Additivity of the Integral

We will show that the integral of a function over an interval $[a,b]$ is the sum of the integrals of the function over $[a,c]$ and $[c,b]$ for any point $c \in (a,b)$.

If

$$F : S \to U$$

is a function, and T is a non-empty subset of S, then

$$F|T$$

denotes the *restriction* of F to the smaller domain T, i.e. $F|T$ is the function whose domain is T and whse values are given through F:

$$(F|T)(x) = F(x) \qquad \text{for all } x \in T \tag{3.79}$$

Theorem 27 *Let a, c, b be real numbers with $a < c < b$, and consider any function*

$$f : [a,b] \to \mathbb{R}$$

which is intergable over $[a,c]$ and over $[c,b]$. Then $f \in \mathcal{R}[a,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f \tag{3.80}$$

Proof. Let $\varepsilon > 0$. By Darboux, there is a partition Y of $[a,c]$ and a partition Z of $[c,b]$ such that

$$U(f|[a,c],Y) - L(f|[a,c],Y) < \varepsilon/2 \tag{3.81}$$

and

$$U(f|[c,b],Z) - L(f|[c,b],Z) < \varepsilon/2 \tag{3.82}$$

Now put together the points of Y and Z. This yields a partition X of the combined interval $[a,b]$. Then,

$$U(f,X) = U(f|[a,c],Y) + U(f|[c,b],Z) \geq \int_a^c f + \int_c^b f \tag{3.83}$$

and

$$L(f,X) = L(f|[a,c],Y) + L(f|[c,b],Z) \le \int_a^c f + \int_c^b f \tag{3.84}$$

Consequently,

$$\begin{aligned} U(f,X) - L(f,X) &= U(f|[a,c],Y) + U(f|[c,b],Z) - [L(f|[a,c],Y) + L(f|[c,b],Z)] \\ &= U(f|[a,c],Y) - L(f|[a,c],Y) + U(f|[c,b],Z) - L(f|[c,b],Z) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Therefore, by Darboux, $f \in \mathcal{R}[a,b]$.

Now from (3.83) and (3.84) it follows that the sum

$$\int_a^c f + \int_c^b f$$

lies between $L(f,X)$ and $U(f,X)$, and, of course, so does $\int_a^b f$. Therefore,

$$\textstyle\int_a^c f + \int_c^b f \text{ and } \int_a^b f \text{ differ by less than } \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_a^b f = \int_a^c f + \int_c^b f. \quad \boxed{\text{QED}}$$

Now we prove that if a function is ingtergable on an interval then it is integrable on any sub-interval:

Theorem 28 *If $[a,b] \subset \mathbb{R}$, with $a < b$, and if $s,t \in [a,b]$ with $s < t$ then for any function $f \in \mathcal{R}[a,b]$ we have $f|[s,t] \in \mathcal{R}[s,t]$.*

Proof This is, as always, a matter of applying Darboux using one of the properties of Var. Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a,b]$ there is a partition Y of $[a,b]$ such that

$$U(f,Y) - L(f,Y) < \varepsilon.$$

Add to Y the points s and t, in case they are not in Y, to obtain a partition

$$Z = (z_0, \ldots, z_M)$$

of $[a,b]$. We know that this lowers the upper sum and raises the lower sum and so

$$U(f,Z)-L(f,Z)<\varepsilon.$$

Now let X be the partition of $[s,t]$ obtained by taking the points of Z which are in $[s,t]$. Then

$$U(f,Z)-L(f,Z)=U(f|[s,t],X)-L(f|[s,t],X)+\sum_{j\in J}\left[M_j(f)-m_j(f)\right]\Delta x_j \tag{3.85}$$

where J consists of those $j\in\{1,\ldots,M\}$ for which the interval $[z_{j-1},z_j]$ is not contained in $[s,t]$. Therefore,

$$U(f,Z)-L(f,Z)\le U(f|[s,t],X)-L(f|[s,t],X)<\varepsilon,$$

and we are done. QED

Monotone Functions are Riemann Integrable

We have made the remark that not every Riemann integrable function is continuous. We will now prove that every monotone function is Riemann integrable on any compact interval.

Let

$$f:[a,b]\to\mathbb{R}$$

be a monotone function, where $[a,b]\subset\mathbb{R}$ and $a<b$. Let

$$X=(x_0,x_1,\ldots,x_N)$$

be any partition of $[a,b]$. Then

$$\begin{aligned} U(f,X)-L(f,X) &= \sum_{j=1}^{N}\mathrm{Var}_j(f)\Delta x_j \\ &\le \left(\sum_{j=1}^{N}\mathrm{Var}_j(f)\right)\|X\| \end{aligned}$$

where

$$\|X\|=\max_{1\le j\le N}\Delta x_j,$$

is the maximum width of the intervals making up the partition.

Suppose for convenience that f is monotone non-decreasing, i.e.

$$f(x) \leq f(y) \text{ for all } x, y \in [a,b] \text{ with } x \leq y$$

Then, by the property of Var for monotone functions given in (3.41) we have

$$\mathrm{Var}_j(f) = f(x_j) - f(x_{j-1}) \qquad \text{for all } j \in \{1, \ldots, N\}$$

Therefore, the sum of the variations over all the intervals is simply the variation over the full interval:

$$\sum_{j=1}^{N} \mathrm{Var}_j(f) = \mathrm{Var}(f, [a,b]) \tag{3.86}$$

The same conclusion holds even if f is monotone non-increasing, i.e. if

$$f(x) \leq f(y) \text{ for all } x, y \in [a,b] \text{ with } x \geq y$$

Thus, in either case, we have

$$U(f,X) - L(f,X) = \mathrm{Var}(f, [a,b])\|X\| \tag{3.87}$$

To make this less than any chosen $\varepsilon > 0$ all we have to do is take a partition X with all the interval sizes less than

$$\varepsilon / [1 + \mathrm{Var}(f, [a,b])].$$

For example, we could divide $[a,b]$ into N equal pieces, with N chosen large enough that

$$\frac{b-a}{N} < \frac{\varepsilon}{1 + \mathrm{Var}(f, [a,b])}.$$

Thus we have proved:

Theorem 29 *If f is a monotone function on a compact interval $[a,b] \subset \mathbb{R}$, with $a < b$, then $f \in \mathcal{R}[a,b]$.*

Riemann Sums and the Riemann Integral

We have used the Archimedean strategy of capturing the value of the integral between upper sums and lower sums. This approach led to a smooth development of the central results of the theory. However, this method is not the most intuitive in understanding concepts such as arc length. It is therefore useful to understand the Riemann integral in terms of Riemann sums as well. This method is also amenable to generalizations such as the notion of line integrals. Furthermore, the Riemann sum approach motivates the construction of more advanced notions such as the stochastic integral of Itô.

So consider a function

$$f : [a,b] \to \mathbb{R}$$

where $[a,b] \subset \mathbb{R}$ with $a < b$. Let

$$X = (x_0, ..., x_N)$$

be any partition of $[a,b]$. Recall that the norm or width of X is the length of the largest interval

$$\|X\| = \max_j \Delta x_j$$

Now consider any sequence

$$X^* = (x_1^*, ..., x_N^*)$$

subordinate to X, i.e. with $x_j^* \in [x_{j-1}, x_j]$ for each j. We denote this by

$$X^* < X$$

Recall the Riemann sum

$$S(f, X, X^*) = \sum_{j=1}^{N} f(x_j^*)\Delta x_j \tag{3.88}$$

Now

$$m_j \le f(x_j^*) \le M_j,$$

for each j, and so

$$L(f,X) \le S(f,X,X^*) \le U(f,X) \tag{3.89}$$

If f is integrable, with $I = \int_a^b$, then for any $\varepsilon > 0$ we can choose partition X such that

$$U(f,X) - L(f,X) < \varepsilon.$$

Since both I and $S(f,X,X^*)$ are squeezed in between the upper and lower sums, it follows that

$$|S(f,X,X^*) - I| < \varepsilon \tag{3.90}$$

The following result is is often used to define the Riemann integral in alternative approaches to the theory.

Theorem 30 *A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if there is a real number I such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|S(f,X,X^*) - I| < \varepsilon \tag{3.91}$$

for every partition X of norm $< \delta$ and every $X^* < X$. In this case,

$$I = \int_a^b f$$

Proof. Suppose the given condition holds. Then there is a real number I such that for any $\varepsilon > 0$ there is a $\delta > 0$ for which the condition

$$|S(f,X,X^*) - I| < \varepsilon/4 \tag{3.92}$$

holds for all partitions X of $[a,b]$ of width $< \delta$ and all $X^* < X$. Thus,

$$I - \varepsilon/4 < \sum_{j=1}^{N} f(x_j^*)\Delta x_j < I + \varepsilon/4$$

for *every* sequence $X^* < X$. Then, taking the supremum over all possible x_1^* in the first interval $[x_0, x_1]$, we see that

$$M_1 \Delta x_1 + \sum_{j=2}^{N} f(x_j^*)\Delta x_j \leq I + \varepsilon/4$$

and, taking the infimum over all possible x_1^* in $[x_0, x_1]$, we have

$$I - \varepsilon/4 \leq m_1 \Delta x_1 + \sum_{j=2}^{N} f(x_j^*)\Delta x_j$$

Carrying this successively for $j = 2, 3, \ldots, N$, we conclude that

$$I - \varepsilon/4 \leq L(f,X) \leq U(f,X) \leq I + \varepsilon/4$$

Consequently,

$$U(f,X) - L(f,X) \leq \varepsilon/2 < \varepsilon$$

and so, by Darboux, $f \in \mathcal{R}[a,b]$. Moreover, since both I and the integral $\int_a^b f$ are trapped in the interval $[L(f,X), U(f,X)]$ whose width is ε it follows that I and $\int_a^b f$ differ by less than ε. But, ε is any positive real number. Thus,

$$I = \int_a^b f.$$

For the converse, suppose $f \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$. By Darboux, there is a partition

$$Y = (y_0, \ldots, y_N)$$

of $[a,b]$ such that $U(f,Y)$ and $L(f,Y)$ differ by less than ε:

$$U(f,Y) - L(f,Y) < \varepsilon/2 \tag{3.93}$$

Now let

$$\delta = \frac{\varepsilon/2}{1 + 2N\|f\|_{\sup}}. \tag{3.94}$$

(Where we get this from will be clear later.) Consider any partition

$$X = (x_0, \ldots, x_T)$$

of $[a,b]$ of norm less than δ:

$$\|X\| < \delta$$

We will compare $U - L$ for X with that for Y and conclude that $U - L$ for X is indeed less than ε. Using our standard trick, let Z be the partition of $[a,b]$ obtained by combining X and Y. Then

$$U(f,Z) - L(f,Z) \leq U(f,Y) - L(f,Y) < \varepsilon/2 \tag{3.95}$$

We also know by Lemma 1 that $U - L$ for Z differs from that for X by at most $2N\|f\|_{\sup}\|X\|$, because at most N points were added to X to obtain Z. Thus, the most $U - L$ for X could be is

$$U(f,Z) - L(f,Z) + 2N\|f\|_{\sup}\|X\| \tag{3.96}$$

Thus,

$$U(f,X) - L(f,X) < \varepsilon/2 + 2N\|f\|_{\sup}\delta \tag{3.97}$$

In (3.94) we chose δ just so this right side now works out to ε:

$$U(f,X) - L(f,X) < \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{3.98}$$

Now take any $X^* < X$. Then the Riemann sum $S(f,X,X^*)$ is sandwiched between the lower sum $L(f,X)$ and the upper sum $U(f,X)$, and so it the integral $\int_a^b f$. Therefore, $S(f,X,X^*)$ and $\int_a^b f$ both lie in the interval

$$[L(f,X), U(f,X)]$$

whose width is $< \varepsilon$. Thus,

$$\left| S(f,X,X^*) - \int_a^b f \right| < \varepsilon \tag{3.99}$$

We have shown that for any $\varepsilon > 0$ there is a $\delta > 0$ such that (3.99) holds for any partition X of width $< \delta$ and any $X^* < X$. $\boxed{\text{QED}}$