Sets and Functions

Introduction

The concept of a *set* permeates every aspect of Mathematics. Set theory underlies the language and concepts of modern Mathematics. The term *set* refers to a well-defined collection of objects that share a certain property or certain properties. The term "well-defined" here means that the set is described in such a way that one can decide whether or not a given object belongs in the set. If A is a set, then the objects of the collection A are called the *elements* or *members* of the set A. If x is an element of the set A, we write $x \in A$. If x is NOT an element of the set A, we write $x \notin A$.

As a convention, we use capital letters to denote the names of sets and lowercase letters for elements of a set.

There are several ways of describing sets, but two are common:

[1] **The Roster method**:- listing the elements of a set, separated by commas and enclosed in braces; e.g., $A = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$. There are two important facts to bear in mind: (1) the order in which the elements are listed is irrelevant, (2) each element should be listed only once in the roster.

[2] **The rule or description method**:- we describe a set in terms of one or more properties that the objects in the set must satisfy. We use set-builder notation to write such a set, e.g., $A = \{x \mid x \text{ satisfies some property or properties}\}$. The vertical bar " | " is read as "such that". Other people use " : " instead of the bar " | ".

If a set A consists of a large (or infinite) number of elements, it is general practice to list a few of its elements followed with ellipsis (...). This method requires recognition of the pattern in the list of elements of A. This practice tends to introduce some ambiguity as the list may be continued in many different ways. It is safer practice to define such a set by spelling out the pattern that determines membership of the set.

Examples

(a) The set $\mathbb{E} = \{2,\ 4,\ 6,\ 8,\ \ldots\}$ is best described as

$$\mathbb{E} = \{n \in \mathbb{N}\ :\ n = 2k \ \text{ for some } \ k \in \mathbb{N}\}.$$

(b) The set $B = \{1,\ 4,\ 9,\ 16,\ \ldots\}$ is best described as

$$B = \{n \in \mathbb{N}\ :\ n = k^2 \ \text{ for some } \ k \in \mathbb{N}\}.$$

Some sets come up often in Mathematics and they have special names assigned to them.

$\mathbb{N}_0 = \{0,\ 1,\ 2,\ 3, \ldots\}$	Natural numbers
$\mathbb{N} = \{1,\ 2,\ 3, \ldots\}$	Positive natural numbers
$\mathbb{Z} = \{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2, \ldots\}$	Integers
$\mathbb{Q} = \{p/q\ :\ p,\ q \in \mathbb{Z}, q \neq 0\}$	Rational numbers
$\mathbb{Q}^+$	Positive rational numbers

$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{C} = \{z = a + bi \ : \ a, b \in \mathbb{R} \text{ and } i^2 = -1\}$	Complex numbers.

Definition
Let A and B be sets. We say that

(a) *B is a* **subset** *of A (or is* **contained** *in A), denoted by $B \subseteq A$, if every element of B is an element of A, i.e., $(\forall x)(x \in B \Rightarrow x \in A)$.*

(b) *$A = B$ if $(A \subseteq B) \wedge (B \subseteq A)$, i.e., $(\forall x)(x \in A \Leftrightarrow x \in B)$.*

(c) *If B is a subset of A and $A \neq B$, then B is a* **proper** *subset of A. In this case we write $B \subsetneq A$. It is clear that*

$$B \subsetneq A \Leftrightarrow [(\forall x)(x \in B \Rightarrow x \in A) \wedge (B \neq A)].$$

Example
Let

$$\begin{aligned} A &= \{-1, 0, 1, 2, 3, 4\} \\ B &= \{1, 2, 3\} \\ C &= \{x \in \mathbb{R} \ : \ x^3 - 6x^2 + 11x - 6 = 0\} \\ D &= \{-1, 0, 1, 8\} \\ E &= \{x \in \mathbb{Z} \ : \ -2 < x < 5\}. \end{aligned}$$

Then

$$3 \in A, \qquad -2 \notin B, \qquad B \subseteq A, \qquad D \nsubseteq A,$$

$$B = C, \qquad A = E, \qquad \{8\} \subseteq D, \qquad \{2, 3\} \subseteq B.$$

We say that a set is **empty** if it has no elements. For example,

$$\{x \in \mathbb{R} \ : \ x^2 + 1 = 0\}$$

is an empty set since the equation $x^2 + 1 = 0$ has no solution in $\mathbb{R}$.

Proposition

(a) *If B is an empty set, then $B \subseteq A$ for any set A.*

(b) *All empty sets are equal, i.e., if B and C are empty sets, then $B = C$.*

Proof. (a) We must show that every element of B is an element of A; i.e., $(\forall x)(x \in B \Rightarrow x \in A)$. Using the contrapositive method, it suffices to show that $(\forall x)(x \notin A \Rightarrow x \notin B)$. This is vacuously true since if $x \notin A$, then $x \notin B$ (since B contains no elements!)
(b) From (a) we have that $(B \subseteq C) \wedge (C \subseteq B)$. Hence $B = C$. ■

It follows from Proposition 2.1.4(b) that there is a unique empty set.

Axiom of the empty set: *There is a set that contains no elements. This is called the* **empty set**. *It is denoted by $\emptyset$ or $\{ \}$.*

Proposition

(a) *For any set A, $A \subseteq A$.*

(b) *Let $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Russell's Paradox

We have been very casual and informal in our definition of a set. One has to be careful though if one is to avoid some unpleasant surprises. Russell's Paradox is a salutary reminder that one has to exercise care when defining sets.

Consider the set $A = \{1, 2, 3, 4\}$. Then, $3 \in A$, $A \subseteq A$, but $A \notin A$. The set A does not contain itself as an element.

Let us now consider the set S of all sets; i.e., $S = \{B : B \text{ is a set}\}$. Notice that not only is $S \subseteq S$, $S \in S$, since S is a set.

There are therefore sets that contain themselves as elements (e.g., S), and there are sets that do not contain themselves as elements (e.g. A).

Let R be the set of all those sets that do not contain themselves, i.e.,

$$R = \{X \mid (X \text{ is a set}) \wedge (X \notin X)\}.$$

The question is "Does R contain itself an element?"

Well, let's assume $R \notin R$, i.e., R does not contain itself as an element. So by definition of R, R is a member of R. So our assumption that R is not an element of R logically leads to the statement that R is a member of R. This is a contradiction, so our assumption must be wrong.

Let's assuming that R is an element of R, i.e., $R \in R$. But R is the set that has only members that do not contain themselves, so R cannot be a member of R. So our assumption that R is a member of R logically leads to the statement that R is not a member of R. This is a contradiction, so our assumption must be wrong.

In short, we have the situation that $R \in R \Leftrightarrow R \notin R$. □

The main point of Russell's Paradox is that there are properties that do not define sets, i.e., all objects with those properties cannot be collected into one set.

As Russell's Paradox indicates, there are logical difficulties that arise in the foundations of Set Theory if one is not careful. We can avoid such difficulties by assuming that each discussion in which a number of sets are involved is taking place within a context of a fixed set. This set is called the **universal set**.

Some notation...

We use special notation to designate intervals of various kinds on the real line. Let $a, b \in \mathbb{R}$ with

$a \leq b$.

$$\begin{aligned}
[a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \\
[a,b) &= \{x \in \mathbb{R} : a \leq x < b\} \\
(a,b] &= \{x \in \mathbb{R} : a < x \leq b\} \\
(a,b) &= \{x \in \mathbb{R} : a < x < b\} \\
[a,\infty) &= \{x \in \mathbb{R} : x \geq a\}
\end{aligned}$$

$$\begin{aligned}(a,\infty) &= \{x \in \mathbb{R} : x > a\}\\(-\infty,b] &= \{x \in \mathbb{R} : x \le b\}\\(-\infty,b) &= \{x \in \mathbb{R} : x < b\}\\(-\infty,\infty) &= \mathbb{R}.\end{aligned}$$

Operations on Sets

Definition
Let A be a set. The **power set** *of A, denoted by $\mathcal{P}(A)$, is the set whose elements are all subsets of A. That is,*

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

Example
Let $A = \{x, y, z\}$. Then

$$\begin{aligned}\mathcal{P}(\emptyset) &= \{\emptyset\}\\\mathcal{P}(\{x\}) &= \{\emptyset, \{x\}\}\\\mathcal{P}(\{x, y\}) &= \{\emptyset, \{x\}, \{y\}, \{x, y\}\}\\\mathcal{P}(A) &= \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}.\end{aligned}$$

$\bigotimes$: Note that $\emptyset$ is not the same as $\{\emptyset\}$.

Definition
Let A and B be subsets of a universal set U.

(a) *The* **union** *of A and B, denoted by $A \cup B$, is the set of all elements in U that are either in A or in B (or in both sets). That is,*

$$A \cup B = \{x \in U : (x \in A) \vee (x \in B)\}.$$

(b) *The* **intersection** *of A and B, denoted by $A \cap B$, is the set of all elements in U that are in A and B. That is,*

$$A \cap B = \{x \in U : (x \in A) \wedge (x \in B)\}.$$

Sets A and B are said to be **disjoint** *if $A \cap B = \emptyset$.*

(c) *The* **complement of** *A* **in** *(or* **relative to***) B, denoted by $B \setminus A$ or $B - A$ and read "B minus A", is the set of all elements of B that are not in A, i.e.,*

$$B - A = \{x \in U : (x \in B) \wedge (x \notin A)\}.$$

(d) *The* **complement of** *A, denoted by A', is the set of all elements in U that are not in A, i.e.,*

$$A' = \{x \in U : x \notin A\}.$$

(e) The **symmetric difference of** A **and** B *denoted by* $A \triangle B$ *is the set*

$$A \triangle B = (B - A) \cup (A - B).$$

Example
Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 9, 10\}$, and $B = \{3, 5, 7, 9\}$. Then

$A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

$A \cap B = \{9\}$.

$B - A = \{3, 5, 7\}$.

$A - B = \{2, 4, 6, 8, 10\}$.

$A \triangle B = \{2, 3, 4, 5, 6, 7, 8, 10\}$.

$A' = \{1, 3, 5, 7\}$.

Proposition
Let A, B, and C be subsets of a universal set U.

(a) $A \cup A = A$ *(idempotent law for union)*

(b) $A \cap A = A$ *(idempotent law for intersection)*

(c) $A \cup \emptyset = A$

(d) $A \cap \emptyset = \emptyset$

(e) $A \cup U = U$

(f) $A \cap U = A$

(g) $A \cup B = B \cup A$ *(commutative law for union)*

(h) $A \cap B = B \cap A$ *(commutative law for intersection)*

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ *(associative law for union)*

(j) $(A \cap B) \cap C = A \cap (B \cap C)$ *(associative law for intersection)*

(k) $A \subseteq A \cup B$ and $B \subseteq A \cup B$

(l) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

(m) $A'' = A$

(n) $A \cup A' = U$

(o) $A \cap A' = \emptyset$

(p) $\emptyset' = U$

(q) $U' = \emptyset$

(r) $(A \cup B)' = A' \cap B'$ *(De Morgan's law)*

(s) $(A \cap B)' = A' \cup B'$ *(De Morgan's law)*

(t) $A \subseteq B$ *if and only if* $B' \subseteq A'$

(u) $A - B = A \cap B'$

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *(intersection distributes over union)*

(w) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *(union distributes over intersection)*

(x) $A \triangle B = (A \cup B) - (A \cap B)$.

Proof. Be sure that you can prove these properties.

(r) In order to show that $(A \cup B)' = A' \cap B'$, we must show that $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$.

$(A \cup B)' \subseteq A' \cap B'$	$A' \cap B' \subseteq (A \cup B)'$
Let $x \in (A \cup B)'$	Let $x \in A' \cap B'$
Then $x \notin A \cup B$	Then $x \in A'$ and $x \in B'$
$\therefore\ \neg[x \in A \cup B]$	$\therefore\ (x \notin A) \wedge (x \notin B)$
$\therefore\ \neg[(x \in A) \vee (x \in B)]$	$\therefore\ \neg[(x \in A) \vee (x \in B)]$
$\therefore\ (x \notin A) \wedge (x \notin B)$	$\therefore\ \neg[x \in A \cup B]$
$\therefore\ (x \in A') \wedge (x \in B')$	$\therefore\ x \in (A \cup B)'$
$\therefore\ x \in A' \cap B'$	$\therefore\ A' \cap B' \subseteq (A \cup B)'$
$\therefore\ (A \cup B)' \subseteq A' \cap B'$	

Proof of (v): Here we use the fact that if P, Q, and R are propositions, then $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.

For each x,

$$
\begin{aligned}
& x \in A \cap (B \cup C) \\
\Leftrightarrow\ & x \in A \text{ and } x \in B \cup C \\
\Leftrightarrow\ & x \in A \text{ and } (x \in B \text{ or } x \in C) \\
\Leftrightarrow\ & (x \in A) \wedge [(x \in B) \vee (x \in C)] \\
\Leftrightarrow\ & [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)] \\
\Leftrightarrow\ & x \in A \cap B \text{ or } x \in A \cap C \\
\Leftrightarrow\ & x \in (A \cap B) \cup (A \cap C).
\end{aligned}
$$

Indexed families of sets

ı Mathematics we often work with large collections of sets. Instead of naming each of those sets using the venty-six letters of the alphabet, we usually index the sets using some convenient indexing set.

Suppose that I is a set and that to each $i \in I$, there corresponds one and only one subset A_i of a universal t U. Then the collection $\{A_i : i \in I\}$ is called an **indexed family of sets** (or an **indexed collection of ts**). The set I is called an **indexing set** for the collection $\{A_i : i \in I\}$. If $I = \{1, 2, 3, \ldots, n\}$, then e indexed collection of sets $\{A_i : i \in I\}$ is called a finite sequence of sets. If $I = \mathbb{N}^+$, the set of positive ıtural numbers, then the indexed collection $\{A_i : i \in \mathbb{N}^+\}$ is called an infinite sequence of sets.

We can extend the definition of *union* and *intersection* discussed earlier to cover an indexed family of ts.

efinition

et $\{A_i : i \in I\}$ be an indexed family of subsets of the universal set U.

(a) The union of the family $\{A_i : i \in I\}$, denoted by $\bigcup_{i \in I} A_i$, is the set of all those elements of U which belong to at least one of the A_i. That is,

$$\bigcup_{i \in I} A_i = \{x \in U : x \in A_i \text{ for some } i \in I\} = \{x \in U : (\exists i \in I)(x \in A_i)\}.$$

(b) The intersection of the family $\{A_i : i \in I\}$, denoted by $\bigcap_{i \in I} A_i$, is the set of all those elements of U which belong to all the A_i. That is,

$$\bigcap_{i \in I} A_i = \{x \in U : x \in A_i \text{ for each } i \in I\} = \{x \in U : (\forall i \in I)(x \in A_i)\}.$$

roposition

et $\{A_i : i \in I\}$ be an indexed family of subsets of the universal set U and let B be a subset of U. Then

(a) $A_k \subseteq \bigcup_{i \in I} A_i$ for each $k \in I$.

(b) $\bigcap_{i \in I} A_i \subseteq A_k$ for each $k \in I$.

(c) $B \cap \left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} (B \cap A_i)$.

(d) $B \cup \left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} (B \cup A_i)$.

(e) $B - \left(\bigcup_{i \in I} A_i\right) = \bigcap_{i \in I} (B - A_i)$.

(f) $B - \left(\bigcap_{i \in I} A_i\right) = \bigcup_{i \in I} (B - A_i)$.

(g) $$\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i' \qquad \text{(de Morgan's law).}$$

(h) $$\left(\bigcap_{i\in I} A_i\right)' = \bigcup_{i\in I} A_i' \qquad \text{(de Morgan's law).}$$

Proof. Be sure that you can prove these statements. We shall prove (c), (e), and (h).

Proof of (c): We should show that $B \cap \left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I}(B\cap A_i)$ and $\bigcup_{i\in I}(B\cap A_i) \subseteq B \cap \left(\bigcup_{i\in I} A_i\right)$. We shall do this in one fell swoop.

$$\begin{aligned}
x \in B \cap \left(\bigcup_{i\in I} A_i\right) &\Leftrightarrow x \in B \text{ and } x \in \bigcup_{i\in I} A_i \\
&\Leftrightarrow x \in B \text{ and } x \in A_i \text{ for some } i \in I \\
&\Leftrightarrow (\exists i \in I)[(x \in B) \wedge (x \in A_i)] \\
&\Leftrightarrow x \in \bigcup_{i\in I} (B \cap A_i)\,.
\end{aligned}$$

Proof of (e): We use the same technique as applied in (c).

$$\begin{aligned}
x \in B - \left(\bigcup_{i\in I} A_i\right) &\Leftrightarrow x \in B \text{ and } x \notin \bigcup_{i\in I} A_i \\
&\Leftrightarrow x \in B \text{ and } \neg[x \in \bigcup_{i\in I} A_i] \\
&\Leftrightarrow x \in B \text{ and } \neg[(\exists i \in I)(x \in A_i)] \\
&\Leftrightarrow (x \in B) \ \wedge \ [(\forall i \in I)(x \notin A_i)] \\
&\Leftrightarrow (\forall i \in I)[(x \in B) \wedge (x \notin A_i)] \\
&\Leftrightarrow (\forall i \in I)(x \in B - A_i) \\
&\Leftrightarrow x \in \bigcap_{i\in I} (B - A_i)\,.
\end{aligned}$$

Proof of (h):

$$
\begin{aligned}
x \in \left(\bigcap_{i\in I} A_i\right)' \quad &\Leftrightarrow \quad x \notin \bigcap_{i\in I} A_i \\
&\Leftrightarrow \quad \neg[x \in \bigcap_{i\in I} A_i] \\
&\Leftrightarrow \quad \neg[(\forall i \in I)(x \in A_i)] \\
&\Leftrightarrow \quad (\exists i \in I)(x \notin A_i) \\
&\Leftrightarrow \quad (\exists i \in I)(x \in A_i') \\
&\Leftrightarrow \quad x \in \bigcup_{i\in I} A_i'.
\end{aligned}
$$

Functions

Definition
Let X and Y be sets. A **function** *f from X to Y, denoted by $f : X \to Y$, is a rule that assigns to each $x \in X$ a unique element $y \in Y$. We write $y = f(x)$ to denote that f assigns the element $x \in X$ to the element $y \in Y$.*

Definition
Let X and Y be sets. A function $f : X \to Y$ is said to be

(i) **injective** *(or* **one-to-one***) if for for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Symbolically,*

$$(\forall x_1, x_2 \in X)[(f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)].$$

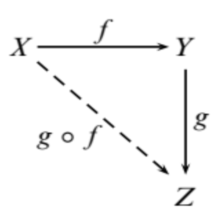
(ii) **surjective** *(or* **onto***) if for each $y \in Y$ there is an $x \in X$ such that $f(x) = y$. Symbolically,we write*

$$(\forall y \in Y)(\exists x \in X)(f(x) = y).$$

(iii) **bijective** *if f is both injective and surjective.*

Definition
Let X, Y, and Z be sets, $f : X \to Y$ and $g : Y \to Z$ be sets. The composition of f and g, denoted by $g \circ f$, is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.

A diagrammatic view of the composition is



Theorem
Let $f : X \to Y$ and $g : Y \to Z$ such that ran$(f) \subseteq$ dom(g). Then

(a) If f and g are onto, then so is the composite function $g \circ f$;

(b) *If f and g are one-to-one, then so is the composite function $g \circ f$;*

(c) *If f and g are bijective, then so is the composite function $g \circ f$;*

(d) *If $g \circ f$ is one-to-one, then so is f;*

(e) *If $g \circ f$ is onto, then so is g;*

(f) *If $g \circ f$ is a bijection, then f is one-to-one and g is onto.*

Proof.

(a) Let $z \in Z$. Since g is onto, there is a $y \in Y$ such that $g(y) = z$. Since f is onto, there is an $x \in X$ such that $f(x) = y$. Therefore $(g \circ f)(x) = g(f(x)) = g(y) = z$. Hence, $g \circ f$ is onto.

(b) Let x_1 and x_2 be in X such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then

$$\begin{aligned} & (g \circ f)(x_1) &=& (g \circ f)(x_2) \\ \Longleftrightarrow\; & g(f(x_1)) &=& g(f(x_2)) \\ \Longleftrightarrow\; & f(x_1) &=& f(x_2) \quad \text{since } g \text{ is one-to-one} \\ \Longleftrightarrow\; & x_1 &=& x_2 \quad \text{since } f \text{ is one-to-one.} \end{aligned}$$

(c) This follows from (a) and (b).

(d) Let x_1 and x_2 be elements of X such that $f(x_1) = f(x_2)$. Then $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = g(f(x_2))$. Since $g \circ f$ is one-to-one, it follows that $x_1 = x_2$. Thus, f is one-to-one.

(e) Let $z \in Z$. We must produce a $y \in Y$ such that $g(y) = z$. Since $g \circ f$ is onto, there is an $x \in X$ such that $(g \circ f)(x) = g(f(x)) = z$. Let $y = f(x)$ $(\in Y)$. Then $g(y) = z$, which proves that g is onto.

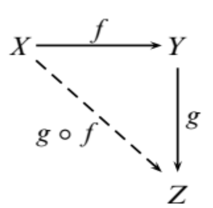
(f) This follows from (d) and (e). ■

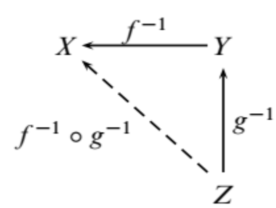
Theorem
Let $f : X \to Y$ be a bijection. Then $f^{-1} : Y \to X$ is a bijection.

Proof. Exercise. ■

Theorem
Let $f : X \to Y$ and $g : Y \to Z$ be bijections. Then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$





Proof. Exercise. ■

Let X and Y be sets, $f : X \to Y$, and $A \subset X$. We denote by $f(A)$ the **image of A in Y**. It is defined by

$$f(A) = \{f(x) \mid x \in A\}.$$

If $B \subset Y$, we denote by $f^{-1}(B)$ the **pre-image** (or **inverse image**) of B in X. It is defined by

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Theorem

Let X and Y be sets, $f : X \to Y$, and $\{A_i : i \in I\}$ an indexed family of subsets of X. Then

(a) $f(\emptyset) = \emptyset$;

(b) $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$;

(c) $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$;

(d) If f is injective, then $f\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} f(A_i)$.

$\prec$Exercise.

Theorem

Let X and Y be sets, $f : X \to Y$, $\{B_i : i \in I\}$ an indexed family of subsets of Y and $D \subset Y$. Then

(a) $f^{-1}(\emptyset) = \emptyset$;

(b) $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$;

(c) $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$;

(d) $f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$.

$\prec$Exercise.

Theorem

Let X and Y be sets and $f : X \to Y$. If $A \subset X$ and $B \subset Y$, then

(a) $A \subseteq f^{-1}(f(A))$;

(b) If f is injective, then $A = f^{-1}(f(A))$;

(c) $f\left(f^{-1}(B)\right) \subseteq B$;

(d) If f is surjective, then $f\left(f^{-1}(B)\right) = B$;

(e) $f(A \cap f^{-1}(B)) = f(A) \cap B$.