The Extended Real Line and Its Topology

In this chapter we study topological concepts in the context of the real line. For technical purposes, it will be convenient to extend the real line $\mathbb{R}$ by adjoining to it a largest element ∞ and a smallest element $-\infty$. No metaphysical meaning need be attached to these infinities. The primary reason for introducing them is to simplify the statements of several theorems.

2.1 The extended real line

The extended real line is obtained by a largest element ∞, and a smallest element $-\infty$, to the real line $\mathbb{R}$:

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\} \tag{2.1}$$

Here ∞ and $-\infty$ are abstract elements. We extend the order relation to $\mathbb{R}$ by declaring that

$$-\infty < x < \infty \qquad \text{for all } x \in \mathbb{R} \tag{2.2}$$

Much of our work will be on $\mathbb{R}^*$, instead of just $\mathbb{R}$.

We define addition on $\mathbb{R}^*$ as follows:

$$x + \infty = \infty = \infty + x \quad \text{for all } x \in \mathbb{R}^* \text{ with } x > -\infty \tag{2.3}$$

$$y + (-\infty) = -\infty = (-\infty) + y \quad \text{for all } y \in \mathbb{R}^* \text{ with } y < \infty. \tag{2.4}$$

Note that

$$\infty + (-\infty) \text{ is } \textit{not defined},$$

i.e. there is no useful or consistent definition for it.

The following algebraic facts continue to hold in $\mathbb{R}^*$:

$$x + y = y + x, \qquad x + (y + z) = (x + y) + z, \tag{2.5}$$

whenever either side of these equations holds (i.e. if one side is defined then so is the other and the equality

2.2 Neighborhoods

A *neighborhood* of a point $p \in \mathbb{R}$ is an interval of the form

$$(p-\delta, p+\delta)$$

where $\delta > 0$ is any positive real number. Thus, the neighborhood consists of all points distance less than δ from p:

$$(p-\delta, p+\delta) = \{x \in \mathbb{R} : |x-p| < \delta\}. \tag{2.6}$$

For example,

$$(1.2, 2.8)$$

is a neighborhood of 2.

A typical neighborhood of 0 is of the form

$$(-\varepsilon, \varepsilon)$$

for any positive real number ε.

A *neighborhood of* ∞ in $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ is a ray of the form

$$(t, \infty] = \{x \in \mathbb{R}^* : x > t\}$$

with t any real number. For example,

$$(5, \infty]$$

is a neigborhood of ∞.

A *neighborhood of* $-\infty$ in $\mathbb{R}^*$ is a ray of the form

$$[-\infty, s) = \{x \in \mathbb{R}^* : x < s\}$$

where $s \in \mathbb{R}$. An example is

$$[-\infty, 4)$$

Observe that *if U and V are neighborhoods of p then $U \cap V$ is also a neighborhood of p*. In fact, either U contains V as a subset of vice versa, and so $U \cap V$ is just the smaller of the two neighborhoods.

Observe also that if N is a neighborhood of a point p, and if $q \in N$ then q had a neighborhood lying entirely inside N. For example, the neighborhood $(2,4)$ of 3 contains 2.5, and we can form the neighborhood $(2,3)$ of 2.5 lying entirely inside $(2,4)$.

Here is a simple but fundamental observation:

$$\textit{Distinct points of } \mathbb{R}^* \textit{ have disjoint neighborhoods.} \tag{2.7}$$

This is called the Hausdorff property of $\mathbb{R}^*$.

For example, 3 and 5 have the neighborhoods

$$(2,4) \qquad \text{and} \qquad (4.5,5.5)$$

The points 2 and ∞ have disjoint neighborhoods, such as

$$(-1,5) \qquad \text{and} \qquad (12,\infty]$$

Exerise Give examples of disjoint neighborhoods of

(i) 2 and -4

(ii) $-\infty$ and 5

(iii) ∞ and $-\infty$

(iv) 1 and -1

2.3 Types of points for a set

Consider a set

$$S \subset \mathbb{R}^*.$$

A point $p \in \mathbb{R}^*$ is said to be an *interior point* of S if it has a neighborhood U lying entirely inside S, i.e. with

$$U \subset S.$$

For example, for the set

$$E = (-4,5] \cup \{6,8\} \cup [9,\infty],$$

the points $-2, 3, 11$ are interior points. The point ∞ is also an interior point of E.

A point p is an *exterior point* if it has a neighborhood U lying entirely outside S, i.e.

$$U \subset S^c.$$

For example, for the set E above, points -5, 7, and $-\infty$ are exterior to E.

A point which is neither interior to S nor exterior to S is a *boundary point* of S. Thus p is a boundary point of S if every neighborhood of p intersects both S and S^c.

In the example above, the boundary points of E are

$$-4, 5, 6, 8, 9, \infty.$$

Next consider the set

$$\{3\} \cup (5, \infty)$$

The boundary points are $3, 5$, and ∞. It is important to observe that if we work with the real line $\mathbb{R}$ instead of the extended line $\mathbb{R}^*$ then we must exclude ∞ as a boundary point, because it doesn't exist as far as $\mathbb{R}$ is concerned.

Example For the set $A = [-\infty, 4) \cup \{5, 9\} \cup [6, 7)$, decide which of the following are true and which false:

(i) -6 is an interior point (T)

(ii) 6 is an interior point (F)

(iii) 9 is a boundary point (T)

(iv) 5 is an interior point (F)

Exerise For the set $B = [-\infty, -5) \cup \{2, 5, 8\} \cup [4, 7)$, decide which of the following are true and which false:

(i) -6 is an interior point

(ii) -5 is an interior point

(iii) 5 is a boundary point

(iv) 4 is an interior point

(v) 7 is a boundary point.

Interior, Exterior, and Boundary of a Set

The set of all interior points of a set S is denoted

$$S^0$$

and is called the *interior* of S.

The set of all boundary points of S is denoted

$$\partial S$$

and is called the *boundary* of S.

The set of all points exterior to S is the *exterior* of S, and we shall denote it

$$S^{\text{ext}}.$$

Thus, the whole extended line $\mathbb{R}^*$ is split up into three disjoint pieces:

$$\boxed{\mathbb{R}^* = S^0 \cup \partial S \cup S^{\text{ext}}} \tag{2.8}$$

Recall that a point p is on the boundary of S if every neighborhood of the point intersects both S and S^c. But this is exactly the condition for p to be on the boundary of S^c. Thus

$$\boxed{\partial S = \partial S^c.} \tag{2.9}$$

The interior of the entire extended line $\mathbb{R}^*$ is all of $\mathbb{R}^*$. So

$$\partial \mathbb{R}^* = \emptyset.$$

Example For the set $A = [-\infty, 4) \cup \{5, 9\} \cup [6, 7)$,

(i) $A^0 = [-\infty, 4) \cup (6, 7)$

(ii) $\partial A = \{4, 5, 9, 7, 6\}$

(iii) $A^c = [4, 5) \cup (5, 6) \cup [7, 9) \cup (9, \infty]$

(iv) the interior of the complement A^c is

$$(A^c)^0 = (4, 5) \cup (5, 6) \cup (7, 9) \cup (9, \infty]$$

For the set

$$G = (3, \infty)$$

the boundary of G, when viewed as a subset of $\mathbb{R}^*$, is

$$\partial G = \{3, \infty\}.$$

But if we decide to work only inside $\mathbb{R}$ then the boundary of G is just $\{3\}$.

Exerise For the set $B = \{-4, 8\} \cup [1, 7) \cup [9, \infty)$,

(i) $B^0 =$

(ii) $\partial B =$

(iii) $B^c =$

(iv) the interior of the complement B^c is

$$(B^c)^0 =$$

2.5 Open Sets and Topology

We say that a set is *open* if it does not contain any of its boundary points. For example,

$$(2, 3) \cup (5, 9)$$

is open.

Also

$$(4, \infty]$$

is open.

But

$$(3, 4]$$

is not open, because the point 4 is a boundary point.

The entire extended line $\mathbb{R}^*$ is open, because it has no boundary points.

Moreove, the empty set $\emptyset$ is open, because, again, it doesn't have any boundary points.

Notice then that every point of an open set is an interior point. Thus, a set S is open means that

$$S^0 = S.$$

Thus for an open set S each point has a neighborhood contained entirely inside S. In other words, *S is made up of a union of neighborhoods.*

Viewed in this way, it becomes clear that the *union of open sets is an open set*.

Now consider two open sets A and B. We will show that $A \cap B$ is open. Take any point

$$p \in A \cap B.$$

Then p is in both A and B. Since $p \in A$ and since A is open, there is a neighborhood U of p which is a subset of A:

$$U \subset A.$$

Similarly there is a neighborhood V of p which is a subset of B:

$$V \subset B$$

But then $U \cap V$ is a neighborhood of p which is a subset of both A and B:

$$U \cap V \subset A \cap B.$$

Thus every point in $A \cap B$ has a neighborhood lying inside $A \cap B$. Consequently, $A \cap B$ is open.

Now consider three open sets A, B, C. The intersection

$$A \cap B \cap C$$

can be viewed as

$$(A \cap B) \cap C$$

But here both $A \cap B$ and C are open, and hence so is their intersection. Thus,

$$A \cap B \cap C$$

is open. This type of argument works for any *finite* number of open sets. Thus:

The intersection of a finite number of open sets is open.

Exerise Check that the intersection of the sets $(4, \infty)$ and (-35) and $(2, 6)$ is open.

The collection of all open subsets of $\mathbb{R}$ is called the *topology* of $\mathbb{R}$.

The set of all open subsets of $\mathbb{R}^*$ is called the *topology* of $\mathbb{R}^*$.

2.6 Closed Sets

A set S is said to be *closed* if it contains all its boundary points.

In other words, S is closed if

$$\partial S \subset S$$

Thus,

$$[4,8] \cup [9,\infty]$$

is closed.

But

$$[4,5)$$

is not closed because the boundary point 5 is not in this set.

The set

$$[3,\infty)$$

is not closed (as a subset of $\mathbb{R}^*$) because the boundary point ∞ is not inside the set. But, viewed as a subset of $\mathbb{R}$ it is closed. So we need to be careful in deciding what is close and what isn't: a set may be closed viewed as a subset of $\mathbb{R}$ but not as a subset of $\mathbb{R}^*$.

The full extended line $\mathbb{R}^*$ is closed.

The empty set $\emptyset$ is also closed.

Note that the sets $\mathbb{R}^*$ and $\emptyset$ are both open and closed.

2.7 Open Sets and Closed Sets

Consider a set $S \subset \mathbb{R}^*$.

If S is open then its boundary points are all outside S:

$$\partial S \subset S^c.$$

But recall that the boundary of S is the same as the boundary of the complement S^c. Thus, for S to be open we must have

$$\partial(S^c) \subset S^c,$$

which means that S^c contains all its boundary points. But this means that S^c is closed.

Thus, *if a set is open then its complement is closed.*

The converse is also true: if a set is closed then its complement is open. Thus,

Theorem 8 *A subset of $\mathbb{R}^*$ is open if and only if its complement is closed.*

Exercise. Consider the open set $(1,5)$. Check that its complement if closed.
Exercise. Consider the closed set $[4,\infty]$. Show that its complement is open.

2.8 Closed sets in $\mathbb{R}$ and in $\mathbb{R}^*$

The set

$$[3,\infty)$$

is closed in $\mathbb{R}$, but is *not closed* in $\mathbb{R}^*$. This is because in $\mathbb{R}$ it has only the boundary point 3, which it contains; in contrast, in $\mathbb{R}^*$ the point ∞ is also a boundary point and is not in the set. Thus, when working with closed sets it is important to bear in mind the distinction between being closed in $\mathbb{R}$ and being closed in $\mathbb{R}^*$. There is no such distinction for open sets.

2.9 Closure of a set

The *closure* of a set S is obtained by throwing in all its boundary points:

$$\boxed{\overline{S} = S \cup \partial S} \tag{2.10}$$

Of course, *if S is closed then its closure is itself.*

For example, the closure of $(3,5)$ is $[3,5]$. The closure of

$$(3,\infty)$$

is $[3,\infty]$. (But, in $\mathbb{R}$ the closure of $(3,\infty)$ is $[3,\infty)$.)

Let us see what the closure of $\mathbf{Q}$ is. Now *every* point in $\mathbb{R}^*$ is a boundary point of $\mathbf{Q}$:

$$\partial\mathbf{Q} = \mathbb{R}^*,$$

because any neighborhood of any point contains both rationals (points in $\mathbf{Q}$) and irrationals (points outside $\mathbf{Q}$).

It is useful to think of the closure

$$\overline{S} = S \cup \partial S$$

in this way:

A point p is in $\overline{S}$ if and only if every neighborhood of p intersects S.

2.10 The closure of a set is closed

Consider the closure $\overline{S}$ of a set S. We will show that $\overline{S}$ is a closed set.

Take any boundary point $p \in \partial\overline{S}$. We have to show that p is actually in $\overline{S}$. Now let N be any neighborhood of p. Then N contains a point q in $\overline{S}$. Choose (as we may) a neighborhood W of q lying entirely inside N. Since $q \in \overline{S}$ it follows that W contains a point of S. Thus, N contains a point of S. So we have seen that every neighborhood of p contains a point of S. This means $p \in \overline{S}$. Thus, we have shown that every boundary point of $\overline{S}$ is in S, and so $\overline{S}$ is closed.

2.11 $\overline{S}$ is the smallest closed set containing S

Consider any closed set K with

$$S \subset K$$

Take any $p \in \overline{S}$. Then every neighborhood of p intersects S, and hence also K. Thus every neighborhood of p intersects K. So, either p is in the interior of K or on its boundary. But K is closed, and so in either case p is in K. Thus,

$$\overline{S} \subset K$$

What we hae shown then is that $\overline{S}$ is the *smallest closed set containing S as a subset.*

2.12 $\mathbb{R}^*$ is compact

There is a special property of $\mathbb{R}^*$ whose full significance is best appreciated at a later stage. However, we have the language and tools to state and prove it now and shall do so.

An *open cover* of $\mathbb{R}^*$ is a collection $\mathcal{U}$ of open sets whose union covers all of $\mathbb{R}^*$. For example,

$$\{[-\infty,5),(-1,8),(7,9)\cup(11,15),(8,\infty]\}$$

is an open cover of $\mathbb{R}^*$. It is best to draw a picture for yourself to make this clear.

More formaly, an open cover of $\mathbb{R}^*$ is a set $\mathcal{U}$ of open subsets of $\mathbb{R}^*$ such that every point $x \in \mathbb{R}^*$ falls inside some open set U in the collection $\mathcal{U}$, i.e. for each $x \in \mathbb{R}^*$ there exists $U \in \mathcal{U}$ such that

$$x \in U \in \mathcal{U}.$$

Another example of an open cover of $\mathbb{R}^*$ is

$$\{[-\infty,50),(2,100),(8,200)\cup(701,800),(150,750),(760,\infty]\}.$$

The examples of open covers of $\mathbb{R}^*$ given above are both *finite* collections of sets. Let us look at an example of a cover which uses infinitely many sets.

We start with an open cover of $\mathbb{R}$, as opposed to $\mathbb{R}^*$: take all intervals of the form $(a,a+2)$, where a runs over all integers:

$$...,(-8,-6),(-7,-5),...(-2,0),(-1,1),(0,2),(1,3),...$$

In short we are looking at the collection

$$\mathcal{U}'=\{(a,a+2):a\in\mathbf{Z}\},$$

where, recall that $\mathbf{Z}$ is the set of all integers. This collection fails to include $-\infty$ and ∞.

To obtain an open cover of $\mathbb{R}^*$ from $\mathcal{U}'$ we could, for example, just throw in the open sets $[-\infty,-4)$ and $(5,\infty]$. Thus, this gives us an open cover of $\mathbb{R}^*$:

$$\mathcal{U}=\{[-\infty,-4),(5\infty]\}\cup\{(a,a+2):a\in\mathbf{Z}\}.$$

If you look at this, however, you realize that even though this collection does contain infinitely many open sets, we don't reall *need* all of them to cover $\mathbb{R}^*$. Indeed, we could just use the sub-collection

$$\mathcal{V}=\{[-\infty,-4),(-3,-1),(-2,0),(-1,1),(0,2),(1,3),(2,4),(3,5),(4,6),(5,\infty]\}$$

and this would cover all of $\mathbb{R}^*$.

This is not just a feature of one particular example. It turns out that

Every open cover of $\mathbb{R}^*$ *has a finite subcover.*

This property is called *compactness*, and so

Theorem 9 $\mathbb{R}^*$ *is compact.*

We can prove the compactness of $\mathbb{R}^*$ using the completeness of the real line.

Proof of compactness of $\mathbb{R}^*$. Let $\mathcal{U}$ be an open cover of $\mathbb{R}^*$. This is a collection of open sets such that every point of $\mathbb{R}^*$ falls inside some set in the collection. In particular, $-\infty$ is in some open set $W\in\mathcal{U}$. Thus W contains a neighborhood of $-\infty$, i.e.

$$[-\infty,t)\subset W$$

for some $t \in \mathbb{R}$. So, for one thing, all the elements of $\mathbb{R}^*$ less than t are covered by just the one set W in $\mathcal{U}$; we would not need any other set from the cover if we are to stick to points to the left of t. Now let

$$S = \{x \in \mathbb{R}^* : \text{finitely many sets in } \mathcal{U} \text{ cover } [-\infty, x]\}$$

This set is not empty because $-\infty \in S$. In fact, $t-1$ is also in S. By completeness, the set S has a supremum $\sup S$. Let

$$m = \sup S.$$

We will argue in three steps:

(i) First we explain that m isn't $-\infty$.

(ii) Next we prove that m is in fact, ∞.

(iii) Finally we prove that $[\infty, m]$, i.e. all of $\mathbb{R}^*$, can be covered by just finitely many sets in $\mathcal{U}$.

For (i), note, as before, that there is a real number t such that $t-1$ is in S, and so m (being an upper bound of S) has to be $\geq t-1$. In particular, m is certainly not $-\infty$.

Now we show that $m = \infty$. Suppose to the contrary that $m < \infty$. Now there is some open set $U \in \mathcal{U}$ such that $m \in U$, because $\mathcal{U}$ covers all points of $\mathbb{R}^*$. Since m is not $-\infty$, and is being assumed to be also not ∞, it is in $\mathbb{R}$ and so there is some neighborhood

$$(m-\varepsilon, m+\varepsilon) \subset U,$$

where ε is a positive real number. Now $m-\varepsilon$ being *less* than the *least* upper bound m of S, there has to be some $x \in S$ which is greater than $m-\varepsilon$. Thus

$$m-\varepsilon < x \leq m \text{ for some } x \in S.$$

So, x being in S, there are finitely many sets, say $U_1, \ldots, U_N$, in $\mathcal{U}$, which cover the ray segment $[-\infty, x]$. The set U covers $(m-\varepsilon, m+\varepsilon)$. Thus, the collection

$$\{U_1, \ldots, U_N, U\}$$

covers all of

$$[-\infty, m+\varepsilon).$$

But this means that, for example, $m+\frac{1}{2}\varepsilon$ is in S, for the ray segment $[-\infty, m+1/2]$ is covered by the finite collection $U_1,\ldots,U_N,U$. But now we have a contradiction, because we have found a number, $m+\varepsilon/2$, greater than m, which is in S. Thus, our original hypothesis concerning m must have been wrong. So $m=\infty$.

Finally we prove that $\mathbb{R}^*$ is covered by finitely many sets in $\mathcal{U}$. The element $\infty \in \mathbb{R}^*$ falls inside some open set V in the collection $\mathcal{U}$. Therefore, there is some 'ray-neighborhood' of ∞

$$(r,\infty] \subset V,$$

for some real number r. Now r being less than $m=\infty$, and the latter being the *least* upper bound of S, there must be some $y \in (r,\infty]$ which is in S. Thus,

$$[-\infty, y]$$

is covered by finitely many open sets, say $V_1,\ldots,V_k$, in $\mathcal{U}$. Note that

$$[-\infty,y] \cup (r,\infty] = \mathbb{R}^*.$$

But then

$$V_1,\ldots,V_k,V$$

cover all of $[-\infty,\infty]$, since V covers the segment $(r,\infty]$. This prove that finitely many sets from $\mathcal{U}$ cover all of $\mathbb{R}^*$.

2.13 Compactness of closed subsets of $\mathbb{R}^*$.

Consider now any closed subset D of $\mathbb{R}^*$. We will prove that it is compact, i.e. that any open cover of D has a finite sub-cover.

Consider any open cover $\mathcal{U}$ of D. This is a collection of open sets such that every point of D is covered by some set in the collection. Now the set D^c, being the complement of a closed set, is an open set. Throw this into the collection, and consider

$$\mathcal{U}' = \mathcal{U} \cup \{D^c\}.$$

This covers *all* of $\mathbb{R}^*$: any point of D would be covered by a set in $\mathcal{U}$ while any point in D^c is, of course, covered by D^c. Then we know that there has to be a finite sub-collection

$$\mathcal{V}' \subset \mathcal{U}'$$

which covers $\mathbb{R}^*$. Now throw out D^c from $\mathcal{V}'$ in case it is there, and consider the collection

$$\mathcal{V} = \mathcal{V}' \setminus \{D^c\}.$$

Of course, this is a finite subcollection of $\mathcal{U}$. Moreover, it covers all points of D, because no point in D could have been covered by the 'rejected' set D^c. Thus, D is covered by a finite sub-collection of sets from $\mathcal{U}$.

Thus, every closed subset of $\mathbb{R}^*$ is compact. The converse is also true, and we can state:

Theorem 10 *A subset of $\mathbb{R}^*$ is compact if and only if it is closed.*

We will leave out the proof of the converse part of this result.

2.14 The Heine-Borel Theorem: Compact subsets of $\mathbb{R}$

A subset B of $\mathbb{R}$ is said to be *bounded* if

$$B \subset [-N, N]$$

for some real number N. Thus, for B to be bounded, there should exist a real number N such that

$$|x| < N \text{ for all } x \in B.$$

Recall that a subset of $\mathbb{R}$ is closed in $\mathbb{R}$ if it contains no boundary point. Equivalently, if a subset of $\mathbb{R}$ is closed if its complement in $\mathbb{R}$ is open.

Consider, for instance, the set

$$[4, \infty) \subset \mathbb{R}.$$

This is a closed subset of $\mathbb{R}$ because, in $\mathbb{R}$, its only boundary point is 4, and this point lies in the set. Note that $[4, \infty)$ is *not* closed when considered as a subset of $\mathbb{R}^*$.

We can now state the **Heine-Borel theorem**:

Theorem 11 *Every closed and bounded subset of $\mathbb{R}$ is compact. Conversely, every compact subset of $\mathbb{R}$ is closed and bounded.*

We shall prove half of this. Suppose $K \subset \mathbb{R}$ is closed and bounded as a subset of $\mathbb{R}$. Boundedness implies that

$$K \subset [-N, N]$$

for some real number N. Then K is closed also as a subset of $\mathbb{R}^*$. Let's check this. We will show that the complement U of K in $\mathbb{R}^*$ is open. Consider any point $p \in U$. If $p \in \mathbb{R}$ then we already know, by closedness of K, that p has a neighborhood lying inside U. If $p = \infty$ then the neighborhood of p given by

$$(N+1, \infty]$$

lies entirely outside K. If $p = -\infty$ then the neighborhood

$$[-\infty, -N-1)$$

is entirely outside K. Thus, in all cases, p has a neighborhood outside K in $\mathbb{R}^*$. So the complement of K in $\mathbb{R}^*$ is open, and hence K is closed in $\mathbb{R}^*$. We know that then K must be compact.