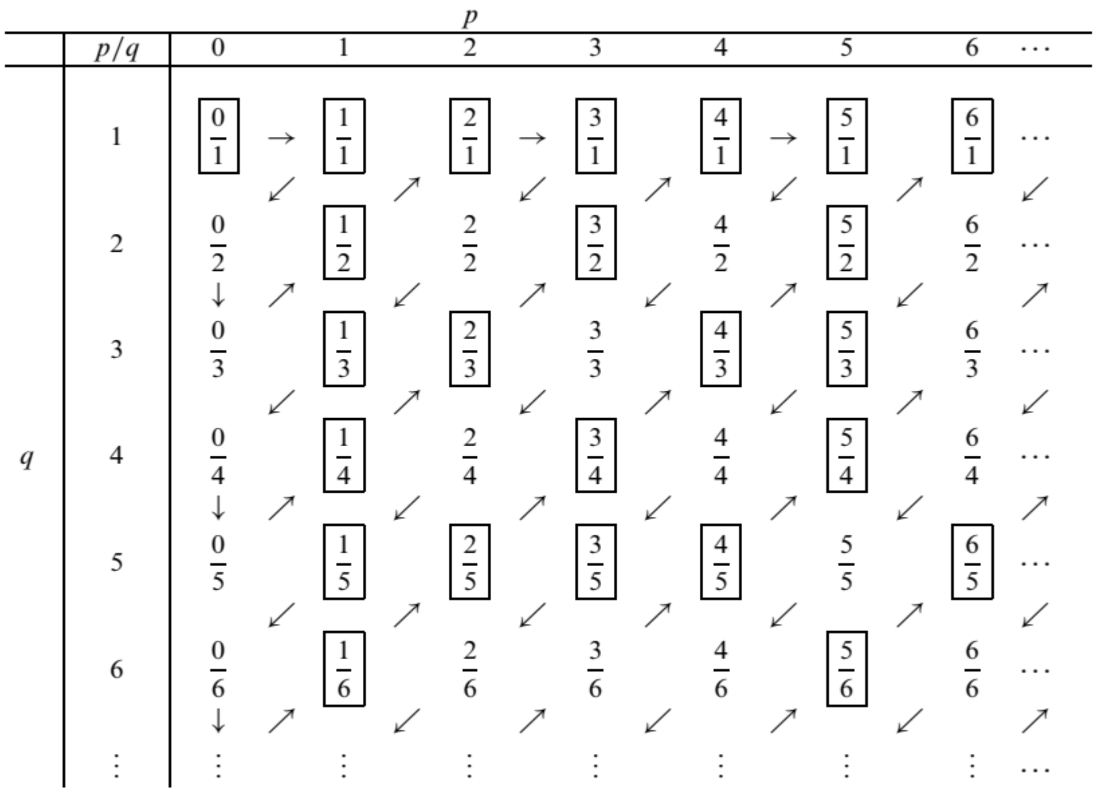
$$f(-2m) = (-1)\frac{(-2m)}{2} = m.$$

If $m \in \mathbb{Z}$ and $m \geq 0$, then $2m + 1$ is in $\mathbb{N}$ and is odd. Therefore

$$f(2m+1) = \frac{(2m+1)-1}{2} = m.$$

[4] $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality.

Proof. We start by listing nonnegative rational numbers in an infinite matrix as follows:



Starting with $\frac{0}{1}$ at the top left corner, we follow the arrows, putting a box around a rational number that occurs for the first time. This assigns a unique natural number to each nonnegative rational number. That is, this defines a function g from $\mathbb{N}_0$ to the set of nonnegative rational numbers $\mathbb{Q}^+ \cup \{0\}$ given by the following table:

n	0	1	2	3	4	5	6	$\cdots$
$g(n)$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{1}{3}$	$\frac{1}{4}$	$\cdots$

Define $f : \mathbb{N}_0 \to \mathbb{Q}$ by

$$f(n) = \begin{cases} -\,g\left(\frac{n}{2}\right) & \text{if} \quad n \text{ is even} \\ g\left(\frac{n+1}{2}\right) & \text{if} \quad n \text{ is odd.} \end{cases}$$

In tabular form

n	0	1	2	3	4	5	6	$\cdots$
$f(n)$	0	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	2	-2	$\cdots$

Then f is a bijection between $\mathbb{N}_0$ and $\mathbb{Q}$. Since $\mathbb{N}$ and $\mathbb{N}_0$ have the same cardinality, there is a bijection $h : \mathbb{N} \to \mathbb{N}_0$. Therefore $f \circ h$ is a bijection from $\mathbb{N}$ onto $\mathbb{Q}$.

[5] $\mathbb{R}$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ have the same cardinality.

Proof. Define $f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by $f(x) = \arctan x$.

Claim: f is one-to-one. Let x_1 and x_2 be elements of $\mathbb{R}$ such that $f(x_1) = f(x_2)$ Then

$$\begin{aligned} \arctan x_1 &= \arctan x_2 \\ \Rightarrow \tan(\arctan x_1) &= \tan(\arctan x_2) \\ \Rightarrow x_1 &= x_2. \end{aligned}$$

Claim: f is onto. Let $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then, $\tan y \in \mathbb{R}$ and, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have that $\arctan(\tan y) = y$. Let $x = \tan y$. Then $f(x) = y$.

[6] The intervals $(0, 1)$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ have the same cardinality.

$\prec$Define $f : (0, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by $f(x) = \pi x - \frac{\pi}{2}$. It is easy to show that f is a well-defined bijection from $(0, 1)$ onto $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We immediately deduce from examples 5 and 6 that $(0, 1)$ and $\mathbb{R}$ have the same cardinality.

Definition

A set S is said to be

(a) **finite** *if $S = \emptyset$ or if there is an $n \in \mathbb{N}$ such that $|S| = |\{1, 2, 3, \ldots, n\}|$.*

(b) **infinite** *if S is not finite.*

(c) **countably infinite** *if $|S| = |\mathbb{N}|$.*

(d) **countable** *if S is finite or is countably infinite.*

(e) **uncountable** *if S is not countable.*

The cardinality of $\mathbb{N}$ is called $\aleph_0$ (aleph nought).
We have shown that the sets $\mathbb{E}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

Theorem

There does not exist a surjection from a set X onto its power set $\mathcal{P}(X)$.

Proof. (By Contradiction). Suppose there were such a surjection $f : X \to \mathcal{P}(X)$. Let A be the subset of X defined by

$$A = \{x \in X \ : \ x \notin f(x)\}.$$

Then $A \in \mathcal{P}(X)$. Since f is assumed to be surjective, there exists an $a \in X$ such that $f(a) = A$. Either $a \in A$ or $a \notin A$. If $a \in A$, then by definition of A, $a \notin f(a) = A$, a contradiction. Therefore, $a \notin A$. But now again by definition of A, it follows that $a \in A$, a contradiction again. We conclude that there is no function from X onto $\mathcal{P}(X)$. ■

Corollary
$\mathcal{P}(\mathbb{N})$ is uncountable.

Theorem
The set of real numbers in the interval $(0, 1)$ is uncountable.

$\prec$(By contradiction). Assume that $(0, 1)$ is countable. Let $\{x_1,\ x_2,\ x_3,\ \ldots\}$ be the enumeration of elements of $(0, 1)$; that is, there is a bijection $f : \mathbb{N} \to (0, 1)$ given by $f(k) = x_k$. Each $x_n \in (0, 1)$ has a decimal expansion of the form

$$\begin{array}{rcl} x_1 & = & 0.a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\ x_2 & = & 0.a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\ x_3 & = & 0.a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\ & \vdots & \\ x_n & = & 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}\cdots \\ \vdots & & \vdots \end{array}$$

where $a_{ij} \in \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$. Let b be the real number that has the decimal expansion:

$$b = 0.b_1b_2b_3b_4b_5\cdots$$

where

$$b_n = \begin{cases} 2 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} \neq 1. \end{cases}$$

Then, clearly, $b \in (0, 1)$ and $b \neq x_k$ for all $k \in \mathbb{N}$ since b and x_k differ at the k-place after the decimal point. Hence, the function $f : k \mapsto x_k$ is **not** surjective. ■

Corollary
The set $\mathbb{R}$ of real numbers is uncountable.

$\prec$This follows immediately from examples 2.5.2 (5 and 6).

In order to establish the next set of important results, we shall need the following result called the Well Ordering Principle or Least Natural Number Principle:

Theorem
Every nonempty subset A of natural numbers has a least member - a number $a_0 \in A$ such that $a_0 \leq a$ for all $a \in A$.

The Least Natural Number Principle is equivalent to the Principle of Mathematical Induction. That is, assuming one principle you can prove the other. Below we prove that the Principle of Mathematical Induction implies the Least Natural Number Principle. We leave the proof of the converse of this statement as an exercise.

Theorem
The Principle of Mathematical Induction implies the Least Natural Number Principle.

Proof. Let T be a subset of $\mathbb{N}$ with no least element. We prove that T is an empty set. Let

$$S = \{n \in \mathbb{N}\ :\ \{1, 2, \ldots, n\} \cap T = \emptyset\}.$$

Claim 1: $1 \in S$. If $1 \notin S$, then $\{1\} \cap T \neq \emptyset$. But then $1 \in T$ and 1 would be the least element of T, contradicting the fact that T has no least element. Hence $1 \in S$.

Claim 2: $k \in S \Rightarrow k+1 \in S$. Since $k \in S$ (by the assumption), it follows that $\{1, 2, \ldots, k\} \cap T = \emptyset$. This says that no positive natural number less than or equal to k belongs to T. We must show that $k + 1$ does not belong to T or equivalently, $k + 1 \in S$. If $k + 1 \notin S$, then $\{1, 2, \ldots, k, k + 1\} \cap T \neq \emptyset$. Since $\{1, 2, \ldots, k\} \cap T = \emptyset$, it follows that $k + 1 \in T$. But then $k + 1$ would be the least element of T, contradicting the fact that T has no least element. Hence $k + 1 \in S$.
By the Principle of Mathematical Induction, we have that $S = \mathbb{N}$. This, of course, means that no natural numbers belongs to T, i.e., $T = \emptyset$. ■

We are now ready to establish some important results.

Theorem
A subset of a countable set is countable.

≺Let A be a subset of a countable set B. If A is finite, then it is obviously countable. Assume that A is infinite. Then B is countably infinite. Let $\{b_1, b_2, b_3, \ldots\}$ be an enumeration of elements of B That is, there is a bijection $f : \mathbb{N} \to B$ given by $f(k) = b_k$.

Let $\mathbb{M} = \{n \in \mathbb{N} \mid b_n \in A\}$. Then $\mathbb{M}$ is a nonempty subset of $\mathbb{N}$. By the Least Natural Number Principle, $\mathbb{M}$ has the least element m_1. Similarly, $\mathbb{M} - \{m_1\}$ has the least element m_2. In general, having chosen $m_1, m_2, \ldots, m_k$, let m_{k+1} be the least element of $\mathbb{M} - \{m_1, m_2, \ldots, m_k\}$. Define $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = m_n$. Since A is infinite, g is defined for each $n \in \mathbb{N}$.

Claim: g is injective. Indeed, if $i < j$, then $m_i \neq m_j$ since $m_j \notin \{m_1, m_2, \ldots, m_i\}$. Thus $g(i) \neq g(j)$.

We have the diagram:

$$\mathbb{N} \xrightarrow{g} \mathbb{N} \xrightarrow{f} B.$$

It now follows that $f \circ g$ is injective. Since each element of A appears somewhere in the enumeration of elements of B, we have that $g(\mathbb{N})$ includes all the subscripts of elements of A. Thus, $f \circ g$ is a bijection from $\mathbb{N}$ onto A. Hence, A is countable. ■

Here is another argument that $\mathbb{R}$ is uncountable: Assume that $\mathbb{R}$ is countable. Then, by Theorem 2.5.10, every subset of $\mathbb{R}$ would be countable. In particular, the set or real numbers in the interval $(0, 1)$ would be countable. This contradicts Theorem 2.5.6. Hence, $\mathbb{R}$ is countable.

Corollary
An intersection of any collection of countable sets is countable.

≺Let $\{A_\lambda \mid \lambda \in I\}$ be a collection of sets such that A_λ is countable for each $\lambda \in I$. Choose and fix $\alpha \in I$. Then

$$\bigcap_{\lambda \in I} A_\lambda \subset A_\alpha.$$

Since A_α is countable, it follows from Theorem 2.5.10 that $\bigcap_{\lambda \in I} A_\lambda$ is countable. ■

Theorem
Let A be a nonempty set. The following statements are equivalent:

(a) *A is countable;*

(b) *There is a surjection $f : \mathbb{N} \to A$.*

(c) *There is an injection $f : A \to \mathbb{N}$.*

≺(a) $\Rightarrow$ (b): Assume that A is countable. If A is finite, then there is nothing to prove. Assume that A is infinite. Then A is countably infinite. Thus, there is a bijection $f : \mathbb{N} \to A$. Therefore, f is a surjection from $\mathbb{N}$ onto A.

(b) $\Rightarrow$ (c): Assume that there is a surjection $f : \mathbb{N} \to A$. Then the set

$$f^{\leftarrow}(a) := \{n \in \mathbb{N} \mid f(n) = a\} \neq \emptyset$$

for each $a \in A$.

Define $g : A \to \mathbb{N}$ by

$$g(a) = \text{the least element of the set } f^{\leftarrow}(a)$$

for each $a \in A$. By the Least Natural Number Principle, we have that g is well-defined. We show that g is injective. Note first that since $g(a) \in f^{\leftarrow}(a)$, it follows that $f(g(a)) = a$.

Let $a, b \in A$ such that $g(a) = g(b)$. Then

$$a = f(g(a)) = f(g(b)) = b.$$

Thus, g is injective.

(c) $\Rightarrow$ (a): Assume that there is an injection $g : A \to \mathbb{N}$. Then g is a bijection from A onto $g(A) := \{n \in \mathbb{N} \mid g(a) = n \text{ for some } a \in A\}$. Since $\mathbb{N}$ is countable and $g(A) \subset \mathbb{N}$, it follows from Theorem 2.5.10 that $g(A)$ is countable. Thus A is countable. ■

Theorem
$\mathbb{N} \times \mathbb{N}$ is countable.

$\prec$Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$f(n, m) = 2^n \cdot 3^m.$$

We show that f is an injection. To that end, let (n, m) and (k, ℓ) be elements of $\mathbb{N} \times \mathbb{N}$ such that

$$f(n, m) = f(k, \ell).$$

Then

$$2^n \cdot 3^m = 2^k \cdot 3^\ell \iff 2^{n-k} = 3^{\ell-m}.$$

Hence, $n - k = 0$ and $\ell - m = 0$ and, consequently, $n = k$ and $m = \ell$. That is, $(n, m) = (k, \ell)$. This shows that f is injective. By Theorem 2.5.12(c), we conclude that $\mathbb{N} \times \mathbb{N}$ is countable. ■

Corollary
If A and B are countable sets, then $A \times B$ is also countable.

$\prec$Since A and B are countable, there are bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Define $h : \mathbb{N} \times \mathbb{N} \to A \times B$ by

$$h(n, m) = (f(n), g(m)) \text{ for all } (n, m) \in \mathbb{N} \times \mathbb{N}.$$

Clearly, h is well-defined.

Claim 1: h is injective. Assume that $h(n, m) = h(k, \ell)$. Then, by definition of h, $(f(n), g(m)) = (f(k), g(\ell))$. Therefore $f(n) = f(k)$ and $g(m) = g(\ell)$. Since f and g are injective, it follows that $n = k$ and $m = \ell$ and, consequently, $(n, m) = (k, \ell)$.

Claim 2: h is surjective. Let $(a, b) \in A \times B$. Since f and g are surjective, there are natural numbers i and j such that $f(i) = a$ and $g(j) = b$. Hence, $(i, j) \in \mathbb{N} \times \mathbb{N}$ and

$$h(i, j) = (f(i), g(j)) = (a, b).$$

Thus, h is surjective. ■

We give another proof that the set $\mathbb{Q}$ of rational numbers is countable.

Corollary
The set $\mathbb{Q}$ of rational numbers is countable.

$\prec$Since $\mathbb{Z}$ and $\mathbb{N}$ are countable, we have, by Corollary 2.5.14, that $\mathbb{Z} \times \mathbb{N}$ is countable. So there is a surjection $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$. Define $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ by

$$g(p, q) = \frac{p}{q}.$$

Clearly, g is surjective (by the definition of rational numbers). We have the following diagram:

$$\mathbb{N} \xrightarrow{f} \mathbb{Z} \times \mathbb{N} \xrightarrow{g} \mathbb{Q}.$$

Since the function $g \circ f$ is a surjection from $\mathbb{N}$ onto $\mathbb{Q}$, it follows from Theorem 2.5.12 that $\mathbb{Q}$ is countable.■

Theorem
A countable union of countable sets is countable.

≺Let $\{A_n \mid n \in \mathbb{N}\}$ be a collection of sets such that A_n is countable for each $n \in \mathbb{N}$ and let $A = \bigcup_{n=1}^{\infty} A_n$. We show that A is countable. Since A_n is countable for each $n \in \mathbb{N}$, there is a surjection $f_n : \mathbb{N} \to A_n$ for each $n \in \mathbb{N}$. Define $f : \mathbb{N} \times \mathbb{N} \to A$ by

$$f(n, m) = f_n(m).$$

We show that f is surjective. Indeed, if $a \in A$, then $a \in A_n$ for some $n \in \mathbb{N}$. Since f_n is surjective, there is an $m \in \mathbb{N}$ such that $f_n(m) = a$. Therefore $(n, m) \in \mathbb{N} \times \mathbb{N}$ and $f(n, m) = f_n(m) = a$. Thus, f is surjective. Since $\mathbb{N} \times \mathbb{N}$ is countable, there is a surjection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. We have the following diagram:

$$\mathbb{N} \xrightarrow{g} \mathbb{N} \times \mathbb{N} \xrightarrow{f} A.$$

Thus, $g \circ f$ is a surjection from $\mathbb{N}$ onto A. By Theorem 2.5.12, A is countable. ■

Exercise
[1] Show that the set of irrational numbers is uncountable.

2.5.1 The Cantor-Schröder-Bernstein Theorem

When we started the section on cardinality, we said that two sets A and B have the same cardinality if there is a bijection (one-to-one and onto function) between them. It is usually easier to find an injection than a bijection between two sets. The Cantor-Schröder-Bernstein Theorem asserts that if A and B are sets for which we can find an injection from A into B and an injection from B into A, then there is a bijection between A and B.

Lemma
Let A and B be sets such that $B \subseteq A$. If there is an injective function $f : A \to B$, then there is a bijective function $g : A \to B$.

Proof. If $A = B$, then the identity function i_A works. Assume that $B \subsetneqq A$. We inductively define a sequence (C_n) of sets as follows:

$$\begin{array}{rcl}
C_0 & = & A \setminus B \\
C_1 & = & f(C_0) = f(A \setminus B) \\
C_2 & = & f(C_1) = f^2(A \setminus B) \\
C_3 & = & f(C_2) = f^3(A \setminus B) \\
\vdots & \vdots & \vdots \\
C_n & = & f(C_{n-1}) = f^n(A \setminus B) \\
\vdots & \vdots & \vdots
\end{array}$$

Let $C = \bigcup_{n=0}^{\infty} C_n = \bigcup_{n=0}^{\infty} f^n(A \setminus B)$, where f^0 is the identity map on A. Note that $A \setminus B = C_0 \subset C$ and $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} f^n(A \setminus B) \subseteq B$.

Claim 1: If $j,\ k \in \mathbb{N}_0$ and $j \neq k$, then $C_j \cap C_k = \emptyset$; that is, the sets C_n are pairwise disjoint. To prove the claim, assume that $j < k$ and that $C_j \cap C_k \neq \emptyset$. Let $z \in C_j \cap C_k$; that is, $z \in f^j(A \setminus B) \cap f^k(A \setminus B)$. Then there are x and y in $A \setminus B$ such that $f^j(x) = z = f^k(y)$. Therefore

$$f^j(x) = f^k(y) = f^{k-j}\left(f^j(y)\right) = f^j\left(f^{k-j}(y)\right).$$

Since f is injective, so is f^j. Hence $x = f^{k-j}(y)$. But, since $x \in A \setminus B$ and $f^{k-j}(y) \in B$, the equality $x = f^{k-j}(y)$ means that $x = f^{k-j}(y) \in (A \setminus B) \cap B = \emptyset$. This is a contradiction. Hence, $C_j \cap C_k = \emptyset$.

Claim 2: $f(C) \subset C$. Indeed,

$$f(C) = f\left(\bigcup_{n=0}^{\infty} C_n\right) = \bigcup_{n=0}^{\infty} f(C_n) = \bigcup_{n=0}^{\infty} C_{n+1} \subset C.$$

Define $g : A \to B$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ \\ x & \text{if } x \in A \setminus C. \end{cases}$$

Claim 3: g is injective. Let $x, y \in A$ such that $g(x) = g(y)$. If $x, y \in C$, then $f(x) = f(y)$. Since f is injective, it follows that $x = y$. If $x \notin C$ and $y \notin C$, then $x = g(x) = g(y) = y$. That is, $x = y$. If $x \in C$ and $y \in A \setminus C$, then $x \neq y$ and $f(x) \in f(C) \subset C$. Therefore $g(x) = f(x) \in C$ and $g(y) = y \in A \setminus C$. Hence $g(x) \neq g(y)$.

Claim 4: g is surjective. Let $y \in B$. If $y \in C$, then $y \in f^n(A \setminus B)$ for some $n = \{1,\ 2,\ \ldots\}$. Hence, there is an $z \in A \setminus B$ such that $y = f^n(z)$. Let $x = f^{n-1}(z)$. Then $x \in f^{n-1}(A \setminus B) \subset C$. Hence, by definition of g,

$$g(x) = f(x) = f\left(f^{n-1}(z)\right) = f^n(z) = y.$$

If $y \in A \setminus C$, then, by definition of g, $g(y) = y$. ■

Theorem
(Cantor-Schröder-Bernstein Theorem). *Let A and B be sets. If there exist two injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.*

Proof. Since f and g are injective functions, the composite function $g \circ f$ is an injection from A into $g(B)$. Also, $g(B) \subseteq A$. By Lemma 2.5.18, there is a bijection $k : A \to g(B)$. Since g is an injection from B into A, it is a bijection from B onto $g(B)$. The inverse function g^{-1} is a bijection from $g(B)$ onto B. We now have the diagram

$$A \xrightarrow{k} g(B) \xrightarrow{g^{-1}} B.$$

The composite function $h := g^{-1} \circ k$ is a bijection from A onto B. ■

Example
We use the Cantor-Schröder-Bernstein Theorem to show that the sets $[-1, 1]$ and $\mathbb{R}^+$ have the same cardinality. Let $f : [-1, 1] \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to [-1, 1]$ be given by $f(x) = x + 3$ and $g(x) = \dfrac{1}{x+1}$ respectively. The function f is clearly injective and maps the interval $[-1, 1]$ onto the interval $[2, 4]$. This function is not onto - for example, for 5, which is in $\mathbb{R}^+$, there is no $x \in [-1, 1]$ such that $f(x) = 5$.

The function g is also injective and maps $\mathbb{R}$ onto the interval $(0, 1)$. This function is not onto - for example, for 0, which is in $[-1, 1]$, there is no $x \in \mathbb{R}$ such that $g(x) = 0$.

By the Cantor-Schröder-Bernstein Theorem, there is a bijection between $[-1, 1]$ and $\mathbb{R}$. Hence, these sets have the same cardinality.