Logic and Methods of Proof

1.1 Logic

In this course you will be expected to read, understand and construct proofs. The purpose of these notes is to teach you the language of Mathematics. Once you have understood the language of Mathematics, you will be able to communicate your ideas in a clear, coherent and comprehensible manner.

1.1.1 Definition
A **proposition** (or **statement**) is a sentence that is either true or false (not both).

1.1.2 Examples

[1] South Africa was beaten by New Zealand in the 2003 cricket world cup.

[2] February 17, 2003 was on a Tuesday.

[3] $3 + 6 = 11$.

[4] $\sqrt{2}$ is irrational.

1.1.3 Examples
(Examples of non-propositions).

[1] Jonty is handsome.

[2] What is the date?

[3] This statement is true.

There are two types of propositions: **atomic** and **compound** propositions.

- An atomic proposition is a proposition that cannot be divided into smaller propositions.
- A compound proposition is a proposition that has parts that are propositions. Compound propositions are built by using **connectives**.

1.1.4 Examples
(Examples of atomic propositions).

[1] John's leg is broken.

[2] Our universe is infinite.

[3] 2 is a prime number.

[4] There are infinitely many primes.

1.1.5 Examples
(Examples of compound propositions).

[1] Jim and Anne went to the movies.

[2] $3 \leq 7$.

[3] n^2 is odd whenever n is an odd integer.

[4] If a function is differentiable, then it is continuous.

[5] If $f' > 0$, then f is increasing.

[6] If f is increasing and f' exists, then $f' > 0$.

Let us look at some of the most commonly used connectives:

Name	English name	Symbol
Conjunction	and	$\wedge$
Disjunction	or	$\vee$
Implication	If ... then	$\Rightarrow$
Biconditional	if and only if	$\Leftrightarrow$
Negation	not	$\neg$

One has to be careful when using everyday English words in Mathematics as they may not carry the same meaning in Mathematics as they do in everyday non-mathematical usage. One such word is *or*. In everyday parlance, the word *or* means that you have a choice of one thing or the other but **not both** - *exclusive* disjunction. In Mathematics, on the other hand, the word *or* stands for an *inclusive* disjunction, i.e., you have a choice of one thing or the other or both.

We shall use the capital letters $P,\ Q,\ R,\ \ldots$ to denote atomic propositions.

1.1.6 Examples
(Using symbols to represent compound statements).

[1] If Lucille has credit for MAT 1E1 and MAT1E2, then she cannot get credit for MAT101.

Let P stand for the statement "Lucille has credit for MAT 1E1", Q stand for the statement "Lucille has credit for MAT 1E2", and R stand for the statement "Lucille can get credit for MAT 101." Then the above statement can represented symbolically as $(P \wedge Q) \Rightarrow \neg R$.

[2] If Lucille has credit for MAM100W or has credit for MAM105H and MAM106H, then she do MAM200W.

Let P stand for the statement "Lucille has credit for MAM100W", Q stand for the statement "Lucille has credit for MAM105H", R stand for the statement "Lucille has credit for MAM106H", and S stand for the statement "Lucille can do MAM200W." Then the above statement can represented symbolically as $[P \vee (Q \wedge R)] \Rightarrow S$.

[3] Either you pay your rent or I will kick you out of the apartment.

Let P stand for the statement "You pay your rent", and Q stand for the statement "I will kick you out of the apartment." Then the above statement can represented symbolically as $P \vee Q$.

[4] Joe will leave home and not come back again.

Let P stand for the statement "Joe will leave home", and Q stand for the statement "Joe will come back again." Then the above statement can represented symbolically as $P \wedge \neg Q$.

[5] The lights are on if and only if either John or Mary is at home.

Let P stand for the statement "The lights are on", Q stand for the statement "John is at home", and S stand for the statement "Mary is at home." Then the above statement can represented symbolically as $P \Leftrightarrow (Q \vee S)$.

A **truth table** is a convenient device to specify all of the possible truth values of a given atomic or compound proposition. We use truth tables to determine the truth or falsity of a compound proposition based on the truth or falsity of its constituent atomic propositions.

When we evaluate the truth or falsity of a statement, we assign to it one of the labels T for "true" and F for "false". We also use 1 for "true" and 0 for "false".

Let us construct truth tables for the above connectives.

[1] **Conjunction**: Let P and Q be two propositions. The proposition $P \wedge Q$ is called the conjunction of P and Q. The proposition $P \wedge Q$ is true if and only if both atomic propositions P **and** Q are true. In other words, if either or both atomic propositions P and Q are false, then the conjunction $P \wedge Q$ is false.

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

1.1.7 Examples

P: Cape Town is in the Western Cape and $\sqrt{3}$ is irrational.
Q: $\sqrt{5} < 3$ and $f(x) = |x|$ is differentiable at $x = 0$.
R: Harare is the capital of Botswana and $f(x) = \cos x$ is continuous on $\mathbb{R}$.
S: $-2 < -10$ and 8 is an odd number.

Only P is true; all the others are false.

[2] **Disjunction**: Let P and Q be two propositions. The proposition $P \vee Q$ is called the disjunction of P and Q. The proposition $P \vee Q$ is true if and only if at least one of the atomic propositions P or Q is true.

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

It is clear from this truth table that the proposition $P \vee Q$ will be false only when **both** P and Q are false.

1.1.8 Examples

(a) $\pi > 2$ or π is an irrational number.
(b) $\pi > 2$ or π is a rational number.
(c) $\pi < 2$ or π is an irrational number.
(d) $\pi < 2$ or π is a rational number.

All these propositions, except (d), are true.

[3] **Implication**: Let P and Q be two propositions. The proposition $P \Rightarrow Q$ is referred to as a *conditional* proposition. It simply means that P implies Q. In the statement $P \Rightarrow Q$, P is called the *hypothesis* (or *antecedent or condition*) and Q is called the *conclusion* (or *consequent*).

There are various ways of stating that P implies Q:

- If P, then Q.
- Q if P.
- P is sufficient for Q.
- Q is necessary for P.
- P only if Q.
- Q whenever P.

P	Q	$P \Rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

It is clear from this truth table that the proposition $P \Rightarrow Q$ will be false only when P is true and Q is false.

In order to have some appreciation of why the above truth table is reasonable, consider the following: If you pass MAM200W exam, I will buy you a cell-phone.

Let P: You pass MAM200W exam.

Let Q: I will buy you a cell-phone.

At the end of MAM200W exam, there are various scenarios that may arise.

(a) You have passed MAM200W exam and then I buy you a cell-phone. You will be happy and feel that I was telling the truth . Therefore $P \Rightarrow Q$ is true.

(b) You have passed MAM200W exam but I refuse to buy you a cell-phone. You will feel cheated and lied to. Therefore $P \Rightarrow Q$ is false.

(c) You have failed MAM200W, but I still buy you a cell-phone. You are unlikely to question that, are you? We did not cover this contingency in my conditional statement.

(d) You have failed MAM200W and, consequently I do not buy you a cell-phone. You will not feel that I have been unfair to you and that I have not kept my promise.

1.1.9 Examples

(a) If $\pi > 2$, then π is an irrational number.

(b) If $\pi > 2$, then π is a rational number.

(c) If $\pi < 2$, then π is an irrational number.

(d) If $\pi < 2$, then π is a rational number.

All these propositions, except (b), are true.

1.1.10 Definition

Let P and Q be propositions. The **converse** of the proposition $P \Rightarrow Q$ is the proposition $Q \Rightarrow P$.

1.1.11 Examples

(Examples of converse statements).

(a) If it is cold, then the lake is frozen.

Converse: If the lake is frozen, then it is cold.

(b) Johny is happy if he is healthy.

Converse: If Johny happy, then he is healthy.

(c) If it rains, Zinzi does not take a walk.

Converse: If Zinzi does not take a walk, then it rains.

The truth table of a proposition and its converse:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
1	1	1	1
1	0	0	1
0	1	1	0
0	0	1	1

Note that the truth tables of $P \Rightarrow Q$ and $Q \Rightarrow P$ are not the same.

Consider the following conditional proposition and its converse:

Proposition: If $\pi > 2$, then $\sqrt{3}$ is rational.

Converse: If $\sqrt{3}$ is rational, then $\pi > 2$.

In this example the conditional statement is false whereas its converse is true. Hence this conditional proposition and its converse are not equivalent.

Consider the following conditional proposition and its converse:

Proposition: If $\pi > 2$, then $\sqrt{3}$ is irrational.

Converse: If $\sqrt{3}$ is irrational, then $\pi > 2$.

Here both that conditional proposition and its converse are true. If, in this example, we let P stand for the proposition "$\pi > 2$" and Q for "$\sqrt{3}$ is irrational", then we have that both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true.

[4] **Biconditional Proposition**: Let P and Q be propositions. The proposition $P \Leftrightarrow Q$ is referred to as a *biconditional* proposition. It simply means that $P \Rightarrow Q$ and $Q \Rightarrow P$. It is called a "biconditional proposition" because it represents two conditional propositions.

There are various ways of stating the proposition $P \Leftrightarrow Q$:

- P if and only if Q (also written as P iff Q).

- P implies Q and Q implies P.
- P is necessary and sufficient for Q.
- Q is necessary and sufficient for P.
- P is equivalent to Q.

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Note that the statement $P \Leftrightarrow Q$ is true precisely in the cases where P and Q are both true or P and Q are both false.

[5] **Negation**: Let P be a proposition. The proposition $\neg P$, meaning "not P", is used to denote the negation of P. If P is true, then $\neg P$ is false and vice versa.

P	$\neg P$
1	0
0	1

Let us construct a few more truth tables.

1.1.12 Examples

[1] Let P and Q be propositions. Construct a truth table for the proposition $(P \wedge Q) \Rightarrow (P \vee Q)$.

Solution:

P	Q	$P \wedge Q$	$P \vee Q$	$P \wedge Q \Rightarrow P \vee Q$
1	1	1	1	1
1	0	0	1	1
0	1	0	1	1
0	0	0	0	1

[2] Let P, Q and R be propositions. Construct the truth table for the proposition $\neg(P \wedge Q) \vee R$.

Solution:

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg(P \wedge Q) \vee R$
1	1	1	1	0	1
1	1	0	1	0	0
1	0	1	0	1	1
0	1	1	0	1	1
1	0	0	0	1	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	1	1

1.2 Tautologies, Contradictions and Equivalences

Some compound propositions are always true while others are always false.

1.2.1 Definition
A compound proposition is a **tautology** if it is always true regardless of the truth values of its atomic propositions. If, on the other hand, a compound proposition is always false regardless of its atomic propositions, we say that such a proposition is a **contradiction**.

1.2.2 Example
The statement $P \vee \neg P$ is always true while the statement $P \wedge \neg P$ is always false.

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
1	0	1	0
0	1	1	0

1.2.3 Remark
In a truth table, if a proposition is a tautology, then every line in its column will have 1 as its entry; if a proposition is a contradiction, every line in its column will have 0 as its entry.

1.2.4 Definition
Let P and Q be propositions. The **contrapositive** of the proposition $P \Rightarrow Q$ is the proposition $\neg Q \Rightarrow \neg P$.

1.2.5 Examples
(Examples of contrapositive statements).

[1] If it is cold, then the lake is frozen.

Contrapositive: If the lake is not frozen, then it is not cold.

[2] If Johny is healthy, then he is happy.

Contrapositive: If Johny not happy, then he is not healthy.

[3] If it rains, Zinzi does not take a walk.

Contrapositive: If Zinzi takes a walk, then it does not rain.

DO NOT CONFUSE THE CONTRAPOSITIVE AND THE CONVERSE. Here is the difference:

Converse: The hypothesis of a converse statement is the conclusion of the conditional statement and the conclusion of the converse statement is the hypothesis of the conditional statement.

Contrapositive: The hypothesis of a contrapositive statement is the *negation* of conclusion of the conditional statement and the conclusion of the contrapositive statement is the *negation* of hypothesis of the conditional statement.

1.2.6 Examples

[1] If Bronwyne lives in Cape Town, then she lives of South Africa.

Converse: If Bronwyne lives in South Africa, then she lives in Cape Town.

Contrapositive: If Bronwyne does not live in South Africa, then she does not live Cape Town.

[2] If it is morning, then the sun is in the east.

Converse: If the sun is in the east, then it is morning.

Contrapositive: If the sun is not in the east, then it is not morning.

1.2.7 Definition
Two propositions P and Q are said to be **logically equivalent**, written as $P \equiv Q$, if $P \Leftrightarrow Q$ is a tautology. Logically equivalent statements have the same truth values.

1.2.8 Remark
When we write "$P \equiv Q$", we basically say that proposition P *means the same* as proposition Q.

Here is an important example: $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$. That is, the conditional and its contrapositive say the same thing.

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
1	1	1	0	0	1	1
1	0	0	0	1	0	1
0	1	1	1	0	1	1
0	0	1	1	1	1	1

1.2.9 Theorem
Let P, Q and R be propositions. Then

(a) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

(b) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

(c) $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

(d) $P \Rightarrow Q \equiv \neg P \vee Q$

(e) $\neg(\neg P) \equiv P$

(f) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

(g) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(h) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

(i) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

Proof. (a) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$
1	1	1	0	0	0	0	1
1	0	0	1	0	1	1	1
0	1	0	1	1	0	1	1
0	0	0	1	1	1	1	1

(c) $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$	$\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$
1	1	1	0	0	0	1
1	0	0	1	1	1	1
0	1	1	0	0	0	1
0	0	1	0	1	0	1

Try to convince yourself that all the other statements are valid.

Let us analyze the following argument: If girls are blonde, they are popular with boys. Ugly girls are unpopular with boys. Intellectual girls are ugly. Therefore blonde girls are not intellectual.

Is this argument valid?

Solution: Let us use letters and connectives to represent the above statement.

P: Girls are blonde.

Q: Girls are popular with boys.

R: Girls are ugly.

S: Girls are intellectual.

We can represent the above argument as follows:

$$P \Rightarrow Q, \ \ R \Rightarrow \neg Q, \ \ S \Rightarrow R.$$

Since $S \Rightarrow R$ and $R \Rightarrow \neg Q$, we can conclude that $S \Rightarrow \neg Q$.

Since $P \Rightarrow Q$, we have, by contrapositive, that $\neg Q \Rightarrow \neg P$. Hence, $S \Rightarrow \neg P$.

Again, by contrapositive, $P \Rightarrow \neg S$, which says that "Blonde girls are not intellectual." Therefore the argument is valid.

1.3 Open Sentences and Quantifiers

In mathematics, one frequently comes across sentences that involve a variable. For example, $x^2+2x-3=0$ is one such. The truth or falsity of this statement depends on the value you assign for the variable x. For example, if $x = 1$, then this sentence is true, whereas if $x = -1$, this sentence is false.

1.3.1 Definition

An **open sentence**(also called a **predicate**) is a sentence that contains variables and whose truth or falsity depends on the values assigned for the variables. We represent an open sentence by a capital letter followed by the variable(s) in parenthesis, e.g., $P(x)$, $Q(x, y)$ etc.

1.3.2 Examples

(Open statements).

[1] $x + 4 = -9$

[2] $x < y$.

[3] She is the queen of jazz.

[4] It has four legs.

1.3.3 Definition

The collection of all allowable values for the variable in an open sentence is called the **universe of discourse**.

Let $P(x)$ be an open sentence containing a free variable x. We want to *quantify* the number of x for which $P(x)$ is true. In particular, we want to say that $P(x)$ is true for at least one x or for all x in the universe of discourse.

Universal Quantifier ($\forall$): To say that $P(x)$ is true for all x in the universe of discourse, we write $(\forall x)P(x)$. Think of the symbol $\forall$ as an inverted A (representing *all*). $\forall$ is called the **universal quantifier**.

$$\forall \text{ means } \begin{cases} \text{all} \\ \text{for all} \\ \text{for every} \\ \text{for each} \end{cases}$$

Existential Quantifier ($\exists$): To say that there is (at least one) x in the universe of discourse for which $P(x)$ is true, we write $(\exists x)P(x)$. Think of the symbol $\exists$ as the backwards capital E (representing *exists*). $\exists$ is called the **existential quantifier**.

$$\exists \text{ means } \begin{cases} \text{there is} \\ \text{there exists} \\ \text{for some} \end{cases}$$

Symbolic Statement	*Translation*
$(\forall x)P(x)$	For all x, $P(x)$ is true
$(\forall x)(\neg P(x))$	For all x, $P(x)$ is false (There is no x for which $P(x)$ is true)
$(\exists x)P(x)$	There exists an x for which $P(x)$ is true
$(\exists x)(\neg P(x))$	There is an x for which $P(x)$ is false
$(\forall x)(\forall y)P(x, y)$	$P(x, y)$ is true for all pairs (x, y)
$(\exists x)(\exists y)P(x, y)$	There is a pair (x, y) for which $P(x, y)$ is true
$(\forall x)(\exists y)P(x, y)$	For each x, there is a y for which $P(x, y)$ is true
$(\exists x)(\forall y)P(x, y)$	There is an x for which $P(x, y)$ is true for every y

1.3.4 Remark
Quantifying an open sentence makes it a proposition.

1.3.5 Examples
Write the following statements using quantifiers.

(a) For each real number $x > 0$, $x^2 + x - 6 = 0$.

Solution: $(\forall x > 0)(x^2 + x - 6 = 0)$.

(b) There is a real number $x > 0$ such that $x^2 + x - 6 = 0$.

Solution: $(\exists x > 0)(x^2 + x - 6 = 0)$.

(c) The square of any real number is nonnegative.

Solution: $(\forall x \in \mathbb{R})(x^2 \geq 0)$.

(d) For each integer x there is an integer y such that $x + y = -1$.

Solution: $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x + y = -1)$.

(e) There is an integer x such that for each integer y, $x + y = -1$.

Solution: $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x + y = -1)$.

Do examples (d) and (e) convey the same message?
The answer is NO. Statement (d) is true: given any integer x, there is an integer, namely, $y = -1 - x$, such that $x + y = -1$. Statement (e) is false.
ORDER DOES MATTERS AFTER ALL!

1.3.6 Remark
In the statement $(\forall x)(\exists y)P(x, y)$, the choice of y is allowed to depend on x - the y that works for one x need not work for another x. On the other hand, in the statement $(\exists y)(\forall x)P(x, y)$, the y must work for all x, i.e., y is *independent* of x.

1.3.7 Examples
Translate the following into English.

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x = y^2)$.

Solution: Every real number is a perfect square.

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 0)$.

Solution: Every real number has an additive inverse.

Negation of Quantifiers

Symbolic Statement	*Translation*
$\neg[(\forall x)P(x)] \equiv (\exists x)(\neg P(x))$	There is an x for which $P(x)$ is false
$\neg[(\exists x)P(x)] \equiv (\forall x)(\neg P(x))$	$P(x)$ is false for every x
$\neg[(\forall x)(\exists y)P(x, y)] \equiv (\exists x)(\forall y)(\neg P(x, y))$	There is an x for which $P(x, y)$ is false for every y
$\neg[(\exists y)(\forall x)P(x, y)] \equiv (\forall y)(\exists x)(\neg P(x, y))$	For each y there is an x for which $P(x, y)$ is false
$\neg[(\forall x)(\forall y)P(x, y)] \equiv (\exists x)(\exists y)(\neg P(x, y))$	There is a pair (x, y) for which $P(x, y)$ is false
$\neg[(\exists x)(\exists y)P(x, y)] \equiv (\forall x)(\forall y)(\neg P(x, y))$	$P(x, y)$ is false for every pair (x, y)

1.3.8 Remark
To negate a statement that involves the quantifiers $\forall$ and $\exists$, change each $\forall$ to $\exists$, change each $\exists$ to $\forall$, and negate the open sentence (predicate).

1.3.9 Examples

[1] All birds can fly.

Negation: There is (at least one) bird that cannot fly.

1.3.10 Exercise
Write the following statements using quantifiers.

(a) A function f has *limit* L at a point a, denoted by $\lim_{x\to a} f(x) = L$, if and only if given any $\epsilon > 0$, there is a $\delta > 0$ such that for each x in the domain of f, we have that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Ans. $\left(\lim_{x\to a} f(x) = L\right) \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \text{dom}(f))[0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon]$.

(b) Write down the negation of (a).

Ans. $\left(\lim_{x\to a} f(x) \neq L\right) \Leftrightarrow (\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \text{dom}(f))[(0 < |x-a| < \delta) \wedge (|f(x)-L| \geq \epsilon)]$.

(c) A function f is *continuous* at $x = a$ if and only if given any $\epsilon > 0$, there is a $\delta > 0$ such that for each x in the domain of f, we have that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Ans. (f is continuous at $x = a$) $\Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \text{dom}(f))[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon]$.

(d) Write down the negation of (c).

(f is discontinuous at $x = a$) $\Leftrightarrow (\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \text{dom}(f))[(|x - a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon)]$.

Overgeneralization and Counterexample

Overgeneralization occurs when a pattern searcher discovers a pattern among finitely many cases and then claim that the pattern holds in general (when in fact it doesn't).

To disprove a general (universally quantified) statement such as $(\forall x)P(x)$, we must exhibit one x for which $P(x)$ is false. That is, $(\exists x)\neg P(x)$. This particular x is called a **counterexample** to the statement that $(\forall x)P(x)$ is true.

1.3.11 Examples

[1] *Statement*: $(\forall x \in \mathbb{R})(x < x^2)$.

The above statement is false. $x = \frac{1}{2}$ is a counterexample since $\frac{1}{2} \in \mathbb{R}$ but $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$.

[2] For all real numbers x and y, $|x + y| = |x| + |y|$.

This is false. Counterexample: take $x = 1$ and $y = -1$. Then $0 = |0| = |-1 + 1| \neq |-1| + |1| = 2$.

The statement $(\forall x)[P(x) \Rightarrow Q(x)]$ occurs frequently in Mathematics. Recall that

$$\neg[(\forall x)(P(x) \Rightarrow Q(x))] \equiv (\exists x)[P(x) \wedge \neg Q(x)].$$

Therefore, to show that the implication $P(x) \Rightarrow Q(x)$ is false, all that you have to do is produce ONE x for which $P(x)$ is true but $Q(x)$ is false.

1.3.12 Examples

[1] If a function f is continuous, then it is differentiable.

This statement is false since $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

[2] For all real numbers $a,\ b,$ and c, if $ac = bc$, then $a = b$.

This statement is false. Take $a = 1,\ b = 7,$ and $c = 0$. Then $0 = ac = bc = 0$, but $1 = a \neq b = 7$.

[3] For all prime numbers p, $2p + 1$ is prime.

While this statement is true for $p = 2, 3, 5$, it is false for $p = 7$ since $2 \times 7 + 1 = 15$ which is not prime. So $p = 7$ is a counterexample to the given statement.