Methods of Proof in Mathematics

In Mathematics we make assertions about systems, e.g. number system. The process of establishing the truth of an assertion is called a **proof**. That is, a **proof** in Mathematics is a sequence of logically sound arguments which establish the truth of a statement in question.

Theorem statements are normally in conditional form ($P \Rightarrow Q$) or biconditional form ($P \Leftrightarrow Q$).

Suppose that we wish to establish the truth of the assertion $P \Rightarrow Q$.

Direct Method

In this method of proof, we assume that P is true and proceed through a sequence of logical steps to arrive at the conclusion that Q is also true.

Examples

(a) Show that if m is an even integer and n is an odd integer, then $n + m$ is an odd integer.

Solution: Assume that m is an even integer and m is an odd integer. Then $m = 2k$ and $n = 2\ell + 1$ for some integers k and ℓ. Therefore

$$m + n = 2k + 2\ell + 1 = 2(k + \ell) + 1.$$

Since $k + \ell$ is an integer whenever k and ℓ are integers, we conclude that $m + n$ is an odd integer.

(b) Show that if n is an even integer, then n^2 is also an even integer.

Solution: Assume that n is an even integer. Then $n = 2k$ for some integer k. Now,

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since $2k^2$ is an integer, it follows that n^2 is an even integer.

Contrapositive Method

Associated with the implication $P \Rightarrow Q$ is the logically equivalent statement $\neg Q \Rightarrow \neg P$, the contrapositive of the conditional $P \Rightarrow Q$. Therefore one way of proving the conditional $P \Rightarrow Q$ is to give a direct proof of its contrapositive $\neg Q \Rightarrow \neg P$. The first step in the proof is to write down the negation of the conclusion. Then you show by a series of logical steps that this leads to the negation of the hypothesis of the original conditional statement.

Examples

(a) Show that if n^2 is an even integer, then n is an even integer.

Solution: We will show the contrapositive -if n is an odd integer, then n^2 is an odd integer. To that end, assume that n is an odd integer. Then, $n = 2\ell + 1$ for some integer ℓ. Now,

$$n^2 = (2\ell + 1)^2 = 4\ell^2 + 4\ell + 1 = 2(2\ell^2 + 2\ell) + 1.$$

Since $2\ell^2 + 2\ell$ is an integer, we conclude that n^2 is an odd integer.

(b) Show that if $3n$ is an odd integer, then n is an odd integer.

Solution: We will show the contrapositive - if n is an even integer, then $3n$ is an even integer. To that end, assume that n is an even integer. Then $n = 2k$ for some integer k. Therefore $3n = 3(2k) = 2(3k)$. It follows that $3n$ is an even integer.

Contradiction Method

Proof by contradiction, also called *reductio ad absurdum*, is one of the most powerful methods of proof in Mathematics. It also tends to be harder to understand than the direct or contrapositive methods. Here is how it works: assume that the P is true and Q is false, i.e. assume that the statement $P \wedge \neg Q$ is true. Then show, in a series of logical steps, that this leads to a contradiction, impossibility or absurdity e.g., $R \wedge \neg R$. This will then mean that the assumption that $P \wedge \neg Q$ must have been fallacious, and therefore its negation $\neg(P \wedge \neg Q)$ must be true. Since $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$, it follows that $(P \Rightarrow Q) \equiv \neg(P \wedge \neg Q)$, and hence $P \Rightarrow Q$ must be true.

Before giving some examples, let us define what it means for a number to be rational.

Definition
A real number r is said to be **rational** if there are integers m and n $(n \neq 0)$ such that $r = m/n$. We denote the set of all rational numbers by the letter $\mathbb{Q}$. A real number that is not rational is said to be **irrational**.

Examples
(Proof by Contradiction).

(a) Show that $\sqrt{2}$ is irrational. That is, there do not exist integers p and q such that $\dfrac{p}{q} = \sqrt{2}$.

Solution: Proceeding by contradiction, assume that there are integers p and q such that $\dfrac{p}{q} = \sqrt{2}$. By cancelling any common factors, we may suppose that p and q have no common factors. Then squaring both sides, we have that

$$\frac{p^2}{q^2} = 2 \;\Leftrightarrow\; p^2 = 2q^2.$$

Hence p^2 is even. By Example 1.4.2(a), we have that p is even. Hence we can express p as $p = 2k$ for some integer k. So,

$$2q^2 = p^2 = (2k)^2 = 4k^2 \text{ and, consequently, } q^2 = 2k^2.$$

This means that q^2 is even and so, again by Example 1.4.2(a), we have that q is even. Hence p and q are both even, contradicting the assumption that p and q have no factors in common. Therefore $\sqrt{2}$ is not of the form $\dfrac{p}{q}$ for some integers p and q. That is, $\sqrt{2}$ is irrational.

(b) Show that if $3n$ is an odd integer, then n is an odd integer.

Solution: We will use contradiction: Assume that $3n$ is an odd integer and n is an even integer. Then $3n = 2k + 1$ and $n = 2\ell$ for some integers k and ℓ. Thus

$$2k + 1 = 3n = 3(2\ell) = 2(3\ell).$$

This shows that $3n$ is both odd and even,which is absurd. Hence n is an odd integer.