Sequences

A sequence of elements in a set A is a string

$$a_1, a_2, a_3, \ldots$$

of elements of A.

More precisely, the sequence is actually a *mapping*

$$a : \mathbb{P} \to A : n \mapsto a_n.$$

We will often be concerned with sequences in $\mathbb{R}^*$.

Sometimes out sequence will be specified explicitly as a string of numbers; for example,

$$-5, -3, 5, 6, 9, 12, \ldots$$

Sometimes we may have a formula for the n-th term:

$$a_n = \frac{(-1)^n}{n+1}.$$

Sometimes we have a sequence specified *recursively*. For example, we might know that

$$b_1 = 1, b_2 = 1$$

and then

$$b_n = b_{n-1} + b_{n-2},$$

for all $n \geq 3$. This generates the *Fibonacci numbers*

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Sometimes it is better to label a sequence starting with the index 0:

$$a_0, a_1, a_2, \ldots$$

For example, for a fixed real number r we can form the sequence

$$r^0 = 1, r^1 = r, r^2, r^3, \ldots$$

Limits points of a sequence

Consider a sequence

$$x_1, x_2, x_3, \ldots.$$

in $\mathbb{R}^*$. A point $p \in \mathbb{R}^*$ is said to be a *limit point* of this sequence if every neighborhood of p is visited infinitely often by the sequence.

For example, for the sequence

$$\frac{1}{2}, 4, \qquad \frac{1}{3}, 4+\frac{1}{3}, \qquad \frac{1}{4}, 4+\frac{1}{4}, \qquad \frac{1}{5}, 4+\frac{1}{5}, \qquad \frac{1}{6}, 4+\frac{1}{6}, \ldots$$

it is intuitively clear that both 0 and 4 are limit points.

Suppose a sequence (x_n) lies entirely inside a set $S \subset \mathbb{R}$:

$$x_n \in S, \qquad \text{for all } n \in \mathbb{P}$$

Consider then any limit point p of this sequence. Let U be any neighborhood of p. We know that this neighborhood is visited infinitely often by the sequence. Therefore, at least one point of S must be in U. Thus, every neigborhood of p contains a point of S, and so

$$p \in \overline{S}.$$

Thus, *any limit point of a sequence which lies always in a set S must be in the closure of S.*

Bolzano-Weierstrass Theorem

Consider any sequence in $\mathbb{R}^*$. We shall prove that it must have at least one limit point.

Suppose to the contrary that no point is a limit of the given sequence. Then each point p of $\mathbb{R}^*$ has a neighborhood U_p which is visited only finitely often by the sequence. The neighborhoods U_p form an open cover of all of $\mathbb{R}^*$. Then, by compactness of $\mathbb{R}^*$, there are finitely many of them, say

$$U_{p_1}, \ldots, U_{p_N}$$

which cover all of $\mathbb{R}^*$. But this is impossible: we have covered all of the extended real line $\mathbb{R}^*$ with finitely many sets each of which is visited only finitely many times by the sequence! Thus, we have a contradiction and so there exist a limit of the sequence.

Notice that all we needed above was the compactness of $\mathbb{R}^*$. Thus, we have the more general **Bolzano-Weierstrass theorem**:

Theorem 12 *Every sequence in a compact set has at least one limit point.*

Of course, we have seen that a sequence may well have more than one limit point. For example, the sequence,

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$

visits every natural number infinitely often and so every natural number is a limit point of the sequence.

In the next subsection we will show how to actually obtain a limit point of a sequence.

Limit points and suprema and infima

Consider a sequence

$$x_1, x_2, x_3, \ldots$$

Let p be any limit point of this sequence.

Now let S_1 be the least upper bound of the set $\{x_1, x_2, \ldots\}$:

$$S_1 = \sup\{x_1, x_2, \ldots\}$$

In particular,

$$S_1 \geq x_n \text{ for each } n.$$

Then it is clear that no point 'to the right' of S_1 could possibly be a limit of the sequence: indeed any point $r > S_1$ would have a neighborhood lying entirely to the 'right' of S_1 and this would never be visited by the sequence. Thus, p cannot be $> S_1$. So

$$p \leq S_1$$

Now we can apply the same argument to

$$S_2 = \sup\{x_2, x_3, x_4, \ldots\}$$

Again, p could not be greater than S_2 for then it would have a neighborhood to the right of S_2 and this would have to be visited infinitely often by $x_1, x_2, \ldots$ and so at least once by $x_2, x_3, \ldots$. Thus,

$$p \leq S_2$$

In this way, we can form

$$S_3 = \sup\{x_3, x_4, \ldots\}$$

and have again

$$p \leq S_3$$

. Thus, p is a *lower bound* for all the S_k's:

$$p \leq S_k \text{ for all } k.$$

Now the inf of a set is the *greatest* lower bound of the set. Therefore,

$$p \leq \inf\{S_1, S_2, S_3, \ldots\}.$$

The infemum on the right is an important object and is called the *limsup* of the given sequence:

$$\boxed{\limsup_{n\to\infty} x_n \stackrel{\text{def}}{=} \inf\{\sup_{n\geq k} x_n\} \,:\, k \in \mathbb{P}\}} \qquad (2.11)$$

So we have proved that

Every limit point is $\leq$ the limsup

for any sequence.

Similarly, we have the notion of liminf: it is the supermum of the sequence of infemums:

$$\liminf_{n\to\infty} x_n \stackrel{\text{def}}{=} \sup\{\inf_{n\geq k} x_n\} : k \in \mathbb{P}\} \tag{2.12}$$

In just the same way as before, we can prove

Every limit point is $\geq$ the limsup

Thus, for any limit point p if any sequence (x_n) we have

$$\liminf_{n\to\infty} x_n \leq \text{any limit point} \leq \limsup_{n\to\infty} x_n. \tag{2.13}$$

In fact, the limsup and liminf of the sequence are also limit points, but we won't prove this here.

Limit of a sequence

A sequence (s_n) is said to lie in a set S *eventually* if after a certain value of n, all the s_n lie in S. Put another way, we say that s_n *lies in S for large n*, if $s_n \in S$ for all values of n beyond some value, say n_0.

For example, the sequence

$$-5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots$$

will lie in the set $(20, \infty)$ eventually.

Let us note again: *a sequence (s_n) lies in a set S eventually if there is an $n_0 \in \mathbb{P}$ such that*

$$s_n \in S$$

for all $n \in \mathbb{P}$ with $n \geq n_0$.

A sequence (x_n) in $\mathbb{R}^*$ is said to have *limit $L \in \mathbb{R}^*$* if *for any neighborhood U of L the sequence lies in this neighborhood eventually.*

We denote this symbolically as

$$x_n \to L, \qquad \text{as } n \to \infty.$$

We shall see later that if a limit exists then it is *unique*; the limit L is denoted

$$\lim_{n\to\infty} x_n.$$

The notion of limit is one of the central notions in mathematics.

For example, the sequence

$$1, 2, 3, 4, \ldots.$$

has limit ∞. For, if we take any neighborhood of ∞, say

$$(t, \infty]$$

then eventually n will exceed t (by the archimidean property) and so the sequence will stay in $(t, \infty]$ from the n-th term onwards.

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

has limit 0. We will look at this more carefully soon.

Simple Examples of limits

Let us look at some examples of sequences and try to see what their limits are.

The simplest sequence is a *constant sequence* which keeps repeating the same value. For example,

$$3, 3, 3, 3, \ldots.$$

The limit of the this sequence is 3: clearly every neighborhood of 3 is hit eventually by the sequence (indeed it is hit every time, since the sequence is stuck at 3).

The sequence

$$-1, 4, 5, 7, 8, 8, 8, 8, 8, \ldots.$$

which *eventually* stabilizes at the constant value 8 has limit 8. For, again, given any neighborhood of 8 the sequence falls inside this neighborhood eventually and stays there.

In contrast, the sequence

$$1, 3, 4, 1, 3, 4, 1, 3, 4, 1, 3, 4, \ldots$$

does not have a limit. For example, the point 3 cannot be the limit of the sequence because, for instance,

$$(2.5, 3.5)$$

is a neighborhood of 3, but the sequence keeps falling outside this neighborhood (when it hits 1 or 4).

The sequence

$$1, 3, 5, 7, \ldots$$

has limit ∞. If you take any neighborhood of ∞, an interval of the form

$$(t, \infty]$$

then eventually the sequence falls inside the nighborhood and stays in there.

The sequence $1/n$

Consider again the sequence

$$1, 1/2, 1/3, 1/4, \ldots$$

It is intuitively clear that this sequence has limit 0. But let us *prove* this.

Note first that the n-th term of the sequence is

$$x_n = 1/n.$$

We have to show that given any neighborhood of 0, our sequence will eventually lie inside this neighborhood. So consider any neighborhood of 0:

$$(-\varepsilon, \varepsilon),$$

where ε is a positive real number. We have to show that x_n lies in $(-\varepsilon, \varepsilon)$ for all n beyond some value. Thus we should show that

$$\frac{1}{n} < \varepsilon$$

for all n beyond some initial value. Now the condition $1/n < \varepsilon$ is equivalent to

$$n\varepsilon > 1.$$

The archimedean property guarantees the existence of an $n_0 \in \mathbb{P}$ for which

$$n_0 \varepsilon > 1.$$

Therefore, $n\varepsilon > 1$ for all integers $n \geq n_0$. This proves that the sequence does indeed tend to the limit 0:

$$\frac{1}{n} \to 0, \qquad \text{as } n \to \infty$$

The sequence R^n

Consider the sequence

$$3^0, 3^1, 3^2, 3^3, \ldots$$

It is clear that in this sequence the terms get large very quickly, and intuitively it is clear that the sequence has limit ∞.

Consider now the sequence of powers

$$R^n$$

where $R > 1$.

Let us see by how much each term exceeds the preceding:

$$R^n - R^{n-1} = R^{n-1}(R-1)$$

But R is > 1, and so the multiplier R^{n-1} is ≥ 1 (it is equal to 1 when n is actually 1). Thus,

$$R^n - R^{n-1} \geq R - 1.$$

Let us write x for $R-1$, and note that

$$x > 0$$

and we have

$$R^n - R^{n-1} > x.$$

Now it takes n 'steps' to climb from R^0 to R^n, and so

$$R^n \geq 1 + nx,$$

because each step is at least x. Now it is clear that $R^n \to \infty$ as $n \to \infty$. We just have to show that for any given t,

$$1 + nx > t$$

when n is large enough. But this is just Archimedes again: some multiple $n_0 x$ of $x$exceeds $t - 1$, and then

$$nx > t - 1$$

for all $n \geq n_0$. Note again that it is important that $x > 0$, and this comes from the fact that $R > 1$.

Thus,

$$\lim_{n\to\infty} R^n = \infty \qquad \text{for all } R > 1.$$

Now consider the case $R = 1$. In this case the sequence is just all powers of 1, and so it is

$$1, 1, 1, 1, \ldots.$$

Thus

$$\lim_{n\to\infty} R^n = 1 \qquad \text{if } R = 1.$$

Next consider the case

$$0 < R < 1.$$

As an example, we have $R = 1/3$ and the sequence

$$1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \cdots$$

We have here a sequence with denomiantor going to ∞ and numerator fixed at 1. It seems then clear that the sequence goes to 0. Indeed, for any real $\varepsilon > 0$ we just have to wait till

$$3^n > 1/\varepsilon$$

and this would ensure that

$$\frac{1}{3^n} < \varepsilon,$$

and so

$$\frac{1}{3^n} \in (-\varepsilon, \varepsilon) \qquad \text{for large } n.$$

Now consider the general case of

$$R^n$$

where $R > 1$. Observe that

$$R = \frac{1}{r},$$

where $r = R^{-1}$ is > 1. Thus

$$R^n = \frac{1}{r^n},$$

where $r > 1$. Now, as we have seen above, the denominator r^n goes to infinity, and so it seems clear that $1/r^n$ should decrease to 0. So we will prove that

$$R^n \to 0$$

in this case. Consider any neighborhood of 0:

$$(-\varepsilon, \varepsilon)$$

where $\varepsilon > 0$ is a positive real number. We want R^n to be in this neighborhood, i.e. R^n should be $< \varepsilon$. But this means we should show that r^n is $> 1/\varepsilon$ for all n large enough. But we know that

$$r^n \to \infty$$

So, after some n_0, we have

$$r^n > \varepsilon^{-1}$$

for all $n \geq n_0$. Consequently,

$$R^n = \frac{1}{r^n} < \varepsilon$$

and so

$$R^n \in (\varepsilon, \varepsilon),$$

for all $n \geq n_0$. Thus

$$\lim_{n\to\infty} R^n = 0 \qquad \text{if } 0 < R < 1.$$

Now it is also clear that

$$\lim_{n\to\infty} R^n = 0 \qquad \text{if } R = 0.$$

Next consider negative values. Suppose

$$-1 < R < 0.$$

For example, for $R = -1/3$ we have the sequence

$$1, -\frac{1}{3}, \frac{1}{3^2}, -\frac{1}{3^3}, \frac{1}{3^4}, \ldots$$

It is clear that this is the same sequence as $1/3^n$ except it swings back and forth between negative and positive values. Thus, this sequence has limit 0.

More generally, if

$$-1 < R < 0$$

we can look at the distance between R^n and 0:

$$|R^n - 0| = |R^n| = |R|^n$$

and

$$|R|^n \to 0 \qquad \text{as } n \to \infty,$$

because

$$0 < |R| < 1.$$

Thus, the conclusion is that

$$\lim_{n\to\infty} R^n = 0 \qquad \text{if } -1 < R < 0.$$

In short,

$$\boxed{\lim_{n\to\infty} R^n = 0 \qquad \text{if } |R| < 1.} \tag{2.14}$$

Lastly, one should look at the case

$$R \leq -1.$$

You should examine a few examples and convince yourself that then there is no limit, for the sequence swings back and forth between widely separated positive and negative values.

Exerise Examine the sequence given by powers of -2:

$$1, -2, 4, -8, 16, \ldots$$

Does this sequence visit every neighborhood of ∞? Does it visit every neighborhood of $-\infty$?