1. Steady state has $\frac{1}{4}Q(1-a) = \sigma T^4 \Rightarrow T = \left[\frac{Q(1-a)}{4\sigma}\right]^{1/4}$.

$Q = \frac{Q_{Earth}}{R^2}$ where R is distance from sun to planet in astronomical units (since energy flux from sun falls off according to inverse square law)

So plugging in numbers,

$T_{Earth} \approx 255\ K$

$T_{Venus} \approx 228\ K$

$T_{Mars} \approx 217\ K$

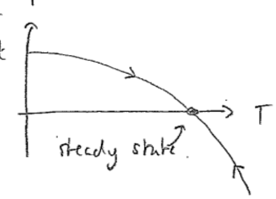
$T_{Jupiter} \approx 98\ K$.

Discrepancies are likely to be due to the greenhouse effect. and perhaps (in the case of Jupiter) to internal heat generation.

The stability of the steady state $\bar{T}$ can be examined by writing $T = \bar{T} + \theta$ where $\theta \ll \bar{T}$, then $\rho c d \frac{d\theta}{dt} \approx -4\sigma\bar{T}^3\theta$ (linearising)

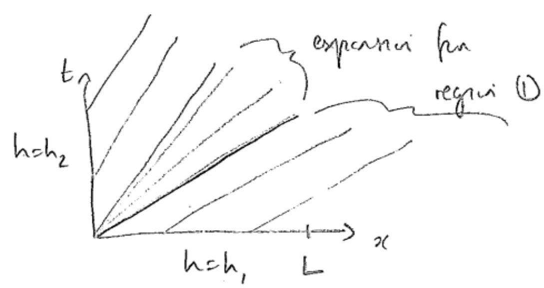
$\Rightarrow$ stable (since any initial perturbation θ will decay to zero)

Alternatively, plot $\frac{dT}{dt}$ as function of T



Trajectories approach steady state

2. $\frac{\partial h}{\partial t} + h^m \frac{\partial h}{\partial x} = 0$



Characteristic equations $\dot{t} = 1 \quad \dot{x} = h^m \quad \dot{h} = 0 \qquad \dot{} =$ derivative along characteristic

Initial conditions for characteristics from $t=0$: $t=0,\ x = x_0,\ h = h_1$

$\Rightarrow \boxed{h = h_1},\ x = x_0 + h_1^m t$

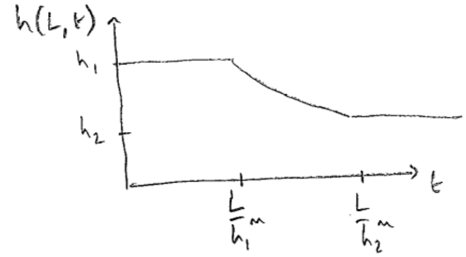
so these characteristics sweep out region ① on diagram,

'Initial' conditions for characteristics from $x=0$: $t = t_0,\ x = 0,\ h = h_2$

$\Rightarrow \boxed{h = h_2},\ x = h_2^m (t - t_0)$

sweep out region ② on diagram.

In between regions ① and ② there must be an expansion fan, where characteristics come from the origin. From the characteristic equation h is constant along each characteristic, which therefore have equations $x = h^m t$. So $\boxed{h = \left(\frac{x}{t}\right)^{\frac{1}{m}}}$



If $h_2 > h_1$, a shock will form (because the two sets of characteristics intersect)

This will move with speed given by Rankine-Hugoniot conditions

$$[h]_-^+ V = \left[\frac{1}{m+1} h^{m+1}\right]_-^+ \quad \text{ie} \quad V = \frac{h_2^{m+1} - h_1^{m+1}}{(m+1)(h_2 - h_1)}$$

3. $\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}$ $\quad \frac{\partial T}{\partial z} \to 0$ as $z \to \infty$. $\quad T = T_0 - \Delta T \cos \omega t$.

Scale $\quad T = T_0 + \Delta T \hat{T}$ $\qquad$ ^ variables are dimensionless.

$z = [z]\hat{z}$

$t = [t]\hat{t}$ $\qquad$ Choose $[t] = \frac{1}{\omega}$. $\quad [z] = \left(\frac{k}{\rho c \omega}\right)^{1/2}$

(to achieve balance of terms indicated by arrows).

$$\frac{\rho c [T]}{[t]} \frac{\partial \hat{T}}{\partial \hat{t}} = \frac{k[T]}{[z]^2} \frac{\partial^2 \hat{T}}{\partial \hat{z}^2}$$

$\Rightarrow \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2}$ with $\frac{\partial T}{\partial z} \to 0$ as $z \to \infty$ $\quad T = -\cos t$. $\qquad$ (dropped hats).

$T = -\mathrm{Re}(e^{it})$ at $z=0$, so look for solution of form $T = -\mathrm{Re}(e^{it} f(z))$

$\Rightarrow -if = f''$ with $f' \to 0$ as $z \to \infty$, $f = 1$ at $z = 0$.

$\Rightarrow f = e^{-\frac{1+i}{\sqrt{2}}z} = e^{-z/\sqrt{2}} e^{-iz/\sqrt{2}}$.

so $$\boxed{T = -e^{-z/\sqrt{2}} \cos\left(t - z/\sqrt{2}\right).}$$

Temperature decays over dimensionless distance $\sim \sqrt{2}$, so dimensional distance is $z \sim \left(\frac{2k}{\rho c \omega}\right)^{1/2} \sim \left(\frac{kP}{\rho c \pi}\right)^{1/2}$ where $P = \frac{2\pi}{\omega}$ is the period of the forcing.

For $\rho = 1600\ \mathrm{kg\,m^{-3}}$, $c = 800\ \mathrm{J\,kg^{-1}K^{-1}}$, $k = 1\ \mathrm{W m^{-1} K^{-1}}$,

$P = 1d \to z \sim 0.15\,m$

$P = 1y \to z \sim 2.8\,m$

For ice ($\rho = 900\ \mathrm{kg\,m^{-3}}$ $c = 2000\ \mathrm{J\,kg^{-1}K^{-1}}$, $k = 2\ \mathrm{W m^{-1} K^{-1}}$) the numbers change by only a small amount (a factor of $\left(\frac{K_{ice}}{K_{soil}}\right)^{1/2} \approx 1.2$) to $z \sim 0.17\,m$, $z \sim 3.3\,m$.

This would not be a good model for the temperature near the surface of a lake because it is likely that the temperature variables cause convection (due to density changing with temperature), which the heat equation ignores.

4. $$B_\nu = \frac{2h\nu^3}{c^2\left(e^{h\nu/kT} - 1\right)} \qquad u = \frac{h\nu}{kT} \qquad du = \frac{h}{kT}\,d\nu.$$

$$B = \int_0^\infty B_\nu\,d\nu = \int_0^\infty \frac{2h}{c^2}\left(\frac{kT}{h}\right)^3 \frac{u^3}{e^u - 1}\left(\frac{kT}{h}\right) du$$

$$= \frac{2k^4T^4}{c^2h^3} \underbrace{\int_0^\infty \frac{u^3}{e^u - 1}\,du}_{I}$$

$$I = \int_0^\infty \frac{u^3 e^{-u}}{1 - e^{-u}}\,du = \int_0^\infty \sum_{n=0}^\infty u^3 e^{-u} e^{-nu}\,du$$

(Binomial expansion on $\frac{1}{1-e^{-u}}$) converges uniformly.

$$= \sum_{n=1}^{\infty} \int_0^{\infty} u^3 e^{-nu}\, du \qquad nu = v \qquad du = \frac{1}{n}\, dv$$

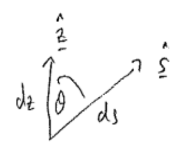
$$= \sum_{n=1}^{\infty} \frac{1}{n^4} \underbrace{\int_0^{\infty} v^3 e^{-v}\, dv}_{= 3! = 6.}$$

$$= 6 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$= \frac{\pi^4}{15}.$$

Hence $B = \frac{2k^4T^4}{c^2h^3}\,\frac{\pi^4}{15} = \frac{\sigma T^4}{\pi}$ where $\sigma = \frac{2\pi^5 k^4}{15h^3c^2}$

SECTION B



$dz = \cos\theta\, ds$

1. Two stream approximation.

Radiative transfer equation $\boxed{\frac{\partial I}{\partial s} = -k\rho\,(I - B)}$ for intensity of radiation travelling in direction $\hat{s}$

B: average intensity (local radiative equilibrium) $= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I \sin\theta\, d\theta\, d\phi = \frac{1}{2}(I_+ + I_-)$ [using definition below].

$= \frac{\sigma T^4}{\pi}$ [from integrating Planck's function].

Writing in terms of z, $\frac{\partial}{\partial s} = \cos\theta \frac{\partial}{\partial z} = \mu \frac{\partial}{\partial z}$, so $\boxed{\mu \frac{\partial I}{\partial z} = -k\rho\,(I - B) \qquad \mu = \cos\theta}$

For the two stream approximation, define $I_+ = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} I \sin\theta\, d\theta\, d\phi = \int_0^1 I\, d\mu.$

$$I_- = \frac{1}{2\pi}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} I \sin\theta\, d\theta\, d\phi = \int_{-1}^{0} I\, d\mu.$$

then approximate $\int_0^1 \mu \frac{\partial I}{\partial z}\, d\mu \approx \frac{1}{2}\frac{dI_+}{dz}$ and $\int_{-1}^0 \mu \frac{\partial I}{\partial z}\, d\mu \approx -\frac{1}{2}\frac{dI_-}{dz}$, and $F_+ = 2\pi\int_0^1 \mu I_+\, d\mu \approx \pi I_+$

$F_- = 2\pi \int_{-1}^0 \mu I_-\, d\mu \approx \pi I_-$

so the radiative transfer equation is integrated to give.

$$\frac{1}{2}\frac{dI_+}{dz} \approx -\kappa\rho\left(I_+ - B\right) = -\frac{1}{2}\kappa\rho\left(I_+ - I_-\right)$$
$$-\frac{1}{2}\frac{dI_-}{dz} \approx -\kappa\rho\left(I_- - B\right) = -\frac{1}{2}\kappa\rho\left(I_- - I_+\right)$$

Subtracting one equation from the other, we see $\frac{d}{dz}(I_+ - I_-) = 0$ so $I_+ - I_- = \frac{F}{\pi}$ is constant.

Hence $\frac{dI_+}{dz} = -\kappa\rho\frac{F}{\pi}$ and $\frac{dI_-}{dz} = -\kappa\rho\frac{F}{\pi}$.

with boundary conditions $\boxed{\begin{array}{l} I_- = 0 \text{ at } z = \infty \\ I_+ = \frac{\sigma T_s^4}{\pi} \text{ at } z = 0 \end{array}}$ (top of atmosphere) (Stefan-Boltzmann law for $F_+ = \pi I_+$).

Integrating $\Rightarrow$ $I_- = \frac{F}{\pi}\int_z^\infty \kappa\rho\, dz = \frac{F\tau}{\pi}$ and hence $I_+ = \frac{F}{\pi}(1+\tau)$

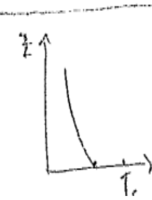
surface boundary condition $\Rightarrow$ $\sigma T_s^4 = F(1+\tau_s)$ $\Rightarrow$ $F = \frac{\sigma T_s^4}{1+\tau_s}$.

Since $F = \sigma T_e^4$ this means $\boxed{T_e^4 = \frac{T_s^4}{1+\tau_s}}$ so T_s is larger than T_e, and increases with increasing τ_s.

The air temperature is given by $B = \frac{1}{2}(I_+ + I_-) = \frac{F}{\pi}\left(\frac{1}{2}+\tau\right) = \frac{\sigma T_e^4}{\pi}\left(\frac{1}{2}+\tau\right)$ $\Rightarrow$ $\boxed{T = T_e\left(\frac{1}{2}+\tau\right)^{1/4}}$

"$\frac{\sigma T^4}{\pi}$"

so the ground air temperature $T|_{z=0} = T_e\left(\frac{1}{2}+\tau_s\right)^{1/4} < T_e(1+\tau_s)^{1/4} = T_s$.



2. Runaway greenhouse effect.

Clausius Clapeyron equation $\frac{\partial p_{sv}}{\partial T} = \frac{\rho_v L}{T}$ $\qquad p_{sv} = p_{sv0}$ at $T = T_0$.

$$= \frac{p_{sv}}{T^2}\frac{M_v L}{R} \qquad \left[\text{using } p_{sv} = \frac{\rho_v R T}{M_v}\right]$$

[integrate] $\Rightarrow$ $\frac{dp_{sv}}{p_{sv}} = \frac{M_v L}{R}\frac{dT}{T^2}$

$$\ln\frac{p_{sv}}{p_{sv0}} = \frac{M_v L}{R}\left\{\frac{1}{T_0} - \frac{1}{T}\right\}.$$

$$\Rightarrow \boxed{p_{sv} = p_{sv0}\, e^{a\left(1-\frac{T_0}{T}\right)}} \qquad \boxed{a = \frac{M_v L}{R T_0}}$$

If $T - T_0 \ll T_0$ write $1 - \frac{T_0}{T} = 1 - \frac{1}{1+\frac{T-T_0}{T_0}} = 1 - \left(1 - \frac{T-T_0}{T_0} + \dots\right) \doteq \frac{T-T_0}{T_0} + O\left(\frac{T-T_0}{T_0}\right)^2$

Energy balance $\frac{1}{4}Q = \sigma\gamma T^4$ $\Rightarrow$ $\boxed{T = \left(\frac{Q}{4\sigma\gamma}\right)^{1/4} = \left(\frac{Q}{4\sigma}\right)^{1/4}\left(1 + b\left(\frac{p_v}{p_{sv0}}\right)^c\right)}$

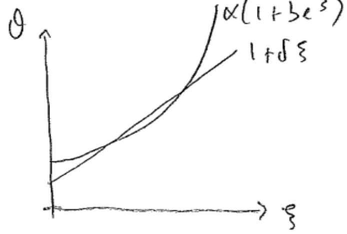
Write $\boxed{\theta = \frac{T}{T_0}}$, then energy balance gives $\theta = \underbrace{\left(\frac{Q}{4\sigma T_0^4}\right)^{1/4}}_{\alpha}\left(1 + b\left(\frac{p_v}{p_{sv0}}\right)^c\right)$

$$\Rightarrow \boxed{\theta = \alpha(1 + b e^{\xi_v})} \qquad \text{defining } \boxed{\xi_v = c\ln\frac{p_v}{p_{sv0}}}$$

With the same notation, the saturation curve becomes $\frac{\xi_{sv}}{c} = a(\theta - 1)$

(but $\xi_{sv} = c\ln\frac{p_{sv}}{p_{sv0}}$) $\qquad \Rightarrow \boxed{\theta = 1 + \delta\xi_{sv}} \qquad \delta = \frac{1}{ac}$.

The runaway greenhouse effect occurs if the temperature calculated from energy balance remains above the saturation curve for all p_v (i.e. ξ_v). So its occurrence depends on the non-intersection of $\theta = \alpha(1 + be^{\xi})$ with $\theta = 1 + \delta\xi$.



Clearly from the graph, non intersection will happen if α is large enough.

The critical α is found from when the curves meet tangentially, i.e. $\boxed{\begin{aligned} &\alpha(1+be^{\xi}) = 1 + \delta\xi \\ \&\ &\alpha b e^{\xi} = \delta. \end{aligned}}$

$$\Rightarrow \boxed{\alpha + \delta = 1 + \delta\xi = 1 + \delta\ln(\delta/\alpha b)}$$

If δ is small then, $\alpha \hat{=} 1$, and putting this back into the right hand side gives improved estimate

$\alpha \approx 1 + \delta\ln(\delta/b) - \delta$ $\qquad$ [the corrections are $O(\delta^2\ln\delta)$]

For the Earth $Q = 1370\ \mathrm{W m^{-2}}$, , $\sigma = 5.67 \times 10^{-8}\ \mathrm{W m^{-2} K^{-1}}$, $T_0 = 273\ \mathrm{K}$.

so $\boxed{\alpha = \left(\dfrac{Q}{4\sigma T_0^4}\right)^{\frac{1}{4}} \approx 1.02119.}$

$R = 8.3\ \mathrm{J mol^{-1} K^{-1}}$, $M_v = 18 \times 10^{-3}\ \mathrm{kg mol^{-1}}$, $L = 2.5 \times 10^6\ \mathrm{J kg^{-1}}$ $\Rightarrow$ $a \approx 19.9.$

$\Rightarrow$ $\delta \approx 0.2$ (using $c = 0.25$).

so $\boxed{\alpha_c \approx 1.05}$

So $\alpha < \alpha_c$, suggesting runaway greenhouse effect does not occur.

For Venus, Q is twice as large so α is increased by a factor of $2^{1/4}$ $\Rightarrow$ $\boxed{\alpha \approx 1.21}$

This is larger than α_c, so runaway greenhouse effect does occur. (ie the saturation vapour pressure is never reached).

If solar radiation were 30% smaller when the atmospheres were forming, this would not make a difference, since decreasing α by $(0.7)^{1/4} \approx 0.91$ does not change the conclusion that $\alpha < \alpha_c$ for Earth and $\alpha > \alpha_c$ for Venus.

3. Lapse rates

$$\rho_a c_p \frac{dT}{dz} - \frac{dp}{dz} + \rho_a L \frac{dm}{dz} = 0 \quad (1) \qquad \frac{dp}{dz} = -\rho_a g \quad (2) \qquad m = \frac{\rho_v}{\rho_a} \quad (3) \qquad p = \frac{\rho_a R T}{M_a} \quad (4) \qquad p_v = \frac{\rho_v R T}{M_v} \quad (5)$$

Clausius-Clapeyron eqn $\dfrac{dp_{sv}}{dT} = \dfrac{\rho_v L}{T} = \dfrac{p_{sv}}{T^2}\dfrac{M_v L}{R}$ (6) (using (5) under saturated conditions).

$$\Rightarrow \ln \frac{p_{sv}}{p_{sv0}} = \frac{M_v L}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) \approx \frac{M_v L}{R T_0}\left(\frac{T - T_0}{T_0}\right) \quad (*).$$

(i) dry adiabat. $\dfrac{dm}{dz} = 0$. So (1) $\Rightarrow \rho_a c_p \dfrac{dT}{dz} = \dfrac{dp}{dz} = -\rho_a g$ (using (2))

$$\Rightarrow \boxed{\frac{dT}{dz} = -\frac{g}{c_p}} = \Gamma_d$$

(ii) moist adiabat. $p_v = p_{sv}$. so $\quad m = \frac{\rho_v}{\rho_a} = \frac{M_v}{M_a}\frac{p_{sv}}{p}$ (using (4),(5))

$$\Rightarrow \frac{dm}{dz} = \frac{M_v}{M_a}\frac{d}{dz}\left(\frac{p_{sv}}{p}\right) = \frac{M_v}{M_a}\left[\frac{1}{p}\left(\frac{\rho_v L}{T}\right)\frac{dT}{dz} - \frac{p_{sv}}{p^2}(-\rho_a g)\right] \quad \text{(using (6), (2))}$$

$$= \frac{M_v}{M_a}\frac{\rho_v L}{p}\frac{1}{T}\frac{dT}{dz} + \frac{\rho_v g}{p}. \quad \text{(using (5))}.$$

so (1) $\Rightarrow$ $\rho_a c_p \frac{dT}{dz} + \rho_a g + \frac{\rho_a L}{T}\frac{M_v}{M_a}\frac{\rho_v L}{p}\frac{dT}{dz} + \rho_a L \frac{\rho_v g}{p} = 0$

$$\Rightarrow \boxed{\frac{dT}{dz} = -\frac{g}{c_p}\left(1 + \frac{\rho_v L}{p}\right) \Big/ \left(1 + \frac{\rho_v L}{p}\frac{M_v}{M_a}\frac{L}{c_p T}\right)} = \Gamma_m.$$

If $RH < 1$, we must be in the 'dry' case, since $p_v < p_{sv}$, so $T = T_0 - \frac{g}{c_p}z$.

$p_{sv}(T)$ is given approximately by the saturation curve (7), so

$$\boxed{p_{sv} \approx p_{sv0}\exp\left[-\frac{M_v L}{R T_0^2}\frac{g}{c_p}z\right]}$$

Since m is constant, $\frac{p_v}{p} = \frac{M_a}{M_v}\frac{\rho_v}{\rho_a} = \frac{M_a}{M_v}m$ (from (4),(5)) implies p_v is proportional to p.

Moreover $\frac{dp}{dz} = -\rho_a g \approx -\frac{M_a g}{R T_0}p \Rightarrow p \approx p_0 e^{-\frac{M_a g}{R T_0}z} \Rightarrow \boxed{p_v = p_{v0}\exp\left[-\frac{M_a g}{R T_0}z\right]}$

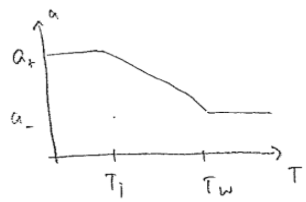
scale height

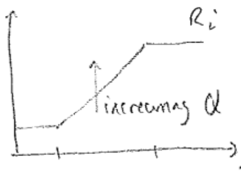
using (4), approximating $T \approx T_0$.

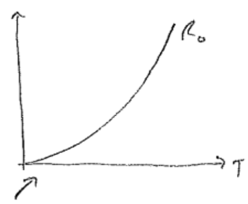
Combining, $\underbrace{\frac{p_v}{p_{sv}}}_{RH} = \underbrace{\frac{p_{v0}}{p_{sv0}}}_{RH_0}\exp\left[\frac{M_a g}{R T_0}\left(\frac{M_v}{M_a}\frac{L}{c_p T_0} - 1\right)z\right]$.

Clouds form where $RH = 1 \Rightarrow \boxed{z = \dfrac{\ln 1/RH_0}{\frac{M_a g}{R T_0}\left(\frac{M_v}{M_a}\frac{L}{c_p T_0} - 1\right)}}$ (eg. for $RH_0 = 0.5$, $z \approx 1.2$ km)

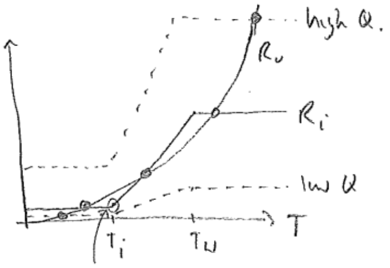
4. Ice-albedo feedback. $c\frac{dT}{dt} = R_i - R_o$. $R_i = \frac{1}{4}Q(1-a)$ $R_o = \sigma\gamma T^4$







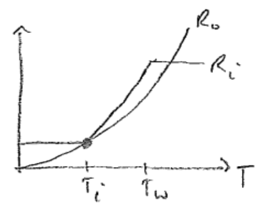
Steady states are intersection of these



From the graph it is clear that there can be multiple intersections for intermediate values of Q, provided the slope of the central section of the R_i curve is sufficiently steep.

In particular, multiple intersections require $R_i(T_i) < R_o(T_i)$, and the slope $R_i'(T_i) = \frac{1}{4}Q\frac{a_+-a_-}{T_w-T_i}$ must be larger than $R_o'(T_i) = 4\sigma\gamma T_i^3$. The largest value that $R_i'(T_i)$ takes, while this point remains below the $R_o(T_i)$ curve is when $R_i(T_i) = R_o(T_i) \Rightarrow \frac{1}{4}Q(1-a_+) = \sigma\gamma T_i^4$

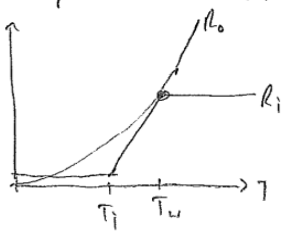
$\Rightarrow \left| Q = \frac{4\sigma\gamma T_i^4}{1-a_+} \right.$, so we require $\frac{\sigma\gamma T_i^4}{1-a_+}\frac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_i^3$



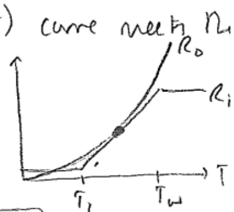
$$\Leftrightarrow \boxed{\frac{T_w-T_i}{T_i} < \frac{a_+-a_-}{4(1-a_+)}}$$

(equality would occur when the two slopes are equal)

If this condition on the slopes hold, then for smaller Q than this value $\left(\boxed{Q_+ = \frac{4\sigma\gamma T_i^4}{1-a_+}}\right)$ there will clearly be multiple intersections (as in diagram above). As Q is reduced, the multiple intersections cease to occur either when $R_i(T_w)$ drops below $R_o(T_w)$:



or when the $R_i(T)$ curve meets the $R_o(T)$ curve tangentially:



In the first case, the lower bound is $\boxed{Q_- = \frac{4\sigma\gamma T_w^4}{1-a_-}}$ and this applies if $R_i'(T_w) > R_o'(T_w)$ then

i.e. $\frac{\sigma\gamma T_w^4}{1-a_-}\frac{a_+-a_-}{T_w-T_i} > 4\sigma\gamma T_w^3$

$$\Leftrightarrow \boxed{\frac{T_w-T_i}{T_w} < \frac{a_+-a_-}{4(1-a_-)}}$$

In the second case, we must find the value of Q for which the curves meet tangentially.

This happens when $\frac{1}{4}Q(1-a) = \sigma\gamma T^4$. $\left.\begin{array}{l} \\ \\ \end{array}\right\}$ solve for T and Q. $\quad 1-a = 1-a_+ + \frac{a_+-a_-}{T_w-T_i}(T-T_i)$

$$\frac{1}{4}Q\,\frac{a_+-a_-}{T_w-T_i} = 4\sigma\gamma T^3$$

Write $\lambda = \frac{a_+-a_-}{T_w-T_i}$. Then $\frac{1}{4}Q\left(1-a_+ + \lambda(T-T_i)\right) = \sigma\gamma T^4$

$$\frac{1}{4}Q\lambda = 4\sigma\gamma T^3$$

$$1-a_+ + \lambda(T-T_i) = \frac{\lambda T}{4}.$$

$$\Rightarrow \frac{3}{4}T = T_i - \frac{(1-a_+)}{\lambda}.$$

$$\Rightarrow T = \frac{4}{3}T_i - \frac{4}{3}\frac{(1-a_+)}{\lambda} = \frac{4}{3}\left[\frac{(1-a_-)T_i - (1-a_+)T_w}{a_+-a_-}\right]$$

$$\text{Then } Q = \frac{16\sigma\gamma T^3}{\lambda} = \frac{16\sigma\gamma T^3(T_w-T_i)}{a_+-a_-} = \frac{64\cdot 16\sigma\gamma}{27}\frac{(T_w-T_i)}{(a_+-a_-)^4}\left[(1-a_-)T_i - (1-a_+)T_w\right]^3.$$

$$\Rightarrow \boxed{Q_- = \frac{512}{27}\sigma\gamma\,\frac{(T_w-T_i)\left[(1-a_-)T_i - (1-a_+)T_w\right]^3}{(a_+-a_-)^4}.}$$

(This can be seen to give the same value as in the first case if $\frac{T_w-T_i}{T_w} = \frac{a_+-a_-}{4(1-a_-)}$)