Monotone Sequences

A sequence such as

$$1, 2, 5, 8, 9, 11, \ldots$$

where each term is $\geq$ the preceding term is said to be *monotone increasing*. Another example is the sequence

$$1, 1.1, 1.11, 1.111, 1.1111, \ldots$$

Thus, a sequence (x_n) is

$$\textit{monotone increasing} \text{ if } x_n \leq x_{n+1} \text{ for all } n \in \mathbb{P} \tag{2.15}$$

We say that (x_n) is

$$\textit{monotone decreasing} \text{ if } x_n \geq x_{n+1} \text{ for all } n \in \mathbb{P} \tag{2.16}$$

An example of a monotone decreasing sequence is

$$1, 1/2, 1/3, 1/4, 1/5, \ldots$$

It is intuitively clear that a monotone increasing sequence (x_n) will tend to the limit $\sup\{x_1, x_2, \ldots\}$. This is indeed true:

If (x_n) is a monotone increasing sequence then

$$\lim_{n\to\infty} x_n = \sup_{n\geq 1} x_n.$$

To prove this let

$$L = \sup_{n\geq 1} x_n.$$

The most extreme case is when L is $-\infty$; as will become clear shortly, we need to take care of this case separately. If $L = -\infty$ each x_n must be $-\infty$. But in this case

$$\lim_{n\to\infty} x_n = -\infty,$$

since x_n is just stuck at $-\infty$. Now consider the other situation: L is not $-\infty$. We will show again that $x_n \to L$, i.e. we will show that for any neigborhood U of L, the sequence (x_n) falls eventually in U and stays there. So consider any neighborhood U of L. Pick a point t of U to the left of L, i.e.

$$t < L \text{ and } t \in U.$$

(Note that there would be no such t if L is $-\infty$.) Since L is the *least* upper bound of $\{x_1, x_2, x_3, ...\}$, the number t can't be an upper bound, and so some x_{n_0} is $> t$:

$$x_{n_0} > t.$$

But recall that we are dealing with a monotone *increasing* sequence. Consequently,

$$x_n > t \text{ for all } n \geq n_0.$$

But recall that L is an *upper bound* of $\{x_1, x_2, ...\}$; therefore,

$$\text{all the } x_n \text{ are } \leq t.$$

Those x_n which are $> t$ but $\leq L$ are, of course, in the neighborhood U of L. Thus,

$$x_n \in U \text{ for all } n \geq n_0.$$

This proves that L is indeed the limit of the sequence (x_n).

In a similar way, if x_n is a monotone decreasing sequence then

$$\lim_{n\to\infty} x_n = \inf_{n\geq 1} x_n.$$

The limit of a sequence is unique

Now we shall prove that a sequence can have at most one limit.

Consider a sequence (x_n) and suppose both L and L' are limits of the sequence, and $L \neq L'$. We will arrive at a contradiction. Since L and L' are distinct, they have *disjoint neighborhoods* U and U' respectively. Since $x_n \to L$ we know that $x_n \in U$ eventually. But $x_n \to L'$, and so $x_n \in U'$ eventually. But this is impossible since U and U' are disjoint and thus have no element in common.

Thus, a sequence which has a limit must have a *unique* limit.

Convergent sequences and Cauchy sequences

A sequence (x_n) in $\mathbb{R}$ is said to be *convergent* if it has a limit and the limit is a real number. If $x_n \to L$, and $L \in \mathbb{R}$, we also say that the sequence x_n *converges* to L.

Note that $x_n \to L \in \mathbb{R}$ if for any real $r > 0$ there is an $n_0 \in \mathbb{P}$ such that

$$|x_n - L| < r$$

for all natural numbers $n > n_0$.

This has a consequence: since the x_n's are all accumulating to L *they are also getting close to each other*. More precisely, for any $\varepsilon > 0$ we can find $N \in \mathbb{P}$ such that

$$|x_n - x_m| < \varepsilon \text{ for all natural numbers } n, m > N.$$

To prove this, observe that there is some $N \in \mathbb{P}$ such that

$$|x_k - L| < \varepsilon/2$$

for all $k \in \mathbb{P}$ with $k > N$. But then for any $n, m \in \mathbb{P}$ with $n, m > N$ we have

$$\begin{array}{rcl} |x_n - x_m| & = & |x_n - L + L - x_m| \\ & \leq & |x_n - L| + |L - x_m| \\ & = & |x_n - L| + |x_m - L| \\ & < & \varepsilon/2 + \varepsilon/2 \\ & = & \varepsilon. \end{array}$$

A sequence (x_n) in $\mathbb{R}$ which bunches up on itself in the sense above is said to be a *Cauchy sequence*, i.e. if for any $\delta > 0$ there is an $N \in \mathbb{P}$ such that

$$|x_n - x_m| < \varepsilon \text{ for all natural numbers } n, m > N.$$

Every Cauchy sequence is bounded

Consider a Cauchy sequence (x_n) in $\mathbb{R}$. Then, eventually the points of this sequence are at most distance 1 from each other; in fact, there is an $N \in \mathbb{P}$ such that

$$|x_n - x_m| < 1$$

for all natural numbers $n, m > N$. In particular, fixing a particular $j > N$ we have

$$x_n \in (x_j - 1, x_j + 1),$$

for all $n \geq N$. So

$$x_j - 1 < x_n < x_j + 1$$

for all $n \in \{N+1, N+2, \ldots\}$. Thus, at least from the $(N+1)$-th term on, the sequence *is* bounded. But the terms not counted here,

$$x_1, \ldots, x_N$$

are just finitely many, and so they have a largest B and a smallest A among them. Thus,

$$\text{every } x_n \text{ is } \leq \max\{B, x_j + 1\}$$

and

$$\text{every } x_n \text{ is } \geq \min\{B, x_j - 1\}$$

Thus the entire sequence is bounded.

Every Cauchy sequence is convergent

The completeness property of the real line $\mathbb{R}$ has an equivalent formulation:

Theorem 13 *Every Cauchy sequence in $\mathbb{R}$ converges.*

Let us prove this.

Consider a Cauchy sequence (x_n) in $\mathbb{R}$. We have seen that it is bounded. Thus,

$$a \leq x_n \leq b \qquad \text{for all } n \in \mathbb{P}$$

for some real numbers a and b.

We know that a sequence always has a limit point in $\mathbb{R}^*$. Let L be a limit point of (x_n). We will prove that L is a real number and $x_n \to L$.

First it is clear that L must also be in $[a,b]$. Therefore, L is not ∞ or $-\infty$, and is a real number.

Take any neighborhood

$$(L - \delta, L + \delta)$$

of L, where $\delta > 0$ is a real number.

The sequence (x_n) visits the neighborhood

$$(L - \frac{\delta}{2}, L + \frac{\delta}{2})$$

infinitely often. Now we also know that eventually, the terms of the sequence vary from each other by $< \delta/2$, i.e. there is some $N \in \mathbb{P}$ such that

$$|x_n - x_m| < \delta/2 \qquad \text{for all } n, m \in \mathbb{P} \text{ with } n, m > N.$$

We can choose some $j > N$ such that

$$x_j \in (L - \frac{\delta}{2}, L + \frac{\delta}{2})$$

because the sequence visits this neighborhood infinitely often. Then

$$\begin{aligned} |x_n - L| &= |x_n - x_j + x_j - L| \\ &\leq |x_n - x_j| + |x_j - L| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta. \end{aligned}$$

This means that

$$x_n \in (L - \delta, L + \delta)$$

for all $n > N$. Thus,

$$x_n \to L.$$

The rationals are countable

A set is set to be *countable* if it is finite or if its elements can be enumerated in a sequence. Thus, S is countable if there is a sequence $x_1, x_2, \ldots$ such that

$$S = \{x_1, x_2, x_3, \ldots\}$$

For example, the even numbers form a countable set:

$$\{2, 4, 6, 8, \ldots\}$$

It may seem at first that the set $\mathbf{Z}$ of integers is not countable, but we can certainly lay out all the integers in a sequence:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

It is much harder to see that the rationals $\mathbf{Q}$ are also countable. This is what we shall prove now.

We will constuct a sequence which enumerates all the rationals, i.e. a sequence $r_1, r_2, \ldots$ such that

$$\{r_1, r_2, r_3, \ldots\} = \mathbf{Q}.$$

Put another way, we will encode each rational by a unique natural number; thus to each natural number n we would associate a rational r_n, and in this way we will cover *all* rationals.

There can be many such encoding schemes. Here is one: Take any rational $x \geq 0$ and write it as $\frac{p}{q}$ where p and q are non-negative integers (of course $q \geq 1$) and q is the smallest possible denominator among all such ways of writing x. (For example, 0.6 can be written as both $6/10$ and $3/5$, but we would take $3/5$ as our representation.) We know that every non-negative rational can be represented in this way uniquely. Associate to this x the natural number

$$f(x) = 2^q \times 3^p$$

For example,

$$f(0) = f(0/1) = 2^1 \times 3^0 = 2$$

$$f(1) = f(1/1) = 2^1 3^1 = 6, \qquad f(4/5) = 2^5 \times 3^4 = 32 \times 81 = 2592.$$

This associates a unique natural number $f(x)$ to each non-negative rational x. If $x \in \mathbf{Q}$ is negative, $x < 0$, let us define $f(x)$ to be just 5 times $f(-x)$:

$$f(x) = f(-x) \times 5 \qquad \text{if } x < 0$$

Now we have labeled each rational by a unique natural number. To enumerate the rationals in a sequence all we have to do then is to run this in reverse: let r_1 be the rational number x with the smallest value for $f(x)$ (so r_1 is in fact 0 because $f(0) = 2$ is the lowest value possible for f); let r_2 be the rational number with the *next* lowest value for $f(x)$, and so on. Thus, r_n is the rational for which $f(r_n)$ is the first value of $f(x)$ greater than $f(r_1), \ldots, f(r_{n-1})$. For example,

$$r_1 = 0$$

and

$$r_2 = 1 \text{ because } f(1) = 2^1 3^1 = 6, \text{ the next value for } f(x) \text{ after } 2$$

Thus we have produced a sequence of *distinct* elements $r_1, r_2, \ldots$ such that

$$\{r_1, r_2, \ldots\} = \mathbf{Q}$$

The real numbers are uncountable

We will prove that $\mathbb{R}$ is not countable, i.e. there is no sequence which touches on all the real numbers.

In fact we will show that even the real numbers between 0 and 1 cannot be enumerated in a sequence.

The argument used here is the celebrated *diagonal method* due to Georg Cantor (1845-1912). The strategy is to make a list of strings, then take the main diagonal string, and then form a new string by altering each element of the diagonal.

Consider any sequence $x_1, x_2, \ldots$ lying in $(0,1)$. Now each real number y in $(0,1)$ can be expressed uniquely in the form

$$y = 0.y_1 y_2 y_3 \ldots = \frac{y_1}{10} + \frac{y_2}{10^2} + \frac{y_3}{10^3} + \cdots,$$

where $y_1, y_2, \ldots \in \{0,1,2,3,\ldots,9\}$ and we exclude all such representations which use an infinite string of 9's at the end (for example, instead of 0.19999.... we would use 0.20000....). Thus each of the numbers x_1, x_2, also has such an expansion:

$$x_1 = 0.x_{11} x_{12} x_{13} \ldots$$

$$x_2 = 0.x_{21} x_{22} x_{23} \ldots$$

$$x_3 = 0.x_{31} x_{32} x_{33} \ldots$$

$$\vdots = \cdots$$

Now form a number

$$w = 0.w_1 w_2 w_3 \ldots$$

as follows: take w_1 to be any number in $\{1,2,\ldots,8\}$ other than x_{11} ; then choose $w_2 \in \{1,2,\ldots,8\}$ other than x_{22}, and so on. This way we make the n-th decimal place of w different from the n-th decimal place of x_n, for every n. Then w cannot be equal to any of the x_n, and $w \in (0,1)$. Thus the original sequence cannot possibly have covered all of $(0,1)$. [The reason we excluded 0 and 9 from our choices for w_n was to avoid ending up at 0 or 1.]

Connected Sets

Consider a subset S of $\mathbb{R}^*$. If U is an open set then the part of U in S, i.e. $U \cap S$ is said to be *open in S*.

For example, in $[2,5]$ the set

$$[2,4)$$

is open because, for example,

$$[2,4) = (1,4) \cap [2,5]$$

Put another way, a subset J of S is said to be *open in S* if every $p \in J$ has a neighborhood N such that every point of N which is in S is in fact in J, i.e.

$$N \cap J \subset S$$

A set S is said to be *connected* if it cannot be split into two non-empty disjoint pieces A and B each of which is open in S. The main result for connected sets is:

Theorem 14 *A subset of $\mathbb{R}$, or of $\mathbb{R}^*$, is connected if and only if it is an interval.*