

Variation of a Function

For any non-empty subset $S \subset \mathbb{R}$ the *diameter* of S is

$$\text{diam}(S) = \sup\{a - b : a, b \in S\} \quad (3.31)$$

It is easy to believe and not hard to prove that the diameter of S is the difference between $\sup S$ and $\inf S$:

$$\text{diam}(S) = \sup S - \inf S \quad (3.32)$$

Now consider a function f on an interval $[s, t]$. It will be useful to have a measure of the fluctuation of f over this interval.

The simplest such measure is given by the diameter of the range of f :

$$\text{Var}(f) = \text{diam}(\text{Range of } f) \quad (3.33)$$

More explicitly, the *variation* of f on the interval $[s, t]$ is:

$$\text{Var}(f, [s, t]) = \sup_{x, y \in [s, t]} (f(x) - f(y)) \quad (3.34)$$

It is also equal to

$$\text{Var}(f, [s, t]) = M(f, [s, t]) - m(f, [s, t]) \quad (3.35)$$

Sometimes we may drop $[s, t]$ and just write $\text{Var}(f)$:

$$\text{Var}(f) = M(f) - m(f) \quad (3.36)$$

We record the following algebraic facts about the variation of functions, which will be very useful in proving corresponding facts about Riemann integration.

Lemma 2 For any functions f and g on an interval $[a, b] \subset \mathbb{R}$ with $a < b$, we have:

- (i) $\text{Var}(f) \geq 0$;
- (ii) $\text{Var}(k) = 0$ if and only if k is constant;
- (iii) the variation scales like length, i.e. the variation of a constant times a function is the absolute value of the function time the variation in the function:

$$\text{Var}(kf) = |k| \text{Var}(f) \text{ for any } k \in \mathbb{R}.$$

- (iv) the variation satisfies the triangle inequality:

$$\text{Var}(f + g) \leq \text{Var}(f) + \text{Var}(g) \quad (3.37)$$

- (v) the variation of the product of two functions is bounded above by the sum of their variations, weighted by their sup-norms:

$$\text{Var}(fg) \leq \|f\|_{\text{sup}} \text{Var}(g) + \|g\|_{\text{sup}} \text{Var}(f) \quad (3.38)$$

- (vi) the variations in f and in g differ by at most the variation in $|f - g|$:

$$\left| \text{Var}(f) - \text{Var}(g) \right| \leq \text{Var}(f - g) \quad (3.39)$$

- (vii) if f is not equal to zero anywhere then

$$\text{Var}(1/f) \leq M(|f|^{-1})^2 \text{Var}(f) \quad (3.40)$$

- (viii) the variation of a function increases monotonically with the interval of variation:

$$\text{Var}(f, [s, t]) \leq \text{Var}(f, [a, b]) \text{ if } [s, t] \subset [a, b] \quad (3.41)$$

- (ix) The variation in the absolute value of f is bounded by the variation in f :

$$\text{Var}(|f|) \leq \text{Var}(f) \quad (3.42)$$

- (x) The variation of f is bounded above by twice $\|f\|_{\text{sup}}$:

$$\text{Var}(f) \leq 2\|f\|_{\text{sup}} \quad (3.43)$$

- (xi) If $s, t, u \in [a, b]$ with $s < t < u$ then

$$\text{Var}(f, [s, u]) \leq \text{Var}(f, [s, t]) + \text{Var}(f, [t, u]) \quad (3.44)$$

- (xii) If f is monotone on an interval $[s, t]$ then

$$\text{Var}(f, [s, t]) = \begin{cases} f(t) - f(s) & \text{if } f(t) \geq f(s) \\ f(s) - f(t) & \text{if } f(t) \leq f(s) \end{cases} \quad (3.45)$$

We also record the following result on variations and partitions for continuous functions:

Lemma 3 *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, where $[a, b] \subset \mathbb{R}$ and $a < b$, then for any $\varepsilon > 0$ there is a partition $X = (x_0, \dots, x_N)$ of $[a, b]$ such that*

$$\text{Var}(f, [x_{j-1}, x_j]) < \varepsilon$$

for every $j \in \{1, \dots, N\}$.

Proof. Since f is continuous on the compact set $[a, b]$ it is *uniformly* continuous. So for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon/2$$

whenever $x, x' \in [a, b]$ with $|x - x'| < \delta$. Take any partition $X = (x_0, \dots, x_N)$ with all the intervals having length less than δ . Then, for each $j \in \{1, \dots, N\}$,

$$f(x) - f(x') < \varepsilon/2$$

for every $x, x' \in [x_{j-1}, x_j]$ since these intervals all have length $< \delta$. Consequently,

$$\text{Var}(f, [x_{j-1}, x_j]) \leq \varepsilon/2$$

and we are done. QED

The sup-norm bound on the variation has the following consequence:

Proposition 1 *If $f_1, f_2, \dots : [a, b] \rightarrow \mathbb{R}$ converge uniformly to a function $f : [a, b] \rightarrow \mathbb{R}$, then*

$$\lim_{n \rightarrow \infty} \text{Var}(f_n) = \text{Var}(f) \tag{3.46}$$

It will be convenient to use the notation

$$\text{Var}_j(f) \stackrel{\text{def}}{=} \text{Var}(f, [x_{j-1}, x_j]) \tag{3.47}$$

The reason why we are interested in the variation is summarized by

Theorem 20 For any function $f : [a, b] \rightarrow \mathbb{R}$, where $[a, b] \subset \mathbb{R}$ and $a < b$, and every partition X of $[a, b]$ we have

$$U(f, X) - L(f, X) = \sum_{j=1}^N \text{Var}_j(f) \Delta x_j \quad (3.48)$$

Proof. Observe that

$$\text{Var}_j(f) = M_j(f) - m_j(f). \quad (3.49)$$

Multiplying this by Δx_j and adding up over all $j \in \{1, \dots, N\}$ gives the result (3.48). QED

The algebra $\mathcal{R}[a, b]$

Recall that

$$\mathcal{R}[a, b]$$

is the set of all Riemann integrable functions on $[a, b]$.

Our main objective now is

Theorem 21 The set $\mathcal{R}[a, b]$ has the following properties:

- (i) Every constant function belongs to $\mathcal{R}[a, b]$
- (ii) If $f, g \in \mathcal{R}[a, b]$ then $f + g \in \mathcal{R}[a, b]$
- (iii) If $f, g \in \mathcal{R}[a, b]$ then $fg \in \mathcal{R}[a, b]$
- (iv) If $f \in \mathcal{R}[a, b]$, and f is never equal to zero and $1/f$ is bounded, then $1/f \in \mathcal{R}[a, b]$.
- (v) If $f \in \mathcal{R}[a, b]$ then $|f| \in \mathcal{R}[a, b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f| \quad (3.50)$$

Properties (i)-(iii) say that $\mathcal{R}[a,b]$ is an *algebra* under pointwise addition and multiplication of functions. It is important to note that the converse of (v) does not hold, i.e. there are functions which are not Riemann integrable but whose absolute values are Riemann integrable.

Proof For a constant function k on $[a,b]$ we have

$$L(f,X) = k(b-a) = U(f,X)$$

for every partition X and so $k(b-a)$ is the unique real number lying between all upper sums and all lower sums. Thus

$$\int_a^b k = k(b-a)$$

Now suppose $f, g \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$. By the Darboux condition, there are partitions Y and Z of $[a,b]$ such that

$$U(f,Y) - L(f,Y) < \varepsilon/2 \quad \text{and} \quad U(g,Z) - L(g,Z) < \varepsilon/2$$

Let X be the partition obtained by combining Y and Z . Then, because upper sums decrease and lower sums increase when points are added to a partition, we have

$$U(f,X) - L(f,X) < \varepsilon/2 \quad \text{and} \quad U(g,X) - L(g,X) < \varepsilon/2$$

Then, with $X = (x_0, \dots, x_N)$,

$$\begin{aligned} U(f+g,X) - L(f+g,X) &= \sum_{j=1}^N [M_j(f+g) - m_j(f+g)] \Delta x_j \\ &= \sum_{j=1}^N \text{Var}(f+g, [x_{j-1}, x_j]) \Delta x_j \\ &\leq \sum_{j=1}^N \text{Var}(f, [x_{j-1}, x_j]) \Delta x_j + \sum_{j=1}^N \text{Var}(g, [x_{j-1}, x_j]) \Delta x_j \end{aligned}$$

$$\begin{aligned}
& \text{(by Lemma 2 (ii))} \\
& = U(f, X) - L(f, X) + U(g, X) - L(g, X) \quad \text{by (3.48)} \\
& < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Thus, by the Darbox criterion, $f + g$ is Riemann integrable.

The other results follow in a similar way by applying other parts of Lemma 2. QED

Consider the function g on $[0, 1]$ given by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Then g is *not* Riemann integrable, because no matter what partition X we take of $[0, 1]$ the upper sum is always

$$U(g, X) = 1$$

and the lower sum is always

$$L(g, X) = -1$$

But the absolute value of g is the constant function 1:

$$|g| = 1$$

and so $|g|$ is Riemann integrable.

3.9 $C[a, b] \subset \mathcal{R}[a, b]$

Every continuous function on a compact interval is integrable. This is a central result of integration theory.

Theorem 22 *For any interval $[a, b] \subset \mathbb{R}$, with $a < b$, the set $C[a, b]$ of continuous functions on $[a, b]$ is contained in the set $\mathcal{R}[a, b]$ of Riemann integrable functions on $[a, b]$:*

$$C[a, b] \subset \mathcal{R}[a, b]$$

The proof has been presented in class. The key result used is Lemma 3 which says that the variation of a continuous function can be controlled suitably to apply the Darboux criterion for integrability.

It should be noted that discontinuous functions might also be integrable. Indeed, any function which is discontinuous at only finitely many points is integrable.

The Integral as a Non-negative Linear Functional

We have seen that the set

$$\mathcal{R}[a, b]$$

of all Riemann integrable functions on a compact interval $[a, b] \subset \mathbb{R}$, with $a < b$, is a *linear space*, i.e. that sums and constant multiples of Riemann integrable functions are again Riemann integrable. Our next objective is to show that the Riemann integral viewed as a function

$$\mathcal{R}[a, b] \rightarrow \mathbb{R} : f \mapsto \int_a^b f \quad (3.51)$$

is linear:

Theorem 23 For interval $[a, b] \subset \mathbb{R}$, where $a < b$, and any $f, g \in \mathcal{R}[a, b]$, we have

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g \quad (3.52)$$

and, for any $k \in \mathbb{R}$,

$$\int_a^b (kf) = k \int_a^b f \quad (3.53)$$

Thus, the Riemann integral

$$\mathcal{R}[a, b] \rightarrow \mathbb{R} : f \mapsto \int_a^b f \quad (3.54)$$

is a linear functional on the linear space $\mathcal{R}[a, b]$.

Proof Let $f, g \in \mathcal{R}[a, b]$, and $k \in \mathbb{R}$. We have already seen, in Theorem 21, that $f + g$ and kf are also in $\mathcal{R}[a, b]$. Now let

$$X = (x_0, \dots, x_N)$$

be any partition of $[a, b]$. As usual, let $\text{Var}_k(h)$ denote the variation of a function h over the interval $[x_{j-1}, x_j]$:

$$\text{Var}_k(h) = \sup_{s,t \in [x_{j-1}, x_j]} (h(s) - h(t)) = M_j(h) - m_j(h) \quad (3.55)$$

Then

$$\begin{aligned} U(f + g, X) &= \sum_{j=1}^N \text{Var}_k(f + g) \Delta_j x \\ &\leq \sum_{j=1}^N \text{Var}_k(f) \Delta_j x + \sum_{j=1}^N \text{Var}_k(g) \Delta_j x \\ &= U(f, X) + U(g, X) \end{aligned}$$

Similarly,

$$L(f + g, X) \geq L(f, X) + L(g, X) \quad (3.56)$$

Thus, $L(f + g, X)$ and $U(f + g, X)$ are squeezed into the interval

$$[L(f, X) + L(g, X), U(f, X) + U(g, X)] \quad (3.57)$$

Now let $\varepsilon > 0$. By the usual trick of combining partitions, there is a partition X of $[a, b]$ such that

$$U(f, X) - L(f, X) < \varepsilon/2$$

and

$$U(g, X) - L(g, X) < \varepsilon/2$$

So

$$U(f + g, X) - L(f + g, X) \leq U(f, X) + U(g, X) - [L(f, X) + L(g, X)] < \varepsilon \quad (3.58)$$

which, by the Darboux criterion implies that $f + g \in \mathcal{R}[a, b]$. (Of course, we have just repeated the proof of Theorem 21 (ii).)

Let

$$I(f) = \int_a^b f, \quad I(g) = \int_a^b g, \quad (3.59)$$

$$I(f+g) = \int_a^b (f+g) \quad (3.60)$$

Then $I(f)$ lies between $U(f, X)$ and $L(f, X)$, and $I(g)$ lies between $U(g, X)$ and $L(g, X)$. Consequently,

$$L(f, X) + L(g, X) \leq I(f) + I(g) \leq U(f, X) + U(g, X)$$

Moreover,

$$L(f+g, X) \leq I(f+g) \leq U(f+g, X)$$

Putting all this together, we see that $I(f+g)$ and the sum $I(f) + I(g)$ both lie in the interval

$$[L(f, X) + L(g, X), U(f, X) + U(g, X)] \quad (3.61)$$

and the width of this interval is $< \varepsilon$. Therefore, $I(f) + I(g)$ and $I(f+g)$ differ by less than ε . But ε is any positive real number. Therefore,

$$I(f+g) = I(f) + I(g) \quad (3.62)$$

Next, consider the function kf , where $k \in \mathbb{R}$. Let

$$X = (x_0, \dots, x_N)$$

be a partition of $[a, b]$ such that

$$U(f, X) - L(f, X) < \frac{\varepsilon}{1 + |k|} \quad (3.63)$$

(The $1+$ in the denominator is to avoid trouble if k happens to be 0 .) Then

$$\begin{aligned} U(kf, X) - L(kf, X) &= \sum_{j=1}^N \text{Var}_j(kf) \Delta x_j \\ &= \sum_{j=1}^N |k| \text{Var}_j(f) \Delta x_j \quad (\text{by Lemma 2 (iii)}) \\ &= |k| [U(f, X) - L(f, X)] \\ &\leq |k| \frac{\varepsilon}{1 + |k|} \\ &< \varepsilon. \end{aligned}$$

Thus, the interval

$$[L(kf, X), U(kf, X)]$$

has width less than ϵ . The Darboux criterion implies that kf is integrable. Let

$$I(kf) = \int_a^b (kf) \tag{3.64}$$

Now for $k \geq 0$ we have

$$M(kf, [s, t]) = \sup_{x \in [s, t]} kf(x) = k \sup_{x \in [a, b]} f(x) = kM(f, [s, t]) \tag{3.65}$$

and for $k < 0$ we have

$$M(kf, [s, t]) = \sup_{x \in [s, t]} kf(x) = k \inf_{x \in [a, b]} f(x) = km(f, [s, t]), \tag{3.66}$$

because multiplying by a negative number reverses inequalities and transforms sup into inf, and inf into sup. Thus, also

$$m(kf, [s, t]) = km(f, [s, t]) \quad \text{if } k \geq 0 \tag{3.67}$$

$$m(kf, [s, t]) = kM(f, [s, t]) \quad \text{if } k \leq 0 \tag{3.68}$$

$$\tag{3.69}$$

Doing this for each interval $[x_{j-1}, x_j]$, and multiplying everything by Δx_j and adding up, we see that $I(kf)$ lies in the interval

$$[kL(f, X), kU(f, X)] \quad \text{if } k \geq 0$$

and it lies in

$$[kU(f, X), kL(f, X)] \quad \text{if } k \leq 0$$

Now $I(f)$ lies between $L(f, X)$ and $U(f, X)$, and so $kI(f)$ lies in the same interval mentioned above as $I(kf)$ does. Consequently,

$$\left| I(kf) - kI(f) \right| \leq |k| [U(f, X) - L(f, X)] < \epsilon, \tag{3.70}$$

as before. Now since $\epsilon > 0$ is any positive real number we have

$$I(kf) = kI(f).$$

This completes the proof. QED

The Riemann integral is a *non-negative* linear functional in the sense that it carried non-negative functions into non-negative numbers:

Theorem 24 If $[a, b] \subset \mathbb{R}$ with $a < b$, and $f \in \mathcal{R}[a, b]$ is non-negative, i.e. $f(x) \geq 0$ for all $x \in [a, b]$ then $\int_a^b f \geq 0$:

$$f \in \mathcal{R}[a, b] \text{ and } f \geq 0 \text{ imply that } \int_a^b f \geq 0 \quad (3.71)$$

Consequently, the integral is order-preserving

$$f, g \in \mathcal{R}[a, b] \text{ and } f \geq g \text{ imply that } \int_a^b f \geq \int_a^b g \quad (3.72)$$

Proof. This is simply because if $f \geq 0$ then all the lower sums are ≥ 0 and so the integral $\int_a^b f$, being \geq all lower sums, is also ≥ 0 .

Next, suppose $f, g \in \mathcal{R}[a, b]$ and $f \geq g$. Observe that

$$f - g = f + (-1)g$$

is also in $\mathcal{R}[a, b]$, and is, of course, ≥ 0 . Thus,

$$f = (f - g) + g$$

and so, by linearity,

$$\int_a^b f = \int_a^b (f - g) + \int_a^b g$$

Now we have just shown that the first term on the right is ≥ 0 . Therefore,

$$\int_a^b f \geq \int_a^b g. \quad \boxed{\text{QED}}$$

The linear functional given by the Riemann integral is a *bounded linear functional* on $\mathcal{R}[a, b]$ for the sup-norm in the following sense:

Theorem 25 For any compact interval $[a, b] \subset \mathbb{R}$, with $a < b$, and for any $f \in \mathcal{R}[a, b]$ we have

$$\left| \int_a^b f \right| \leq \|f\|_{\text{sup}}(b - a) \quad (3.73)$$

Proof We know that if $f \in \mathcal{R}[a, b]$ then $|f| \in \mathcal{R}[a, b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f| \quad (3.74)$$

Now the function $|f|$ is bounded above by the constant $\|f\|_{\text{sup}}$ (recall from Theorem 19 that this is finite.) Therefore,

$$\int_a^b |f| \leq \int_a^b \|f\|_{\text{sup}} = \|f\|_{\text{sup}}(b-a)$$

and we are done. QED

A useful but simple consequence of this result is:

Theorem 26 For any compact interval $[a, b] \subset \mathbb{R}$, with $a < b$, if $f_1, f_2, \dots \in \mathcal{R}[a, b]$ converge uniformly to a function f on $[a, b]$ then $f \in \mathcal{R}[a, b]$ and

$$\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f \quad (3.75)$$

Proof First let us show that $f \in \mathcal{R}[a, b]$. Let $\varepsilon > 0$. By uniform convergence, we have an $n \in \mathbb{P}$ such that

$$\|f_n - f\|_{\text{sup}} < \frac{\varepsilon}{4(b-a)} \quad (3.76)$$

By integrability of f_n we know that there is a partition $X = (x_0, \dots, x_N)$ of $[a, b]$ such that

$$U(f_n, X) - L(f_n, X) = \sum_{j=1}^N \text{Var}_j(f_n) \Delta x_j < \frac{\varepsilon}{2} \quad (3.77)$$

Now

$$\begin{aligned} \left| \text{Var}_j(f_n) - \text{Var}_j(f) \right| &\leq \text{Var}_j(f_n - f) \\ &\leq 2\|f_n - f\|_{\text{sup}} \\ &< 2 \frac{\varepsilon}{4(b-a)} = \frac{\varepsilon}{2(b-a)} \end{aligned}$$

and so

$$\text{Var}_j(f) \leq \text{Var}_j(f_n) + \frac{\varepsilon}{2(b-a)} \quad (3.78)$$

Therefore,

$$\begin{aligned} U(f, X) - L(f, X) &= \sum_{j=1}^N \text{Var}_j(f) \Delta x_j \\ &\leq \sum_{j=1}^N \text{Var}_j(f_n) \Delta x_j + \frac{\varepsilon}{2(b-a)} \sum_{j=1}^N \Delta x_j \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $f \in \mathcal{R}[a, b]$.

Next, we have

$$\left| \int f_n - \int f \right| \leq \|f_n - f\|_{\text{sup}}(b - a) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad \boxed{\text{QED}}$$