Integration

Modern integration theory was built from the works of Newton, Riemann, and Lebesgue, but the origins of integration theory lie in the computation of areas and volumes. The idea of measuring area of a curved region by lower and upper bounds is present even Archimedes' study of the area enclosed by a circle.

The area of a rectangle whose sides are 2 units and 3 units is 6 square units, because if you tile the rectangle with unit squares you will need three rows of such squares, each row having 2 tiles. By extension, the area of a rectangle of sides $1/4$ unit and $1/5$ unit should be $1/20$ square units because we would need 20 of these rectangles to cover a unit square. Thus, it is clear that the area enclosed by a rectangle whose sides are a units and b units, where a and b are rational numbers, is ab square units. Consider now a rectangle R whose sides a and b are possibly not rational. Take any rationals a', a'', and b', b'' with

$$a' < a < a'', \quad \text{and} \quad b' < b < b''.$$

Then a rectangle R' of sides a' by b' sits inside R, while a rectangle R'' of sides a'' by b'' contains R. Thus, it makes sense to suppose that

$$\text{area of } R' \leq \text{area of } R \leq \text{area of } R''$$

which is to say:

$$a'b' \leq \text{area of } R \leq a''b''$$

It is intuitively clear, and not hard to prove, that ab is the *unique real number* that lies between all the possible values of $a'b'$ and $a''b''$. Thus,

$$\text{area of } R = ab$$

The rectangle R is made up of two congruent right angled triangles, and so each of these would have area $(ab)/2$. This makes it possible to compute the areas of all kinds of polygonal figures but cutting up these fgures into right angled triangles. However, this strategy fails when we try to compute the area enclosed by a circle. No amount of cutting would turn a disc into a finite number of right angled triangles.

Archimedes computed area enclosed by a circle C by consider polygons P' and P'', where P lies inside the circle C and P'' encloses the circle C; thus

$$\text{area } A' \text{ enclosed by } P' \leq \text{area enclosed by } C \leq \text{area } A'' \text{ enclosed by } P''$$

Some computation shows that for any $\varepsilon > 0$ there are such polygons P' and P'' such that

$$A'' - A' < \varepsilon$$

This implies that there is a *unique real number* A which lies between the area of all the polygons P' and the polygons P''. Clearly then this number should be the area enclosed by the circle C.

Our development of the theory of the Riemann integrals is based on these ideas.

Approaching the Riemann Integral

Consider a function

$$f : [a,b] \to \mathbb{R}$$

and think of its graph. Assume, for convenience of visualization, that $f \geq 0$. Then f specifies a region which lies below its graph and above the x-axis. In general, this is a region whose upper boundary, given by the graph of f, is curved. The integral

$$\int_a^b f$$

which is also, conveniently, written as

$$\int_a^b f(x)\,dx$$

measures the area of this region.

The strategy used for computing the area is to cut up the region into vertical slices. The area of each such slice is between the area of a larger 'upper' rectangle and a smaller 'lower' rectangle. Thus we should expect the actual area to be the unique real number lying between these 'upper rectangle' areas and the 'lower rectangle' areas.

Riemann Sums

We will work with functions on an interval

$$[a,b] \subset \mathbb{R}$$

] where $< b$.

A *partition* X of $[a,b]$ is specified by a sequence

$$X = (x_0, x_1, \ldots, x_N)$$

of points $x_0, x_1, \ldots, x_N \in [a,b]$ with

$$a = x_0 < x_1 < \cdots < x_N = b$$

Here $N \in \{1,2,3,\ldots\}$.

We will often use the notation

$$\Delta x_j \overset{\text{def}}{=} x_j - x_{j-1} \tag{3.1}$$

to denote the length of the j-th interval

$$[x_{j-1}, x_j]$$

marked out by the partition X.

The *width* or *norm* of the partition X is the maximum size of these intervals:

$$\|X\| = \max_{j \in \{1,\ldots,N\}} \Delta x_j \tag{3.2}$$

Consider a function

$$f : [a,b] \to \mathbb{R}$$

on an interval $[a,b] \subset \mathbb{R}$, where $a < b$.

Let

$$M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x) \tag{3.3}$$

and

$$m_j(f) = \inf_{x \in [x_{j-1}, x_j]} f(x) \tag{3.4}$$

Thus, if A_j were the area of the region under the graph of f over $[x_{j-1}, x_j]$ then

$$m_j(f)\Delta x_j \leq A_j \leq M_j(f)\Delta x_j.$$

Our objective is to specify the actual area A under the graph of f, and this would be the sum of the A_j.

The *upper Riemann sum* $U(f,X)$ is defined to be

$$U(f,X) \overset{\text{def}}{=} \sum_{j=1}^{N} M_j(f)\Delta x_j \tag{3.5}$$

and the *lower Riemann sum* $L(f,X)$ is

$$L(f,X) \overset{\text{def}}{=} \sum_{j=1}^{N} m_j(f)\Delta x_j \tag{3.6}$$

Note that

$$L(f,X) \leq U(f,X) \tag{3.7}$$

We will show later that in fact every lower sum if less or equal to every upper sum, i.e. $L(f,X)$ is less or equal to $U(f,Y)$ for every partitions X and Y of $[a,b]$.

Now consider a sequence

$$X^* = (x_1^*, \ldots, x_N^*)$$

obtained by picking a point from each interval $[x_{j-1}, x_j]$::

$$x_j^* \in [x_{j-1}, x_j]$$

We shall indicate this by writing

$$X^* < X$$

The *Riemann sum* $S(f,X,X^*)$ is

$$S(f,X,X^*) = \sum_{j=1}^{N} f(x_j^*)\Delta x_j \tag{3.8}$$

Note that the upper sum is ∞ if and only if one of the M_j is ∞, and this occurs if and only if the superemum of f is ∞:

$$U(f,X) = \infty \qquad \text{if and only if} \qquad \sup_{x\in[a,b]} f(x) = \infty \tag{3.9}$$

However, $U(f,X)$ *cannot be* $-\infty$, because that would mean that at least one of the M_j is $-\infty$ which can only be if f is equal to $-\infty$ on that interval $[x_{j-1}, x_j]$ which contradicts the fact that f is real-valued. (Note that we are working with *non-empty* intervals because $x_{j-1} < x_j$.)

Similarly,

$$L(f,X) = -\infty \qquad \text{if and only if} \qquad \inf_{x\in[a,b]} f(x) = -\infty \tag{3.10}$$

and $L(f,X)$ can never be ∞.

It is useful then to observe that the difference

$$U(f,X) - L(f,X)$$

is always defined, i.e. we never have the $\infty - \infty$ situation.

Definition of the Riemann Integral

Consider a function

$$f : [a,b] \to \mathbb{R}$$

on an interval $[a,b] \subset \mathbb{R}$, where $a < b$. This function is said to be *Riemann integrable* if there is a unique real number I lying between all the lower sums and all the upper sums:

$$L(f,X) \leq I \leq U(f,X) \tag{3.11}$$

for every partition X of $[a,b]$. This number I is the *integral* of f over $[a,b]$ and denoted

$$\int_a^b f \qquad \text{or} \qquad \int_a^b f(x)\,dx \tag{3.12}$$

The set of all Riemann integrable functions on $[a,b]$ is denote

$$\mathcal{R}[a,b] \tag{3.13}$$

Refining Partitions

We work with a function

$$f : [a,b] \to \mathbb{R}.$$

We use the notation

$$M(f,[s,t]) = \sup_{x\in[s,t]} f(x), \quad \text{and} \quad m(f,[s,t]) = \inf_{x\in[s,t]} f(x) \tag{3.14}$$

Let

$$X = (x_0, x_1, \ldots, x_N)$$

be a partition of the interval $[a,b]$ and let X' be a partition obtained by adding one more point x' to X. Let us say that x' lies in the j-th interval

$$x' \in [x_{j-1}, x_j]$$

Thus, X' cuts up $[x_{j-1}, x - j]$ into two intervals

$$[x_{j-1}, x'] \qquad \text{and} \qquad [x', x_j]$$

Let us compare the upper sums $U(f,X)$ and $U(f,X')$. To this end, let us write

$$M_j = M(f,[x_{j-1},x_j]), \quad M_j' = M(f,[x_{j-1},x']), \qquad M_j'' = M(f,[x',x_j])$$

It is important to observe that

$$M_j \geq M_j', \quad \text{and} \quad M_j \geq M_j'' \tag{3.15}$$

These sums differ only in the contribution that comes from $[x_{j-1}, x_j]$:

$$\begin{aligned} U(f,X) - U(f,X') &= M_j(x_j - x_{j-1}) \\ &\qquad - \left[M_j'(x' - x_{j-1}) + M_j''(x_j - x')\right] \\ &= (M_j - M_j')(x' - x_{j-1}) + (M_j - M_j'')(x_j - x') \\ &\geq 0. \end{aligned}$$

Thus

$$U(f,X') \leq U(f,X), \tag{3.16}$$

i.e. *adding a point to a partition lowers upper sums.*

If we keep adding points to the initial partition, we keep lower the upper sums.

Arguing similarly, it follows that *adding points to a partition raises lower sums.*

This observation is important enough to state as a theorem:

Theorem 15 *Adding points to a partition lowers upper sums and raises lower sums. Thus if* X *and* X' *are partitions of* $[a,b]$*, with* X' *containing all the points of* X *and some more, then*

$$L(f,X) \leq L(f,X') \leq U(f,X') \leq U(f,X) \tag{3.17}$$

for every function $f : [a,b] \to \mathbb{R}$.

Using this we can prove that upper sums always dominate lower sums:

Theorem 16 *If* $f : [a,b] \to \mathbb{R}$ *is a function on an interval* $[a,b] \subset \mathbb{R}$*, where* $a < b$*, and if* X *and* Y *are any partitions of* $[a,b]$ *then*

$$L(f,X) \leq U(f,Y) \tag{3.18}$$

Thus, every lower sum is dominated above by every upper sum.

Proof. Here is a useful trick: we can combine the partitions X and Y to form a partition Z which contains all points of X and all the points of Y as well. Therefore,

$$U(f,Z) \leq U(f,Y)$$

and

$$L(f,X) \leq L(f,Z)$$

Stringing these together we have

$$L(f,X) \leq L(f,Z) \leq U(f,Z) \leq U(f,Y)$$

and this gives the desired result. $\boxed{\text{QED}}$

We have used the fact that adding a point to a partition lowers upper sums and raises lower sums. It will be useful to take a closer look at this and estimate by *how much* the upper and lower sums move as points are added to the partition. Recall the formula

$$U(f,X) - U(f,X') = (M_j - M'_j)(x' - x_{j-1}) + (M_j - M''_j)(x_j - x') \tag{3.19}$$

where X' is the partition obtained from X by adding the single point x' to the j-th interval $[x_{j-1}, x_j]$ of the partition $X = (x_0, \ldots, x_T)$. Now suppose instead of adding just the one point x', we add m distinct points $y_1, \ldots, y_m$ to the interval $[x_{j-1}, x_j]$ to form the new partition X'. We label the new points y_i in increasing order

$$y_1 < \cdots < y_m.$$

For convenience of notation we write y_0 for x_{j-1} and y_{m+1} for x_j. Thus

$$x_{j-1} = y_0 < y_1 < \cdots < y_m < y_{m+1} = x_j$$

The intervals

$$[y_{k-1}, y_k]$$

make up the interval

$$[x_{j-1}, x_j]$$

Then:

$$U(f,X) - U(f,X') = \sum_{k=1}^{m+1} \left[M(f,[x_{j-1},x_j]) - M(f,[y_{k-1},y_k])\right](y_k - y_{k-1}) \quad (3.20)$$

Now

$$0 \le M(f,[x_{j-1},x_j]) - M(f,[y_{k-1},y_k]) \le 2\|f\|_{\sup},$$

where $\|f\|_{\sup}$ is supremum of $|f|$ over the full original interval $[a,b]$. The interval lengths $y_k - y_{k-1}$ add up to Δx_j. Consequently,

$$U(f,X) - U(f,X') \le 2\|f\|_{\sup}\Delta x_j \le 2\|f\|_{\sup}\|X\| \quad (3.21)$$

This provides an upper bound for how much the upper sum is decreased by addition of points all in one single interval of the original partition X.

If we add points to each of N' intervals to create a new partition X' then

$$U(f,X) - U(f,X') \le 2N'\|f\|_{\sup}\|X\| \quad (3.22)$$

Similarly,

$$L(f,X') - L(f,X) \le 2N'\|f\|_{\sup}\|X\| \quad (3.23)$$

Note that

$$N' \le N,$$

where N is the total number of new points added. Consequently, we have

Lemma 1 *If*

$$X = (x_0, \ldots, x_T)$$

is a partition of the interval $[a,b] \subset \mathbb{R}$, where $a < b$, and X' is a partition of the same interval containing all the points of X and possibly some more, then

$$U(f,X') - L(f,X') \le U(f,X) - L(f,X) \quad (3.24)$$

but the decrease in the value of $U - L$ is at most $2N\|f\|_{\sup}\|X\|$:

$$[U(f,X) - L(f,X)] - \left[U(f,X') - L(f,X')\right] \le 2N\|f\|_{\sup}\|X\| \quad (3.25)$$

where N is the number of new points added.

The Darboux Criterion

Now that we know that the upper sums dominate the lower sums, it is clear that there will be a unique real number between them if and only if the sup of the lower sums equals the inf of the upper sums:

Theorem 17 *A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if*

$$\sup\{U(f,X) : \textit{all partitions } X \textit{ of } [a,b]\} = \inf\{L(f,X) : \textit{all partitions } X \textit{ of } [a,b]\} \tag{3.26}$$

The common value is $\int_a^b f$.

Thus, the function f is Riemann integrable if and only if there is no ‘gap’ between the lower sums and upper sums. Thus, another equivalent formulation is the **Darboux criterion**:

Theorem 18 *T:Darbouxcrit A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon > 0$ there is a partition X of $[a,b]$ for which*

$$U(f,X) - L(f,X) < \varepsilon \tag{3.27}$$

This is an extremely useful result: virtually all of our results on integrability will use the Darboux criterion.

Proof. Suppose f is integrable. Then there is a unique real number I which lies between all upper sums and all lower sums. Take any $\varepsilon > 0$. Consider the interval

$$[I, I + \varepsilon/2)$$

All the lower sums lie to the left, i.e. $\leq I$. All upper sums lie to the right ($\geq$) of I. So since I is the only real number lying between the upper sums and lower sums, there must be at least one upper sum which lies in $[I, I + \varepsilon/2)$. Thus, there is a partition Y for which

$$I \leq U(f,Y) < I + \varepsilon/2$$

Similarly, there is a partition Z for which

$$I - \varepsilon/2 < L(f,Z) \leq I.$$

Let X be the partition obtained by pooling together Y and Z. Then

$$I - \varepsilon/2 < L(f,Z) \leq L(f,X) \leq U(f,X) \leq U(f,Z) < I + \varepsilon/2 \tag{3.28}$$

Since both $L(f,X)$ and $U(f,X)$ lie in

$$(I-\varepsilon/2, I+\varepsilon/2)$$

it follows that

$$U(f,X)-L(f,X)<\varepsilon. \tag{3.29}$$

Conversely, suppose that for every $\varepsilon>0$ there is a partition for which (3.29) holds. Suppose that I and I' are distinct real numbers which both lie between all upper sums and all lower sums. Let

$$\varepsilon=|I'-I|$$

We know that there is a partition X satifying (3.29). Now I and I' both lie between $U(f,X)$ and $L(f,X)$. So the difference between I and I' is $<\varepsilon$:

$$|I'-I|<\varepsilon$$

But this contradicts the deifnition of ε taken above. Thus I and I' must be equal. So there is a unique real number between all the upper sums and all the lower sums. Thus, f is integrable. $\boxed{\text{QED}}$

Integrable functions are bounded

Suppose $f:[a,b]\to\mathbb{R}$ is integrable. We will prove that f must be bounded.

Theorem 19 *Every Riemann integrable function is bounded.*

Proof. Consider

$$f:[a,b]\to\mathbb{R}$$

where $[a,b]\subset\mathbb{R}$ and $a<b$. Assume that $f\in\mathcal{R}[a,b]$.

Let us suppose that f is unbounded. Then we will reach a contradiction.

Suppose, for example, that f is not bounded above, i.e.

$$\sup_{x\in[a,b]} f(x)=\infty.$$

Consider any partition

$$X=(x_0,\ldots,x_N)$$

of $[a,b]$. Then there is some subinterval, say $[x_{j-1}, x_j]$, on which f is not bounded above, i.e.

$$M_j = \infty.$$

But then the supper sum for f with this partition is also ∞:

$$U(f,X) = \sum_{k=1}^{N} M_k \Delta x_k = \infty$$

Thus, *every upper sum* for f is ∞. Now let

$$I = \int_a^b f$$

and recall that this is the unique real number lying between all upper sums and all lower sums. Then

$$L(f,X) \le I < I+1 < \infty = U(f,X) \tag{3.30}$$

for every partition X. But then $I+1$ would also be a real number lying between all upper sums and all lower sums. Thus, we have a contradiction. $\boxed{\text{QED}}$