FINAL EXAMINATION

ATTEMPT ALL THE QUESTIONS IN SECTION A AND ANY ONE IN SECTION B.

TIME: 3 HOURS. CARRY A SCIENTIFIC CALCULATOR.

ANY OTHER EXAM MATERIAL WILL BE SUPPLIED. SHOW YOUR WORKING.

SECTION A

QUESTION 1. (20 Marks)

Planetary temperatures A simple energy balance model for the temperature T of a planetary atmosphere is

$$\rho c_p d \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4} Q(1-a) - \sigma T^4,$$

where ρ, c_p and d are the average density, specific heat capacity, and depth of the atmosphere, Q is the solar radiation, a is the planetary albedo, all assumed constant, and $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} is the Stefan-Boltzmann constant.

Find the steady state temperature and show that it is a stable equilibrium.

Calculate the steady state temperatures and compare with measured surface temperatures T_m for Earth ($Q = 1370$ W m^{-2}, $a = 0.3$, $T_m = 290$ K), Venus (0.72 a.u., $a = 0.77$, $T_m = 740$ K), Mars (1.52 a.u., $a = 0.15$, $T_m = 220$ K), and Jupiter (5.2 a.u., $a = 0.58$, $T_m = 130$ K). 1 a.u. is the distance from the sun to the Earth.

QUESTION 2. (20 Marks)

Dam A simple dimensionless model for the depth of water in a river $h(x,t)$ is

$$\frac{\partial h}{\partial t} + h^m \frac{\partial h}{\partial x} = 0,$$

where x is distance along the centreline of the river and $m > 0$ is a constant.

A dam at $x = 0$ controls the flow in the river downstream. Suppose that at time $t = 0$ the outflow from the dam is suddenly decreased such that $h(0,t)$ decreases from h_1 (at which it has been held for a long time) to h_2. Use the method of characteristics to solve for the water depth, as a function of time, at a distance L downstream.

QUESTION 3. (20 Marks)

Seasonal wave Ground temperature is governed by the diffusion equation

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2},$$

with boundary conditions $\partial T/\partial z \to 0$ as $z \to \infty$ and $T = T_0 - \Delta T \cos \omega t$ at $z = 0$ (the z coordinate points vertically downwards from the surface).

Non-dimensionalise the model, and solve the resulting dimensionless equation and boundary conditions. Using values $\rho = 1600$ kg m^{-3}, $c = 800$ J kg^{-1} K^{-1}, and $k = 1$ W m^{-1} K^{-1}, estimate the depth over which (a) daily, and (b) annual temperature variations decay.

Why might the same model not apply to calculate the near-surface temperature of a lake?

QUESTION 4. (20 Marks)

Stefan-Boltzmann law Given the Planck function $B_\nu = 2h\nu^3/c^2(e^{h\nu/kT} - 1)$ for the emission of radiation with frequency ν, show that the total emission is

$$B = \int_0^\infty B_\nu \, \mathrm{d}\nu = \frac{\sigma T^4}{\pi},$$

where the Stefan-Boltzmann constant is

$$\sigma = \frac{2\pi k^4}{h^3 c^2} \int_0^\infty \frac{u^3 \, \mathrm{d}u}{e^u - 1} = \frac{2\pi^5 k^4}{15 h^3 c^2}.$$

[*Hint: the last integral can be evaluated by making use of the identity* $\sum_1^\infty 1/n^4 = \pi^4/90$.]

SECTION B

QUESTION 1. (20 Marks)

Two stream approximation Starting from the radiative transfer equation given in lectures, and assuming local radiative equilibrium, derive the two-stream approximation for long wave radiation, given by

$$\tfrac{1}{2} I'_+ = -\kappa\rho(I_+ - B),$$
$$-\tfrac{1}{2} I'_- = -\kappa\rho(I_- - B),$$

where $' = d/dz$, $B = \sigma T^4/\pi = \frac{1}{2}(I_+ + I_-)$, T is the air temperature, and the net upward and downward energy fluxes are $F_\pm = \pi I_\pm$.

Write down appropriate boundary conditions for these equations if there is no incoming longwave radiation from the top of the atmosphere and the surface temperature is T_s (use the Stefan-Boltzman law). Show that the net upwards flux $F = F_+ - F_-$ is constant, and hence solve for $I_\pm$ in terms of T_s and the optical depth $\tau = \int_z^\infty \kappa\rho \, \mathrm{d}z$.

Deduce that the surface temperature is higher than the effective longwave emission temperature T_e (given by $F = \sigma T_e^4$) and that it increases with the optical depth of the atmosphere $\tau_s = \int_0^\infty \kappa\rho \, \mathrm{d}z$.

Deduce also that the surface air temperature at the ground $T\big|_{z=0}$ is lower than the surface temperature T_s.

QUESTION 2. (20 Marks)

Runaway greenhouse effect Show that the solution of the Clausius-Clapeyron equation for saturation vapour pressure p_{sv} as a function of temperature T is

$$p_{sv} = p_{sv0} \exp\left[a\left(1 - \frac{T_0}{T}\right)\right],$$

where $a = M_v L / R T_0$, and we may take $T_0 = 273$ K at $p_{sv0} = 600$ Pa (the *triple point*, where ice, water, and vapour can all exist at equilibrium). If T is close to T_0 show that

$$p_{sv} \approx p_{sv0} \exp\left[a\left(\frac{T - T_0}{T_0}\right)\right].$$

If the long wave radiation from a planet is $\sigma\gamma T^4$, the solar flux is Q, the planetary albedo is zero, and the greenhouse factor is given in terms of vapour pressure p by

$$\gamma^{-1/4} = 1 + b(p_v/p_{sv0})^c,$$

where b and c are constants, find the equilibrium mean surface temperature T in terms of p_v.

Hence show that the occurrence of a runaway greenhouse effect is controlled by the intersection of the two curves

$$\theta = 1 + \delta\xi, \quad \theta = \alpha(1 + be^{\xi}),$$

where $\delta = 1/ac$, $\alpha = (Q/4\sigma T_0^4)^{1/4}$. Show that runaway occurs if $\alpha > \alpha_c$, where

$$\alpha_c + \delta = 1 + \delta \ln(\delta / b\alpha_c),$$

and, if δ is small, that $\alpha_c \approx 1 + \delta \ln(\delta/b) - \delta$.

Estimate values of α and λ appropriate to the present Earth, and comment on the implications of these values for climatic evolution if we choose $b = 0.06$, $c = 0.25$. What are the implications for Venus, if the solar flux is twice as great? What if the solar radiation were 30% lower when the planetary atmospheres were being formed?

Parameter values: $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}, $M_v = 18 \times 10^{-3}$ kg mol^{-1}, $L = 2.5 \times 10^6$ J kg^{-1}, $R = 8.3$ J K^{-1} mol^{-1}.

QUESTION 3. (20 Marks)

Lapse rates An adiabatic atmosphere is described by

$$\rho_a c_p \frac{\mathrm{d}T}{\mathrm{d}z} - \frac{\mathrm{d}p}{\mathrm{d}z} + \rho_a L \frac{\mathrm{d}m}{\mathrm{d}z} = 0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\rho_a g,$$

where

$$m = \frac{\rho_v}{\rho_a}, \quad p = \frac{\rho_a RT}{M_a}, \quad p_v = \frac{\rho_v RT}{M_v},$$

and the saturation curve $p_v = p_{sv}$ is given by

$$\frac{\mathrm{d}p_{sv}}{\mathrm{d}T} = \frac{\rho_v L}{T}, \qquad T = T_0 \quad \text{at} \quad p_{sv} = p_{sv0}.$$

Find expressions for the lapse rate $\Gamma = -\mathrm{d}T/\mathrm{d}z$, for (i) a *dry adiabat*, when $p_v < p_{sv}$ and $\mathrm{d}m/\mathrm{d}z = 0$, and (ii) a *moist adiabat*, when $p_v = p_{sv}$.

Relative humidity is defined by $RH = p_v/p_{sv}$. If the ground surface $z = 0$ has relative humidity $RH_0 < 1$ and temperature T_0, make judicious use of the approximation $T \approx T_0$ to show that

$$RH \approx RH_0 \exp\left[\left(\frac{M_v}{M_a}\frac{L}{c_p T_0} - 1\right)\frac{g M_a}{R T_0} z\right].$$

Hence find the approximate height of the *lifting condensation level*, where the cloud base forms.

[*Hint: calculate (or re-use) the same approximation for $p_{sv}(T)$ as in question 1, and make use of the appropriate lapse rate to infer how p_{sv} varies with z. Then relate p_v to p and use the approximation $T \approx T_0$ to establish how p (and hence p_v) varies with z.*]

QUESTION 4. (20 Marks)

Ice albedo feedback A model for the mean temperature T of the Earth's atmosphere is

$$c\frac{dT}{dt} = R_i - R_o, \qquad R_i = \frac{1}{4}Q(1-a), \quad R_o = \sigma\gamma T^4,$$

where σ and γ are constant, and a varies piecewise linearly with temperature, such that

$a = a_+$ for $T < T_i$, $a = a_-$ for $T > T_w$ (with $a_- < a_+$), and $a(T)$ is linear for $T_i \leq T \leq T_w$.

Show graphically that there can be multiple steady states for some range of Q provided

$$\frac{T_w - T_i}{T_i} < \frac{a_+ - a_-}{4(1 - a_+)}.$$

[*Hint: consider the slopes* $R_i'(T)$ *and* $R_o'(T)$ *at* $T = T_i$.]

Show that in that case the upper and lower solutions are stable, but the intermediate one is unstable.

[*Harder*] Find the range $Q_- \leq Q \leq Q_+$ for which multiple steady states occur (*i.e.* give formulae for $Q_\pm$ in terms of the other parameters), taking care to distinguish the cases

$$\frac{T_w - T_i}{T_w} \gtrless \frac{a_+ - a_-}{4(1 - a_-)}.$$