Topology of Isoparametric Submanifolds

In this chapter we use the Morse theory developed in part II to prove that any non-degenerate distance function on an isoparametric submanifold of Hilbert space is of linking type, and so it is perfect. We also give some restriction for the possible marked Dynkin diagrams of these submanifolds. As a byproduct we are able to generalize the notion of tautness to proper Fredholm immersions of Hilbert manifolds into Hilbert space.

Tight and taut immersions in $\boldsymbol{R}^n$

Let $\varphi : M^n \to \boldsymbol{R}^m$ be an immersed compact submanifold, and $\nu^1(M)$ the the bundle of unit normal vectors of M. The restriction of the normal map N of M to $\nu^1(M)$ will still be denoted by N, i.e., $N : \nu^1(M) \to \boldsymbol{S}^{m-1}$ is defined be $N(v) = v$. There is a natural volume element $d\sigma$ on $\nu^1(M)$. In fact, if dV is a $(m-n-1)$-form on $\nu^1(M)$ such that dV restricts to each fiber of $\nu^1(M)_x$ is the volume form of the sphere of $\nu(M)_x$, then $d\sigma = dv \wedge dV$, where dv is the volume element of M. Let da be the standard volume form on $\boldsymbol{S}^{n+k-1}$ normalized so that $\int_{S^{n+k-1}} da = 1$. Then the Gauss-Kronecker curvature of an immersed surface in $\boldsymbol{R}^3$ can be generalized as follows:

8.1.1. Definition. The Lipschitz-Killing curvature at the unit normal direction v of an immersed submanifold M^n in $\boldsymbol{R}^m$ is defined to be the determinant of the shape operator A_v.

8.1.2. Definition. The total absolute curvature of an immersion $\varphi : M^n \to \boldsymbol{R}^m$ is

$$\tau(M, \varphi) = \int_{\nu^1(M)} |\det(A_v)| \, d\sigma,$$

where $d\sigma$ is the volume element of $\nu^1(M)$, and A_v is the shape operator of M in the unit normal direction v.

8.1.3. Definition. An immersion $\varphi_0 : M^n \to \boldsymbol{R}^m$ is called *tight* if

$$\tau(M, \varphi) \geq \tau(M, \varphi_0),$$

for any immersions $\varphi : M \to \boldsymbol{R}^s$.

Chern and Lashof began the study of tight immersions in the 1950's [CL1,2]. They proved the following theorem:

8.1.4. Theorem. *If $\varphi : M^n \to \boldsymbol{R}^m$ is an immersion then, for any field F,*

$$\tau(M, \varphi) \geq \sum_i b_i(M, F),$$

where $b_i(M, F)$ is the i^{th} Betti number of M with respect to F.

It is a difficult and as yet unsolved problem to determine which manifolds admit tight immersions. An important step towards the solution is Kuiper's [Ku2] reformulation of the problem in terms of the Morse theory of height functions. Given a Morse function $f : M \to \boldsymbol{R}$, let

$$\mu_k(f) = \text{the number of critical points of } f \text{ with index } k,$$
$$\mu(f) = \sum_i \mu_k(f).$$

The Morse number $\gamma(M)$ of M is defined by

$$\gamma(M) = \inf\{\mu(f) | \; f : M \to \boldsymbol{R} \text{ is a Morse function}\}.$$

Let $\varphi : M^n \to \boldsymbol{R}^m$ be an immersion. By Proposition 4.1.8, $dN_v = (-A_v, id)$, so we have

$$N^*(da) = (-1)^n \det(A_v) d\sigma,$$

and the total absolute curvature $\tau(M, \varphi)$ is the total volume of the image $N(\nu^1(M))$, counted with multiplicities but ignoring orientation. Let h_p denote the height function as in section 4.1. Then it follows from Propositions 4.1.1 and 4.1.8 that $p \in \boldsymbol{S}^{m-1}$ is a regular value of N if and only if the height function h_p is a Morse function. In this case $N^{-1}(p)$ is a finite set with $\mu(h_p)$ elements. But by the Morse inequalities we have $\mu(h_p) \geq \sum_i b_i(M, F)$, and in particular:

$$\tau(M, \varphi) \geq \sum_i b_i(M),$$

$$\tau(M, \varphi) \geq \gamma(M).$$

This proves the following stronger result of Kuiper:

8.1.5. Theorem.

(i) $\gamma(M) = \inf\{\tau(M, \varphi) | \; \varphi : M \to \boldsymbol{R}^m \text{ is an immersion }\}$.

(ii) An immersion $\varphi_0 : M \to \boldsymbol{R}^m$ is tight if and only if every non-degenerate height functions h_p has $\gamma(M)$ critical points.

Banchoff [Ba] studied the problem of finding all tight surfaces that lie in a sphere, and later this led to the study of taut immersions by Carter and West [CW1]. Note that if $\varphi : M^n \to \boldsymbol{R}^m$ is a tight immersion and $\varphi(M)$ is contained in the unit sphere $\boldsymbol{S}^{m-1}$, then the Euclidean distance function f_p and the height function h_p have the same critical point theory because $f_p = 1 + \|p\|^2 - 2h_p$. Taut immersions are "essentially" the spherical tight immersions.

A non-degenerate smooth function $f : M \to \boldsymbol{R}$ is called a perfect Morse function if $\mu(f) = \sum b_i(M, F)$ for some field F. If we restrict ourself to the class of manifolds that satisfy the condition that $\gamma(M) = \sum b_i(M, F)$ for some field F, then an immersion $\varphi : M \to \boldsymbol{R}^m$ is tight if and only if every non-degenerate height function h_a is perfect, and it is taut if and only if every non-degenerate Euclidean distance function f_a is perfect. There is a detailed and beautiful theory of tight and taut immersions for which we refer the reader to [CR2].

8.2. Taut immersions in Hilbert space

In Theorem 7.1.13 we showed that the distance functions f_a of PF submanifolds in Hilbert space satisfy Condition C, so the concept of tautness can be generalized easily to PF immersions.

8.2.1. Definition. A smooth function $f : M \to \boldsymbol{R}$ on a Riemannian Hilbert manifold M is called a *Morse function* if f is non-degenerate, bounded from below, and satisfies Condition C.

For a Morse function f on M let

$$M_r(f) = \{x \in M |\ f(x) \leq r\}.$$

Then it follows from Condition C that there are only finitely many critical points of f in $M_r(f)$. Let

$$\begin{aligned} \mu_k(f, r) &= \text{ the number of critical points of index } k \text{ on } M_r(f), \\ \beta_k(f, r, F) &= \dim(H_k(M_r(f), F)), \end{aligned}$$

for a field F. Then the weak Morse inequalities gives

$$\mu_k(f, r) \geq \beta_k(f, r, F)$$

for all r and F.

8.2.2. Definition. A Morse function $f : M \to \boldsymbol{R}$ is *perfect*, if there exists a field F such that $\mu_k(f, r) = \beta_k(f, r, F)$ for all r and k.

It follows from the standard Morse theory in part II that:

8.2.3. Theorem. *Let f be a Morse function. Then f is perfect if and only if there exists a field F such that the induced map on the homology*

$$i_* : H_*(M_r(f), F) \to H_*(M, F)$$

of the inclusion of $M_r(f)$ in M is injective for all r.

8.2.4. Definition. An immersed submanifold M of a Hilbert space is *taut* if M is proper Fredholm and every non-degenerate Euclidean distance function f_a on M is a perfect Morse function.

8.2.5. Remark. If M is properly immersed in $\boldsymbol{R}^n$ then the above definition is the same as section 8.1.

8.2.6. Remark. It is easy to see that the unit sphere $\boldsymbol{S}^{n-1}$ is a taut submanifold in $\boldsymbol{R}^n$. But the unit hypersphere S of an infinite dimensional Hilbert space is not taut. First, S is contractible, but the non-degenerate distance function f_a has two critical points. Moreover S is not PF.

8.2.7. Example. We will see later that, given a simple compact connected group G, the orbits of the gauge group $H^1(\boldsymbol{S}^1, G)$ acting on the space of connections $H^0(\boldsymbol{S}^1, \mathcal{G})$ by gauge transformations as in section 5.8 are taut.

Let $R(f)$ denote the set of all regular values of f, and $C(f)$ denote the set of all critical points of f. The fact that the restriction of the end point map Y of M to the unit disk normal bundle is proper gives a uniform condition C for the Euclidean distance functions as we see in the following two propositions:

8.2.8. Proposition. *Let M be an immersed PF submanifold of V, and $a \in V$. Suppose $r < s$ and $[r, s] \subseteq R(f_a)$. Then there exists $\delta > 0$ such that if $\|b - a\| < \delta$ then $[r, s] \subseteq R(f_b)$.*

PROOF. If not, then there exist sequences b_n in V and x_n in M such that x_n is a critical point of f_{b_n} and

$$b_n \to a, \ \ r \le \|x_n - b_n\| \le s.$$

It follows from Proposition 7.1.10 that $(x_n - b_n) \in \nu(M)_{x_n}$. Since the endpoint map Y of M restricted to the disk normal bundle of radius s is proper and $Y(x_n, b_n - x_n) = b_n \to a$, there is a subsequence of x_n converging to a point x_0 in M. Then it is easily seen that $r \le \|x_0 - a\| \le s$ and x_0 is a critical point of f_a, a contradiction. ∎

8.2.9. Proposition. *Let M be an immersed PF submanifold of V, and $a \in V$. Suppose $r < s$ and $[r,s] \subseteq R(f_a)$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that if $\|b - a\| < \delta_1$ and $x \in M_s(f_b) \setminus M_r(f_b)$ then $\|\nabla f_b(x)\| \geq \delta_2$.*

PROOF. By Proposition 8.2.8 there exists $\delta > 0$ such that $[r,s] \subseteq R(f_b)$ if $\|b - a\| < \delta$. Suppose no such δ_1 and δ_2 exist. Then there exist sequences b_n in V and x_n in M such that $b_n \to a$, $x_n \in M_s(f_{b_n}) \setminus M_r(f_{b_n})$, and $\|\nabla(f_{b_n})(x_n)\| \to 0$. Then

$$\begin{aligned} Y(x_n, -(x_n - b_n)^\nu) &= x_n - (x_n - b_n)^\nu \\ &= b_n + (x_n - b_n)^{TM_{x_n}} \to a, \end{aligned}$$

and $\|x_n - b_n\| \leq s$. Since M is PF, x_n has a subsequence converging to a critical point x_0 of f_a in $M_s(f_a) - M_r(f_a)$, a contradiction. ■

8.2.10. Proposition. *Let M be an immersed, taut submanifold of a Hilbert space V, $a \in V$, and $r \in R$. Then the induced map on homology*

$$i_* : H_*(M_r(f_a), F) \to H_*(M, F)$$

of the inclusion of $M_r(f_a)$ in M is injective.

PROOF. If a is a non-focal point (so f_a is non-degenerate), then it follows from the definition of tautness and Theorem 8.2.3 that i_* is an injection. Now suppose a is a focal point. If there is no critical value of f_a in $(r, r']$, then $M_r(f_a)$ is a deformation retract of $M_{r'}(f_a)$. So we may assume that r is a regular value of f_a, i.e., $r \in R(f_a)$. Then there exists $s > r$ such that $[r,s] \subseteq R(f_a)$. Choose $\delta_1 > 0$ and $\delta_2 > 0$ as in Proposition 8.2.9, and $\epsilon > 0$ such that $\epsilon < \min\{\delta_1,\ \delta_2,\ (s-r)/5\}$. Since the set of non-focal points of M in V is open and dense, there exists a non-focal point b such that $\|b - a\| < \epsilon$. Since f_b is non-degenerate, it follows from the definition of tautness that $i_* : H_*(M_t(f_b), F) \to H_*(M, F)$ is injective for all t. So it suffices to prove that $M_r(f_a)$ is a deformation retract of $M_r(f_b)$. Since $\epsilon < (s-r)/5$, there exist r_1, r_2, s_1 and s_2 such that $r_1 < s_1,\ r_2 < s_2$ and

$$r < r_1 - \epsilon < s_1 + \epsilon < s, \quad r_1 < r_2 - \epsilon < s_2 < s_2 + \epsilon < s_1.$$

From triangle inequality we have

$$M_{s_2}(f_b) - M_{r_2}(f_b) \subset M_{s_1}(f_a) - M_{r_1}(f_a) \subseteq M_s(f_b) - M_r(f_b).$$

Note that $\|\nabla f_a(x)\| \geq \delta_2$ if $x \in M_s(f_a) - M_r(f_a)$ and $\|\nabla f_b(x)\| > \delta_2$ if $x \in M_s(f_b) - M_r(f_b)$. Recall that $\nabla f_a(x) = (x-a)^T$ and $\nabla f_b(x) =$

$(x-b)^T$. Since $\epsilon < \delta_2$, $(a-b)^T$ is the shortest side of the triangle with three sides $(x-a)^T$, $(x-b)^T$ and $(a-b)^T$ for all x in $M_{s_1}(f_a) - M_{r_1}(f_a)$. Using the cosine formula for the triangle we have

$$\langle \nabla f_a(x), \nabla f_b(x) \rangle > \frac{2\delta_2^2 - \epsilon^2}{2} > \frac{\epsilon^2}{2},$$

for x in $M_{s_1}(f_a) - M_{r_1}(f_a)$. Hence the gradient flow of f_a gives a deformation retract of $M_{s_1}(f_a)$ to $M_{s_2}(f_b)$. If $[r,s] \subseteq R(f)$, then $M_r(f)$ is a deformation retract of $M_t(f)$ for all $t \in [r,s]$, which proves our claim. ■

8.2.11. Corollary. *If M is connected and $\varphi : M \to V$ is a taut immersion then φ is an embedding.*

PROOF. Since M is PF, $\varphi = Y|M \times 0$ is proper. So it suffices to prove that φ is one to one. Suppose $\varphi(p) = \varphi(q) = a$. If $p \neq q$ then there exists $\epsilon > 0$ such that $(0, \epsilon) \subseteq R(f_a)$ and p, q are in two different connected components of $M_\epsilon(f_a)$. This contradicts to the fact that $i_0 : H_0(M_\epsilon(f_a), F) \to H_0(M, F)$ is injective. ■

8.2.12. Corollary. *Suppose M is a connected taut submanifold of V, $a \in V$, and let $D_r(a)$ denote the closed ball of radius r and center a in V.*

(i) For any $r \in R$ the set $M_r(f_a)$ is connected, or equivalently, $M \cap D_r(a)$ is connected.

(ii) If x_o is an index 0 critical point of f_a then $f_a(x_o)$ is the absolute minimum of f_a; in particular a local minimum of f_a is the absolute minimum.

(iii) If x_o is an isolate critical point of f_a with index 0 and $r_o = f_a(x_o)$, then $M_{r_o}(f_a) = \{x_o\}$, i.e., $\{x_o\} = M \cap D_{r_o}(a)$.

PROOF. By Proposition 8.2.10, the map

$$i_0 : H_0(M_r(f_a), F) \to H_0(M, F)$$

is injective. Since $H_0(M, F) = F$, $M_r(f_a)$ is connected, which proves (i).

Next we prove (iii) for non-degenerate index 0 critical point. Let x_o be a non-degenerate index 0 critical point of f_a and $r_o = f_a(x_o)$. Then there is an open neighborhood U of x_o such that $M_{r_o}(f_a) \cap U = \{x_o\}$. Since $M_{r_o}(f_a)$ is connected, $M_{r_o}(f_a) = \{x_o\}$. In particular, r_o is the absolute minimum of f_a, i.e.,

$$\|x - a\| \geq \|x_o - a\|.$$

If x_o is a degenerate critical point, then there is $v \in \nu(M)_{x_o}$ such that $a = x_o + v$ and

$$\mathrm{Hess}(f_a, x_o) = I - A_v \geq 0.$$

Let $a_t = a + tv$ for $0 < t < 1$. Then

$$\text{Hess}(f_{a_t}, x_o) = I - tA_v \; > 0.$$

So x_o is a non-degenerate index 0 critical point of f_{a_t}. But we have just shown that

$$\|x - a_t\| \geq \|x_o - a_t\| \tag{8.2.1}$$

for all $x \in M$. Letting $t \to 1$ in (8.2.1), we obtain (ii). ∎

8.3. Homology of isoparametric submanifolds

In this section we use Morse theory to calculate the homology of isoparametric submanifolds of Hilbert spaces and prove that they are taut.

Let f be a Morse function on a Hilbert manifold M, q a critical point of f of index m. In Chapter 10 of Part II we define a pair (N, φ) to be a *Bott-Samelson cycle* for f at q if N is a smooth m-dimensional manifold and $\varphi : N \to M$ is a smooth map such that $f \circ \varphi$ has a unique and non-degenerate maximum at y_0, where $\varphi(y_0) = q$. (N, φ) is $\mathcal{R}$-orientable for a ring $\mathcal{R}$, if $H_m(N, \mathcal{R}) = \mathcal{R}$. We say f is of Bott-Samelson type with respect to $\mathcal{R}$ if every critical point of f has an $\mathcal{R}$-orientable Bott-Samelson cycle. Moreover if $\{q_i | \, i \in I\}$ is the set of critical points of f and (N_i, φ_i) is an $\mathcal{R}$-orientable Bott-Samelson cycle for f at q_i for $i \in I$ then $H_*(N, \mathcal{R})$ is a free module over $\mathcal{R}$ generated by the descending cells $(\varphi_i)_*([N_i])$, which implies that f is of linking type, and f is perfect.

8.3.1. Theorem. *Let M be an isoparametric submanifold of the Hilbert space V, and x_0 a critical point of the Euclidean distance function f_a. Then*

(i) there exist a parallel normal field v on M and finitely many curvature normals v_i such that $a = x_0 + v(x_0)$ and $\langle v, v_i \rangle \; > 1$,

(ii) if

$$\langle v, v_r \rangle \; \geq \cdots \geq \langle v, v_1 \rangle \; > 1 \; > \langle v, v_{r+1} \rangle \; \geq \; \langle v, v_{r+2} \rangle \geq \ldots,$$

then

(1) $\bigoplus \{E_i(x_0) | i \leq r\}$ is the negative space of f at x_0,

(2) (N_r, u_r) is an $\mathcal{R}$-orientiable Bott-Samelson cycle at x_0 for f, where

$$N_r = \{(y_1, \ldots, y_r) | y_1 \in S_1(x_0), y_2 \in S_2(y_1), \ldots, y_r \in S_r(y_{r-1})\},$$

$$u_r : N_r \to M, \;\; u_r(y_1, \ldots, y_r) = y_r,$$

and $S_i(x)$ is the leaf of E_i through x. Here $\mathcal{R} = \boldsymbol{Z}$ if all $m_i > 1$, and $\mathcal{R} = \boldsymbol{Z}_2$ otherwise.

PROOF. Since x_0 is a critical point of f_a, by Proposition 7.1.10 $a - x_0 \in \nu(M)_{x_0}$. Let v be the parallel normal field on M such that $v(x_0) = a - x_0$. Then (i) follows form the fact that the shape operator A_v is compact, the eigenvalues of A_v are $\langle v, v_i\rangle$, and $\nabla^2 f_a(x_0) = I - A_{(a-x_0)}$.

For (ii) it suffices to prove the following three statements:

(a) $y_0 = (x_0, \ldots, x_0)$ is the unique maximum point of $f \circ u_r$.

(b) $d(u_r)_{y_0}$ maps $T(N_r)_{y_0}$ isomorphically onto the negative space of f at x_0.

(c) If all $m_i > 1$, then (N_r, u_r) is $\boldsymbol{Z}$-orientable.

To see (b), let $N = N_r$, we note that N is contained in the product of r copies of M, $TN_{y_0} = \bigoplus\{F_i | i \leq r\}$, where $F_i = (0, , \ldots, E_i(x_0), \ldots, 0)$ is contained in $\bigoplus\{TM_{x_0} | i \leq r\}$ and $d(u_r)_{y_0}$ maps F_i isomorphically onto $E_i(x_0)$.

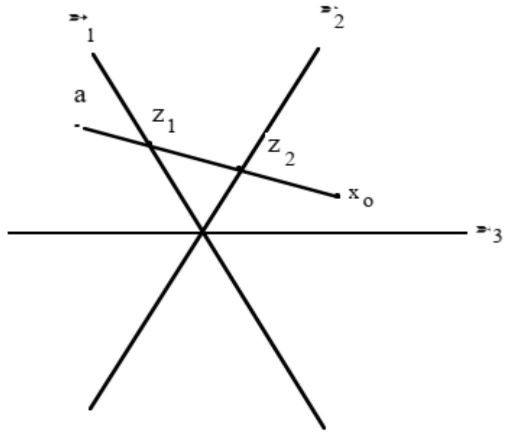
It follows from the definition of N_r that it is an iterated sphere bundle. The homotopy exact sequence for the fibrations implies that if the fiber and the base of a fibration are simply connected then the total space is also simply connected. Hence by induction the iterated sphere bundle N_r is simply connected, which proves (c).

Statement (a) follows from the lemma below. ∎

8.3.2. Lemma. *We use the same notation as in Theorem 8.3.1. Then for any $q = (y_1, , \ldots, y_r)$ in N_r there is a continuous piecewise smooth geodesic α_q in V joining a to y_r such that the length of α_q is $\|x_0 - a\|$, and α_q is smooth if and only if $q = (x_0, \ldots, x_0)$.*

PROOF. Let $[xy]$ denote the line segment joining x and y in V. Let $\{z_i\} = \ell_i(x_0) \cap [ax_0]$. Then

$$[ax_0] = [az_1] \cup [z_1 z_2] \cup \ldots [z_r x_0].$$



Let $a_i = (y_i + v(y_i))$, and $z_j(i) \in \ell_j(y_i) \cap [y_i a_i]$. Then $z_1(1) = z_1$ and $z_j(j-1) = z_j(j)$. Since $y_j \in S_j(y_{j-1})$ and $z_j(j-1) = z_j(j)$,

$$\alpha_q = [az_1] \cup [z_1(1), z_2(1)] \cap [z_2(2), z_3(2)] \cup \ldots \cup [z_r(r), y_r]$$

satisfies the properties of the lemma. ∎

8.3.3. Corollary. *Let M be an immersed isoparametric submanifold in a Hilbert space V with multiplicities m_i, and $a \in V$ a non-focal point of M. Then*

(i) f_a is of Bott-Samelson type with respect to the ring $\mathcal{R} = \boldsymbol{Z}$, if all the multiplicities $m_i > 1$, and with respect to $\mathcal{R} = \boldsymbol{Z}_2$ otherwise,

(ii) M is taut.

It follows from Corollary 8.2.11 that:

8.3.4. Corollary. *An immersed isoparametric submanifold of a Hilbert space V is embedded.*

To obtain more precise information concerning the homology groups of isoparametric submanifolds, we need to know the structure of the set of critical points of f_a. By The Morse Index Theorem (see Part II) we have

8.3.5. Proposition. *Let $M \subseteq V$ be isoparametric, and W its associated Coxeter group. Let $a \in V$, and let $C(f_a)$ denote the set of critical points of f_a.*

(i) If $x_0 \in C(f_a)$ then $W \cdot x_0 \subseteq C(f_a)$, where $W \cdot x_0$ is the W-orbit through x_0 on $x_0 + \nu(M)_{x_0}$,

(ii) If $q \in C(f_a)$ then the index of f_a at q is the sum of the m_i's such that the open line segment (q, a) joining q to a meets $\ell_i(q)$.

Let $\nu_x = x + \nu(M)_x$. Then the closure of a connected component of $\nu_x \setminus \bigcup\{\ell_i(q) | i \in I\}$ is a Weyl chamber for the Coxeter group W-action on ν_x.

In the following we let $\triangle_q$ denote the Weyl chamber of W on ν_q containing q. As a consequence of Proposition 8.3.5 and Corollary 8.2.12, we have

8.3.6. Proposition. *Suppose M is isoparametric in a Hilbert space and let $q \in M$. Let $\triangle_q$ be the Weyl chamber in $\nu_q = (q + \nu(M)q)$ containing q, and $a \in \triangle_q$.*
Then:
(i) q is a critical point of f_a with index 0,
(ii) $f_a(q)$ is the absolute minimum of f_a,
(iii) if a is non-focal with respect to q, then $f_a^{-1}(f_a(q)) = \{q\}$,
(ii) if a is a focal point with respect to q and a lies on the simplex σ of $\triangle_q$, then $f_a^{-1}(f_a(q)) = S_{q,\sigma}$ (as in the Slice Theorem 6.5.9).

8.3.7. Theorem. *Let M be an isoparametric submanifold of V, and $a \in \nu_q \cap \nu_{q'}$. Then a is non-focal with respect to q if and only if a is non-focal with respect to q', and $q' \in W \cdot q$.*

PROOF. There are $p \in W \cdot q$ and $p' \in W \cdot q'$ such that $a \in \triangle_p$ and $a \in \triangle_{p'}$. By Proposition 8.3.5 (ii), both p and p' are critical points of f_a with index 0. So by Proposition 8.3.6, $f_a(p) = f_a(p')$ is the absolute minimum of f_a. If a is non-focal with respect to q then by Proposition 8.3.6 (iii), we have $p = p'$ and a is non-focal with respect to p'. ∎

8.3.8. Corollary. *Let $M \subseteq V$ be isoparametric, W its associated Coxeter group. If $a \in V$ is non-focal with respect to q in M, then $C(f_a) = W \cdot q$.*

8.3.9. Corollary. *Let $M \subseteq V$ be isoparametric. Then $H_*(M, \mathcal{R}))$ can be computed explicitly in terms of the associated Coxeter group W and its multiplicities m_i. Here $\mathcal{R}$ is $\boldsymbol{Z}$ if all $m_i > 1$ and is $\boldsymbol{Z}_2$ otherwise.*

8.3.10. Corollary. *Let $M^n \subseteq \boldsymbol{R}^{n+k}$ be isoparametric. Then*

$$\sum_i \operatorname{rank}(H_i(M, \mathcal{R})) = |W|,$$

the order of W.

8.3.11. Corollary. *Let $M \subseteq V$ be isoparametric. A point $a \in V$ is non-focal with respect to $q \in M$ if and only if a is a regular point with respect to the W-action on ν_q.*

8.3.12. Corollary. *Let $M \subseteq V$ be isoparametric. If f_a has one non-degenerate critical point then f_a is non-degenerate, or equivalently if $a \in \nu_q$ is non-focal with respect to q then a is non-focal with respect to M.*

Let $a \in V$. Since f_a is bounded from below and satisfies condition C on M, f_a assumes its minimum, say at q. So $a \in \nu_q$, i.e.,

8.3.13. Proposition. *Let $M \subseteq V$ be isoparametric, and $Y : \nu(M) \to V$ the endpoint map. Then $Y(\nu(M)) = V$.*

8.4. Rank 2 isoparametric submanifolds in $\boldsymbol{R}^m$

In this section we will apply the results we have developed for isoparametric submanifolds of arbitrary codimension to a rank 2 isoparametric submanifold M^n of $\boldsymbol{R}^{n+2}$. Because of Corollaries 6.3.12 and 6.3.11, we may assume that M is a hypersurface of $\boldsymbol{S}^{n+1}$.

Let $X : M^n \to \boldsymbol{S}^{n+1} \subseteq \boldsymbol{R}^{n+2}$ be isoparametric, and e_{n+1} the unit normal field of M in $\boldsymbol{S}^{n+1}$. Suppose M has p distinct principal curvatures $\lambda_1, \ldots, \lambda_p$ as a hypersurface of $\boldsymbol{S}^{n+1}$ with multiplicities m_i. Then

$$e_\alpha = e_{n+1}, \;\; e_\beta = X$$

is a parallel normal frame on M, and the reflection hyperplanes $\ell_i(q)$ on $\nu_q = q + \nu(M)_q$ (we use q as the origin, $e_\alpha(q)$ and $e_\beta(q)$ are the two axes) are given by the equations:

$$\lambda_i z_\alpha - z_\beta = 1, \;\; 1 \leq i \leq p.$$

The Coxeter group W associated to M is generated by reflections in ℓ_i. By the classification of rank 2 Coxeter groups, W is the Dihedral group of order $2p$. So we may assume that

$$\lambda_i = \cot\left(\theta_1 + \frac{(i-1)\pi}{p}\right), \; 1 \leq i \leq p,$$

for some θ_1, where $-\pi/p < \theta_1 < 0$. This fact was proved by Cartan ([Ca3]). Let R_i denote the reflections of ν_q in $\ell_i(q)$. It is easily seen that

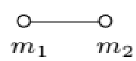
$$R_{i+1}(\ell_i) = \ell_{i+2},$$

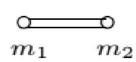
if we let $\ell_{p+i} = \ell_i$ for $1 \leq i < p$. By Theorem 6.3.2, we obtain the following result of Münzner:

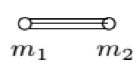
$$m_1 = m_3 = \ldots,$$
$$m_2 = m_4 = \ldots.$$

In particular, if p is odd then all the multiplicities are equal. So the possible marked Dynkin diagrams for a rank 2 isoparametric submanifold of the Euclidean space are

$\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ 

A_2 

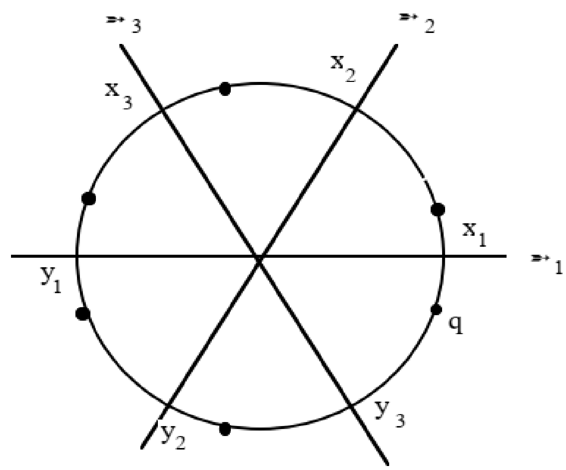
B_2 

G_2 

Note that the intersection of $\ell_i(q)$ and the normal geodesic circle of M in $\boldsymbol{S}^{n+1}$ at q has exactly two points, which will be denoted by x_i and y_i, i.e.,

$$x_i = \cos\,\theta_i\; q + \sin\,\theta_i\; e_\alpha(q),$$
$$y_i = \cos(\pi + \theta_i)\; q + \sin(\pi + \theta_i)\; e_\alpha(q),$$

where $\theta_i = \theta_1 + \frac{(i-1)\pi}{p}$.



Let $\triangle_q$ denote the Weyl chamber of W on ν_q containing q. Then the intersection of $\triangle_q$ and the normal geodesic circle of M in $\boldsymbol{S}^{n+1}$ at q is the arc joining x_1 to y_p. Let M_t denote the parallel submanifold of M through $\cos\,t\; q + \sin\,t\; e_\alpha(q)$. Then

$$\bigcup\{M_t \mid -\pi/p + \theta_1 \leq t \leq \theta_1\} = \boldsymbol{S}^{n+1}.$$

Note that M_t is diffeomorphic to M and is an embedded isoparametric hypersurface of $\boldsymbol{S}^{n+1}$ if $-\pi/p + \theta_1 \leq t \leq \theta_1$. And the focal set Γ of M in $\boldsymbol{S}^{n+1}$ has exactly two sheets, $M_1 = M_{\theta_1}$ and $M_2 = M_{(-\pi/p+\theta_1)}$, so they are also called the focal submanifolds of M. The dimension of M_i is $n - m_i$ for $i = 1, 2$. Let v_i be the parallel normal fields on M such that $x_1 = q + v_1(q)$, $y_p = q + v_2(q)$. Then $M_i = M_{v_i}$ the parallel submanifold. So by Proposition 6.5.1, $\pi_i : M \to M_i$ defined by $\pi_i(x) = x + v_i(x)$ is a fibration and M is a $\boldsymbol{S}^{m_i}$-bundle over M_i.

Let B_i be the normal disk bundle of radius r_i of M_i in $\boldsymbol{S}^{n+1}$, where $r_1 = \theta_1$ and $r_2 = \pi/p - \theta_1$. So

$$B_i = \{\cos\, t\, x + \sin\, t\, v(x) \mid |t| \leq r_i,\ v \text{ is normal to } M \text{ in } \boldsymbol{S}^{n+1}\},$$

and $\partial B_i = M$. Next we claim that $B_1 \cup B_2 = \boldsymbol{S}^{n+1}$. To see this, let $a \in \boldsymbol{S}^{n+1}$. Since M is compact, f_a assumes minimum, say at x_0. So $a = \cos t\, x_0 + \sin t\, e_\alpha(x_0)$ for some t. Because x_0 is the minimum of f_a, a must lie in the Weyl chamber $\triangle_{x_0}$ of W on ν_{x_0}, i.e., $-r_2 < t < r_1$. So $a \in B_1$ if $0 \leq t \leq r_1$ and $a \in B_2$ if $-r_2 \leq t \leq 0$. This proves the following results of Münzner [Mü1]:

$$\begin{aligned}
&B_1 \cup B_2 = \boldsymbol{S}^{n+1},\\
&\partial B_1 = \partial B_2 = B_1 \cap B_2 = M,\\
&B_i \text{ is a } (m_i + 1) - \text{disk bundle over } M_i.
\end{aligned}$$

Using this decomposition of $\boldsymbol{S}^{n+1}$ as two disk bundles and results from algebraic topology, Münzner [Mü2] proved that

$$\sum_i \operatorname{rank}(H_i(M)) = 2p,$$

which is the same as our result in Corollary 8.3.10, because $|W| = 2p$. He also obtained the explicit cohomology ring structure of $H^*(M, \boldsymbol{Z}_2)$. Using the cohomology ring structure, Münzner proved the following:

8.4.1. Theorem. *If M is an isoparametric hypersurface of $\boldsymbol{S}^{n+1}$ with p distinct principal curvatures, then p must be $1, 2, 3, 4$ or 6.*

Next we state some restrictions on the possible multiplicities m_i. The first result of this type was proved by É. Cartan:

8.4.2. Theorem. *If M^n is an isoparametric hypersurface of $\boldsymbol{S}^{n+1}$ with three distinct principal curvatures, then $m_1 = m_2 = m \in \{1, 2, 4, 8\}$.*

Using delicate topological arguments, Münzner [Mü2] and Abresch [Ab] obtained restrictions on the m_i's for the case of $p = 4$ and $p = 6$. First we make a definition:

8.4.3. Definition. A pair of integers (m_1, m_2) is said to satisfy condition (*) if one of the following hold:

(a) 2^k divides $(m_1 + m_2 + 1)$, where $2^k = \min\{2^\sigma \,|\, m_1 < 2^\sigma,\ \sigma \in N\}$,

(b) if m_1 is a power of 2, then $2m_1$ divides (m_2+1) or $3m_1 = 2(m_2+1)$.

8.4.4. Theorem. *Suppose M^n is isoparametric in S^{n+1} with p distinct principal curvatures.*

(i) If $p = 4$ and $m_1 \leq m_2$, then (m_1, m_2) must satisfy condition (*).

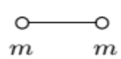
(ii) If $p = 6$, then $m_1 = m_2 \in \{1, 2\}$.

We will omit the difficult proof of these results and instead refer the reader to [Mü2] and [Ab].

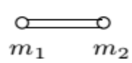
As consequence of Theorem 8.4.1, we have:

8.4.5. Theorem. *If M is a rank 2 isoparametric submanifold of Euclidean space, then the Coxeter group W associated to M is crystallographic, i.e., $W = A_1 \times A_1,\ A_2,\ B_2$, or G_2.*

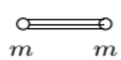
8.4.6. Theorem. *If M is an irreducible rank 2 isoparametric submanifold of Euclidean space, then the marked Dynkin diagram associated to M must be one of the following:*

A_2 

$m \in \{1,2,4,8\}$

B_2 

(m_1, m_2) satisfies $(*)$

G_2 

$m \in \{1,2\}$

8.5. Parallel foliations

Although the proof of the existence of parallel foliation for finite dimensional isoparametric submanifolds does not work in the infinite dimensional case, the topological results of section 8.3 lead to the existence of parallel foliation.

Let M be a PF submanifold of V with flat normal bundle, and Y the end point map of M. In general, the parallel set,

$$M_v = \{Y(v(x)) = x + v(x) \mid x \in M\},$$

defined by a parallel normal field v, may be a singular set, and $\mathcal{F} = \{M_v \mid v \text{ is a parallel normal field on } M\}$ need not foliate V. The main result of this section is that if M is isoparametric, then each M_v is an embedded submanifold of V and $\mathcal{F}$ gives an orbit-like singular foliation on V.

In what follows M is a rank k isoparametric submanifold of a Hilbert space V, $\nu_q = q + \nu(M)_q$ and $\triangle_q$ is the Weyl chamber of W on ν_q containing q.

8.5.1. Proposition. $M \cap \nu_q = W \cdot q$.

PROOF. It is easily seen that $W \cdot q \subseteq \nu_q$. Now suppose that $b \in M \cap \nu_q$. Then $b \in \nu_b \cap \nu_q$. But b is non-focal with respect to b, so it follows from Theorem 8.3.7 that we have $b \in W \cdot q$. ∎

8.5.2. Proposition. *Suppose σ is a simplex of $\triangle_q$ and σ' is a simplex of $\triangle_{q'}$. If $\sigma \cap \sigma' \neq \emptyset$ then $\sigma = \sigma'$, and the slices $S_{q,\sigma}$ and $S_{q',\sigma'}$ are equal.*

PROOF. Suppose $a \in \sigma \cap \sigma'$. Then q and q' are critical points of f_a with index zero, nullities $m_{q,\sigma}$, $m_{q',\sigma'}$, and critical submanifolds $S_{q,\sigma}$, $S_{q',\sigma'}$ of f_a at q and q' respectively. So it follows from Proposition 8.3.6 that $S_{q,\sigma} = S_{q',\sigma'}$. It then follows from the Slice Theorem 6.5.9 that we have $\sigma = \sigma'$. ∎

8.5.3. Proposition. *Let σ be a simplex of a Weyl chamber in ν_q, $\varphi \in W$, and $S_{x,\sigma}$ the slice as in Theorem 6.5.9. Then $\varphi(S_{q,\sigma}) = S_{\varphi(q),\sigma}$.*

PROOF. Using Theorem 6.5.9, we see that $S_{q,\sigma}$ is the leaf of the distribution $\bigoplus\{E_j | \; j \in I(q,\sigma)\}$ through q. But both $\varphi(S_{q,\sigma})$ and $S_{\varphi(q),\sigma}$ are the leaves of the distribution $\bigoplus\{E_j | \; j \in I(\varphi(q),\sigma)\}$ through $\varphi(q)$. So $\varphi(S_{q,\sigma}) = S_{\varphi(q),\sigma}$. ∎

8.5.4. Theorem. *Let M be a rank k isoparametric submanifold of V, σ a simplex of $\triangle_q$ of dimension less than k, and $a \in \sigma$. Then f_a is non-degenerate in the sense of Bott, and the set $C(f_a)$ of critical points of f_a is $\bigcup\{S_{x,\sigma} | x \in W \cdot q\}$.*

PROOF. Let $x \in W \cdot q$. Then x is a critical point of f_a with nullity $m_{x,\sigma}$ and $S_{x,\sigma}$ is the critical submanifold of f_a through x. Hence $S_{x,\sigma} \subseteq C(f_a)$. Conversely, if $y \in C(f_a)$ then $a \in \nu_y$. By Theorem 8.3.7, a is a focal point with respect to y. so there exists $\varphi \in W$ such that $\varphi^{-1}(y) = y_0$, and a simplex σ' in the Weyl chamber $\triangle_{y_0}$ on ν_{y_0} such that $a \in \sigma'$. Then it follows from Proposition 8.5.2 that $\sigma = \sigma'$ and $S_{q,\sigma} = S_{y_0,\sigma}$. Thus we have

$$\varphi(S_{q,\sigma}) = S_{\varphi(q),\sigma} = \varphi(S_{y_o,\sigma}) = S_{\varphi(y_0),\sigma} = S_{y,\sigma}. \quad \blacksquare$$

8.5.5. Theorem. *Let M be an isoparametric submanifold in V, $q \in M$, and $\triangle_q$ the Weyl chamber of W on ν_q containing q. Let v be in $\nu(M)_q$, $\tilde{v}$ the parallel normal vector field on M determined by $\tilde{v}(q) = v$, and M_v the parallel submanifold $M_{\tilde{v}}$, i.e.,*

$$M_v = \{x + \tilde{v}(x) |\ x \in M\}.$$

Then:

(i) if $v \neq w$, and $q + v$ and $q + w$ are in $\triangle_q$, then M_v and M_w are disjoint,

(ii) given any $a \in V$ there exists a unique $v \in \nu(M)_q$ such that $q + v \in \triangle_q$ and $a \in M_v$,

(iii) each M_v is an embedded submanifold of V.

PROOF. Suppose $(q + v)$, $(q + w)$ are in $\triangle_q$, and $M_v \cap M_w \neq \emptyset$. Let $a \in M_v \cap M_w$ then there exist $x, y \in M$ such that $a = x + \tilde{v}(x) = y + \tilde{w}(y)$. Since $a \in \triangle_q$ and $\tilde{v}$, $\tilde{w}$ are parallel, $\langle \tilde{v}, v_i \rangle$ and $\langle \tilde{w}, v_i \rangle$ are constant. So $a \in \triangle_x$ and $a \in \triangle_y$, which imply that x and y are critical points of f_a with index 0. If a is non-focal then $x = y$, so by Proposition 8.3.6 we have $v = w$. If a is focal (suppose a lies in the simplex σ of $\triangle_q$) then the two critical submanifolds $S_{x,\sigma}$ and $S_{y,\sigma}$ are equal. In particular, $y \in S_{x,\sigma}$. Using the same notation as in the Slice Theorem 6.5.9, we note that the slice $S_{x,\sigma}$ is a finite dimensional isoparametric submanifold in $x + \eta(\sigma) \subset a + \nu(M_v)_a$. Let $v = u_1 + u_2$, where u_1 is the orthogonal projection of v along $V(\sigma)$. Then $S_{x,\sigma}$ is contained in the sphere of radius $\|u_1\|$ and centered at $x + u_1$. So $y + \tilde{u}_1(y) = x + u_1$. Since $V(\sigma)$ is perpendicular to $S_{x,\sigma}$, $\tilde{u}_2(y) = u_2$. Therefore we have $y + \tilde{v}(y) = x + \tilde{v}(x) = a = y + \tilde{w}(y)$, which implies that $v = w$

To prove (ii), we note that since f_a is bounded from below and satisfies condition C, there exists $x_0 \in M$ such that $f_a(x_0)$ is the minimum. Then $a \in \triangle_{x_0}$, so there exists a parallel normal field $\tilde{v}$ such that $\tilde{v}(x_0) = a - x_0$.

If $x, y \in M_v$ and $x + \tilde{v}(x) = y + \tilde{v}(y) = b$, then both x and y are critical points of f_b with index 0. Then, by Proposition 8.3.6 (iii), $f_b(x) = f_b(y)$ is the absolute minimum of f_b and if b is non-focal then, by 8.3.6 (ii), $x = y$. If b is a focal point of M, then a is a focal point with respect to both x and y by Theorem 8.3.7. Suppose a lies in a simplex σ of $\triangle_x$. Then by Proposition

8.3.6 again, $y \in N_{x,\sigma}$. Since $S_{x,\sigma}$ is isoparametric in $\eta(\sigma)$, it is an embedded submanifold, i.e., $x = y$. ∎

8.5.6. Corollary. *Let M be an isoparametric submanifold of V and $q \in M$. Then $\mathcal{F} = \{M_v \mid q + v \in \triangle_q\}$ defines an orbit-like singular foliation on V, which will be called the isoparametric foliation of M. The leaf space of $\mathcal{F}$ is isomorphic to the orbit space ν_q/W.*

8.5.7. Corollary. *If $a \in \sigma \subseteq \triangle_q$ and $a = q + v$, then the isoparametric foliation of $S_{q,\sigma}$ in $(a + \nu(M_v)_a)$ is $\{M_u \cap (a + \nu(M_u)_a) \mid M_u \in \mathcal{F}\}$.*

8.6. Convexity theorem

A well-known theorem of Schur ([Su]) can be stated as follows: Let M be the set of $n \times n$ Hermitian matrices with eigenvalues $a_1, \ldots, a_n$, and $u : M \to \boldsymbol{R}^n$ the map defined by $u((x_{ij})) = (x_{11}, \ldots, x_{nn})$. Then $u(M)$ is contained in the convex hull of $S_n \cdot a$, where S_n is the symmetric group acting on $\boldsymbol{R}^n$ by permuting the coordinates. Conversely, A. Horn ([Hr]) showed that the convex hull of $S_n \cdot a$ is contained in $u(M)$. Hence we have

8.6.1. Theorem. $u(M) = \mathrm{cvx}(S_n \cdot a)$, *the convex hull of $S_n \cdot a$.*

Note that Theorem 8.6.1 can be viewed as a theorem about a certain symmetric space, because M is an orbit of the isotropy representation of the symmetric space $\boldsymbol{SL}(n, C)/\boldsymbol{SU}(n, C)$, and u is the orthogonal projection onto a maximal abelian subspace. B. Kostant ([Ks]) generalized this to any symmetric space; his result is :

8.6.2. Theorem. *Let G/K be a symmetric space, $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the corresponding decomposition of the Lie algebra, $\mathcal{T}$ a maximal abelian subspace of $\mathcal{P}$, $W = N(\mathcal{T})/Z(\mathcal{T})$ the associated Weyl group of G/K acting on $\mathcal{T}$, and $u : \mathcal{P} \to \mathcal{T}$ the linear orthogonal projection onto $\mathcal{T}$. Let M be an orbit of the isotropy representation of G/K through z, i.e., $M = Kz$. Then $u(M) = \mathrm{cvx}(W \cdot z)$.*

The isotropy action of the compact symmetric space $G \times G/G$ is just the adjoint action of G on $\mathcal{G}$. Moreover, if we identify $\mathcal{G}$ to its dual $\mathcal{G}^*$ via the Killing form, then these orbits have a natural symplectic structure. In this case, the map u in Theorem 8.6.2 is the moment map. Recently, Theorem 8.6.2 has been generalized in the framework of symplectic geometry by Atiyah ([At]) and independently by Guillemin and Sternberg ([GS]) to the following:

8.6.3. Theorem. *Let M be a compact connected symplectic manifold with a symplectic action of a torus T, and $f : M \to \mathcal{T}^*$ the moment map. Then $f(M)$ is a convex polyhedron.*

The orbits that occur in Kostant's theorem 8.6.2, are isoparametric. Moreover, as we shall now see, it turns out that the convexity result follows just from this geometric condition of being isoparametric. Since there are infinitely many families of rank 2 isoparametric submanifolds that are not orbits of any linear orthogonal representation, the Riemannian geometric proof of Theorem 8.6.2 gives a more general result [Te3].

8.6.4. Main Theorem. *Let $M^n \subseteq \boldsymbol{S}^{n+k-1} \subseteq \boldsymbol{R}^{n+k}$ be isoparametric, $q \in M$, and W the associated Weyl group of M. Let P denote the orthogonal projection of $\boldsymbol{R}^{n+k}$ onto the normal plane $\nu_q = q + \nu(M)_q$, and $u = P|M$ the restriction of P to M. Then $u(M) = \mathrm{cvx}(W \cdot q)$, the convex hull of $W \cdot q$.*

As we said above, our main tool for proving this is Riemannian geometry. However, the basic idea of the proof goes back to Atiyah ([At]), and Guillemin-Sternberg ([GS]). Although there is no symplectic torus action around, the height function of M plays the role of the Hamiltonian function in their symplectic proofs. In section 8.3, we showed that M is taut (Corollary 8.3.3), i.e., every non-degenerate Euclidean distance function f_a on M is perfect. Because $M \subseteq \boldsymbol{S}^{n+k-1}$, the height function h_a and $-1/2 f_a$ differ by a constant, i.e.,

$$f_a = -2h_a + (1 + \|a\|^2).$$

In particular f_a and h_a have the same critical point theory. Using our detailed knowledge of the Morse theory of these height functions, Theorem 8.6.4 can be proved rather easily. It seems that tautness and convexity are closely related, however, the precise relation is not yet clear.

Henceforth we assume that $M^n \subseteq \boldsymbol{S}^{n+k-1} \subseteq \boldsymbol{R}^{n+k}$ is isoparametric, W is its Weyl group, and we use the same notations as in Chapter 6. In particular, for $x \in M$, we let $\triangle_x$ denote the Weyl chamber on $\nu_x = x + \nu(M)_x$ containing x. First we recall following results concerning the height functions.

8.6.5. Theorem. *Let $a \in \boldsymbol{R}^{n+k}$ be a fixed non-zero vector, $h_a : M \to \boldsymbol{R}$ the associated height function, i.e., $h_a(x) = \langle x, a \rangle$, and let $C(h_a)$ denote the set of all critical points of h_a.*

(i) $x \in C(h_a)$ if and only if $a \in \nu_x$.

(ii) If x_0 is an index 0 critical of h_a, then $b = h_a(x_0)$ is the absolute minimum value of h_a on M and $h_a^{-1}(b)$ is connected. Moreover,

(1) $a \in \triangle_{x_0}$,

(2) if $a \in \sigma$, a simplex of $\triangle_{x_0}$, then $h_a^{-1}(b) = S_{x_0,\sigma}$ (the slice through x_0 with respect to σ).

(iii) If $x \in C(h_a)$ and a is regular with respect to the W-action on ν_x, then h_a is non-degenerate and $C(h_a) = W \cdot x$.

(iv) If a is W-singular, then h_a is non-degenerate in the sense of Bott ([Bt]). More specifically, if $x^0 \in C(h_a)$ is a minimum and a lies on the simplex σ of $\triangle_{x^0}$, then

$$C(h_a) = \bigcup\{S_{x,\sigma} | \, x \in W \cdot x^0\}.$$

8.6.6. Lemma. *We use the same notation as in Theorem 8.6.4. Let $u = P|M$, the restriction of P to M, and C the set of all singular points of u. Then C is the union of all slices $S_{x,\sigma}$ for x in $W \cdot q$ and σ a 1-simplex of some Weyl chamber of ν_q.*

PROOF. We may assume that $\nu_q = \boldsymbol{R}^k$. Let $t_1, \ldots, t_k$ be the standard base of $\boldsymbol{R}^k$. Then $u = (u_1, \ldots, u_k)$, where $u_i(x) = h_{t_i}(x) = \langle x, t_i \rangle$. It is easy to see that the following statements are equivalent:

(1) $\text{rank}(du_x) < k$.

(2) $du_1(x), \ldots, du_k(x)$ are linearly dependent.

(3) there exists a non-zero vector $a = (a_1, \ldots, a_k)$ such that

$$a_1 du_1(x) + \ldots + a_k du_k(x) = 0.$$

(4) x is a critical point of some height function h_a.

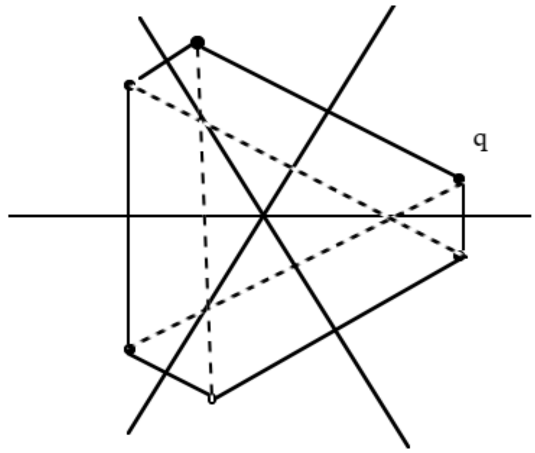
Then the lemma follows from Theorem 8.5.4. ∎

8.6.7. Proof of the Main Theorem. We will use induction on k to show that:

(*) $u(M) = \text{cvx}(W \cdot q)$, if M is isoparametric of rank k.

If $k = 1$, then M is a standard sphere of $\boldsymbol{R}^{n+1}$, so $u(M)$ is the line segment joining q to $-q$. Suppose (*) is true when the codimension is less than k, and M^n is full and isoparametric in $\boldsymbol{R}^{n+k}$. Then we want to show that $u(M) = D$, where $D = \text{cvx}(W \cdot q)$. We divide our proof into five steps.

(i) Let C denote the set of singular points of u. Then $u(C)$ is the union of finitely many $(k-1)$-polyhedra, and $\partial D \subseteq u(C) \subseteq D$. So in particular, $D \setminus u(C)$ is open.



To see this, we note by Theorem 6.5.9 that if σ a 1-simplex on ν_q and $x \in W \cdot q$ then the slice $S_{x,\sigma}$ is a rank $k-1$ isoparametric submanifold. So by the induction hypothesis, $u(S_{x,\sigma})$ is a $(k-1)$-polyhedron. Then using Theorem 8.5.4, we have

$$u(C) = \bigcup \{P(S_{x,\sigma}) \mid x \in W \cdot q,$$
$$\sigma \text{ a } 1-\text{simplexof some Weyl chamber of } \nu_q\}.$$

But it is also easy to see that

$$\partial D = \bigcup \{P(S_{x,\sigma}) | \, x \in W \cdot q, \; \sigma \text{ a } 1-\text{simplex of } \triangle_x\}.$$

(ii) $\partial(u(M)) \subseteq u(C)$. This follows from the Inverse function theorem, because the image of a regular point of u is in the interior of $u(M)$.

(iii) $u(M) \subseteq D$. This follows from the fact that $u(C) \subseteq D$.

(iv) If U_i is a connected component of $D \setminus u(C)$, then either $U_i \subseteq u(M)$, or $U_i \cap u(M) = \emptyset$. To prove this, we proceed as follows: Suppose $U_i \cap u(M)^0$ is a non-empty proper subset of U_i, where $u(M)^0$ denotes the interior of $u(M)$. Then there is a sequence $y_n \in U_i \cap u(M)^o$ such that y_n converges to y, which is not in $U_i \cap u(M)^o$. Since $u(M)$ is compact, $y \in u(M)$. Using step (ii), we have $\partial(u(M)) \subseteq u(C)$. But by definition of U_i, $U_i \cap u(C) = \emptyset$, so we conclude that y is a regular value of u, hence $y \in U_i \cap u(M)^o$, a contradiction.

(v) $U_i \subseteq u(M)$ for all i. Suppose not, then we may assume $U_1 \cap u(M) = \emptyset$. Using step (i), we know that ∂U_1 is the union of $(k-1)$-polyhedra. Let μ be a $(k-1)$-face of ∂U_1, and t the outward unit normal of ∂U_1 at μ. Then by Euclidean geometry the height function h_t on M has local minimum value c_0 on μ, hence by Proposition 8.3.6, c_0 is the absolute minimum of h_t and $\mu \subset \partial D$. But by Euclidean geometry $\mu \subseteq \partial D$ implies that c_0 is also a local maximum value of h_t hence the maximum value of h_t on M, and hence M is contained in the hyperplane $\langle x, t \rangle = c_0$, which contradicts the fact that M is full.

This completes the proof of (*). ∎

If $z \in \nu_q$ is W-regular, then the leaf M_z of the parallel foliation of M through z is isoparametric of codimension k and $\nu(M_z)_z = \nu(M)_q$. Hence (*) implies that $P(M_z) = \mathrm{cvx}(W \cdot z)$. If $z \in \nu_q$ is W-singular, then we may assume that $M_z = M_v$ for a parallel normal field v on M, and M_{tv} is isoparametric for all $0 \leq t < 1$ (or equivalently that $q + tv(q)$ is W-regular for all $0 \leq t < 1$). We define the following smooth map

$$F : M \times [0,1] \to \boldsymbol{R}^k, \quad \text{by } F(x,t) = P(x + tv(x)).$$

Let $u_t(x) = F(x,t)$, then $u_t(M) = P(M_{tv})$. By (*), $P(M_{tv}) = \mathrm{cvx}(W \cdot (q + tv(q)))$ for all $0 \leq t < 1$. But $u_t \to u_1$ uniformly as $t \to 1$, so its image $u_t(M)$ converges in the Hausdorff topology to $u_1(M)$. But $q + tv(q)$ converges to $q + v(q) = z$, so $P(M_{tv})$ converges to the convex hull of $W \cdot (q + v(q)) = W \cdot z$, hence we obtain

8.6.8. Theorem. *With the same assumption as in Theorem 8.6.4. Let $z \in \nu_q$, and M_z the leaf of the parallel foliation of M through z. Then $P(M_z) = \mathrm{cvx}(W \cdot z)$.*

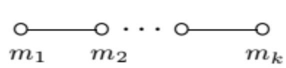
8.7. Marked Dynkin Diagrams for Isoparametric Submanifolds

In this section we determine the possible marked Dynkin diagrams for both the finite and infinite dimensional isoparametric submanifolds.

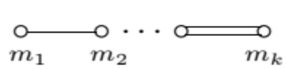
Let M be a rank k irreducible isoparametric submanifold in a Hilbert space V, $\{\ell_i | \, i \in I\}$ the focal hyperplanes, $\{v_i | \, i \in I\}$ the curvature normals, m_i the corresponding multiplicities, and W the associated Coxeter group.

(1) If V is of finite dimension, then we may assume that $\ell_1, \ldots, \ell_k$ form a simple root system for W, and the marked Dynkin diagram has k vertices (one for each ℓ_i, $1 \leq i \leq k$) such that the i^{th} vertex is marked with multiplicities m_i and there are $\alpha(g)$ edges joined the i^{th} and j^{th} vertices if the angle between ℓ_i and ℓ_j is π/g, where $\alpha(g) = g - 2$ if $1 < g \leq 4$ and $\alpha(6) = 3$. So the possible marked Dynkin diagram for rank k finite dimensional irreducible isoparametric submanifolds are:

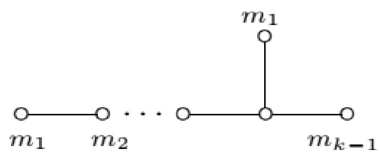
A_k



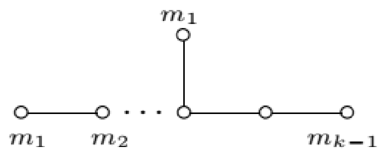
B_k



D_k



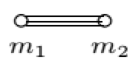
E_k



k=6,7,8

F_4

G_2



(2) If V is an infinite dimensional Hilbert space, then we may assume that $\ell_1, \ldots, \ell_{k+1}$ form a simple root system for W, and the marked Dynkin diagram has $k+1$ vertices (one for each ℓ_i, $1 \leq i \leq k+1$); the i^{th} vertex is marked with the multiplicity m_i. There are $\alpha(g)$ edges joining the i^{th} and j^{th} vertices if the angle between ℓ_i and ℓ_j is π/g with $g > 1$, and there are infinitely many edges joining i^{th} and j^{th} vertices if ℓ_i is parallel to ℓ_j. So using the classification of the affine Weyl groups, we can easily write down the possible marked Dynkin diagrams for rank k infinite dimensional irreducible isoparametric submanifolds.

Let $q \in M$, and $\nu_q = q + \nu(M)_q$. Given $i \neq j \in I$, suppose ℓ_i is not parallel to ℓ_j and the angle between ℓ_i and ℓ_j is π/g. Then there exists a unique $(k-2)$-dimensional simplex σ of the chamber $\triangle$ on ν_q containing q such that $\sigma \subseteq \ell_i(q) \cap \ell_j(q)$. By the Slice Theorem, 6.5.9, the slice $S_{q,\sigma}$ is a finite dimensional rank 2 isoparametric submanifold with the Dihedral group of $2g$ elements as its Coxeter group, and m_i, m_j its multiplicities. So use the classification of Coxeter groups and the results in section 8.4 of Cartan, Münzner, and Abresch on rank 2 finite dimensional case, we obtain some immediate restrictions of the possible marked Dynkin diagrams for rank k isoparametric submanifolds of Hilbert spaces. In particular, we have

8.7.1. Theorem. *If M^n is isoparametric in $\boldsymbol{R}^{n+k}$, then the angle between any two focal hyperplanes ℓ_i and ℓ_j is π/g for some $g \in \{2, 3, 4, 6\}$.*

8.7.2. Corollary. *If M^n is an irreducible rank k isoparametric submanifold in $\boldsymbol{R}^{n+k}$, then the associated Coxeter group W of M is an irreducible Weyl (or crystallographic) group.*

8.7.3. Proposition. *There are at most two distinct multiplicities for an irreducible isoparametric submanifolds M of V.*

PROOF. If the i^{th} and $(i+1)^{th}$ vertices of the Dynkin diagram are joined by one edge, then by Theorem 8.4.2, $m_i = m_{i+1}$. But each irreducible Dynkin graph has at most one i_0 such that the i_0^{th} and $(i_0+1)^{th}$ vertices are joined by more than one edge. So the result follows. ∎

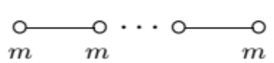
8.7.4. Theorem. *Let M^n be a rank k isoparametric submanifold in $\mathbf{R}^{n+k}$. If all the multiplicities are even, then they are all equal to an integer m, where $m \in \{2, 4, 8\}$.*

PROOF. If the i^{th} and $(i+1)^{th}$ vertices of the Dynkin diagram are joined by two or four edges, then by Theorem 8.4.5, $m_i = m_{i+1} = 2$. ∎

To obtain further such restrictions we need the more information on the cohomology ring of M. The details can be found in [HPT2]. Here we will only state the results without proof.

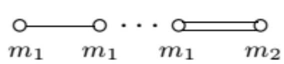
8.7.5. Theorem. *The possible marked Dynkin diagrams of irreducible rank $k \geq 3$ finite dimensional isoparametric submanifolds are as follows:*

A_k



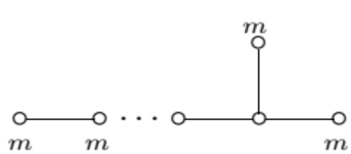
$m\in\{1,2,4\}$

B_k



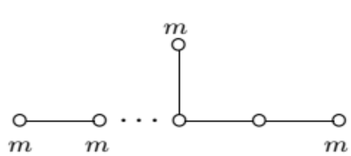
(m_1, m_2) satisfies $(*)$ below.

D_k



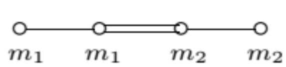
$m\in\{1,2,4\}$

E_k



$m\in\{1,2,4\}$ $k=6,7,8$

F_4



$m_1=m_2=2$ or $m_1=1$, $m_2\in\{1,2,4,8\}$

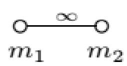
The pair (m_1, m_2) satisfies (*) if it satisfies one of the following conditions:

(1) $m_1 = 1$, *m_2 is arbitrary,*
(2) $m_1 = 2$, $m_2 = 2$ *or* $2r + 1$,
(3) $m_1 = 4$, $m_2 = 1, 5$, *or* $4r + 3$,
(4) $k = 3,\ m_1 = 8,\ m_2 = 1, 3, 7, 11$, *or* $8r + 7$.

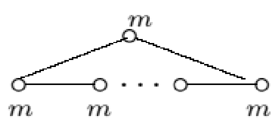
As a consequence of Theorem 8.7.5, Theorem 8.4.4 and the Slice Theorem 6.5.9, we have:

8.7.6. Theorem. *The possible marked Dynkin diagrams of irreducible rank $k \geq 3$ infinite dimensional isoparametric submanifolds are as follows:*

$\tilde{A}_1$

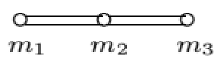


$\tilde{A}_k$



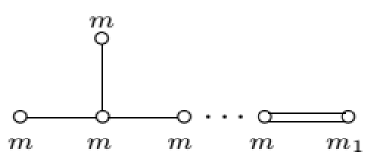
$m \in \{1,2,4\}$

$\tilde{B}_2$



(m_1, m_2), (m_2, m_3) satisfy $(*)$.

$\tilde{B}_k$

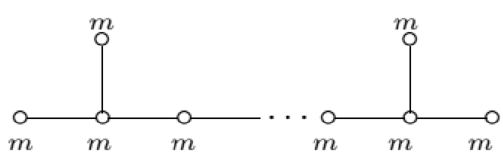


(m, m_1) satisfies $(*)$.

$\tilde{C}_k$

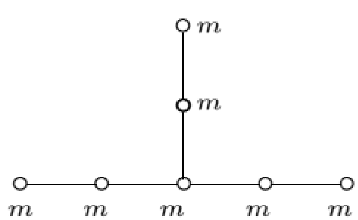
(m, m_1), (m, m_2) satisfy $(*)$.

$\tilde{D}_k$



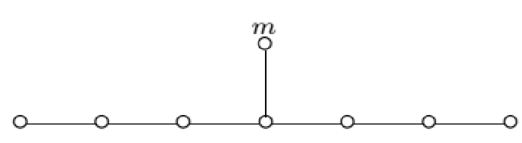
$m \in \{1,2,4\}$

$\tilde{E}_6$



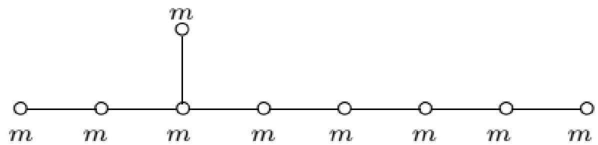
$m \in \{1,2,4\}$

$\tilde{E}_7$



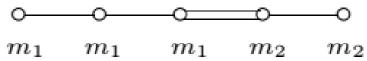
$m \in \{1,2,4\}$

$\tilde{E}_8$



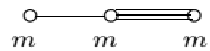
$m \in \{1,2,4\}$

$\tilde{F}_4$



$m_1 = m_2 = 2$ or
$m_2 = 1$, $m_1 \in \{1,2,4\}$ or
$m_1 = 1$, $m_2 \in \{1,2,4,8\}$

$\tilde{G}_2$

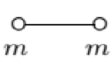


$m \in \{1,2,4\}$

Let G/K be a rank k symmetric space, $\mathcal{G} = \mathcal{K} + \mathcal{P}$, $\mathcal{A}$ the maximal abelian subalgebra contained in $\mathcal{P}$, and $q \in \mathcal{A}$ a regular point with respect to the isotropy action K on $\mathcal{P}$. Then $M = Kq$ is a principal orbit, and is a rank k isoparametric submanifold of $\mathcal{P}$. The Weyl group associated to M as an isoparametric submanifold is the standard Weyl group associated to G/K, i.e., $W = N(\mathcal{A})/Z(\mathcal{A})$. If $x_i \in \ell_i(q)$ and x_i lies on a $(k-1)$-simplex, then $m_i = \dim(M) - \dim(Kx_i)$. It is shown in [PT2] that these principal orbits are the only homogeneous isoparametric submanifolds (i.e., a submanifold which is both an orbit of an orthogonal action and is isoparametric). So from the classification of symmetric spaces, we have (for details see [HPT2])

8.7.7. Theorem. *The marked Dynkin diagrams for rank k, irreducible, finite dimensional, homogeneous isoparametric submanifolds are the following:*

A_2

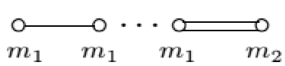


$m \in \{1,2,4,8\}$

A_k

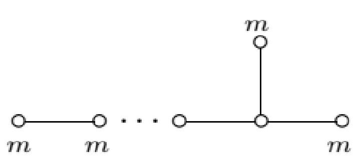
$m \in \{1,2,4\}$

B_k



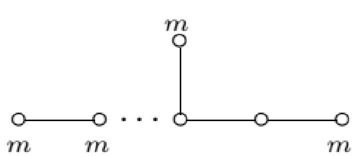
(m_1, m_2) satisfies $(*)$

D_k



$m \in \{1,2\}$

E_k



$m \in \{1,2\}$ $k = 6,7,8$

F_4 

$m_1{=}m_2{=}2$ or $m_1{=}1$, $m_2{\in}\{1,2,4,8\}$

G_2 

$m{\in}\{1,2\}$

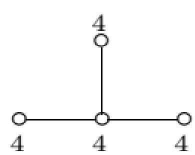
The pair (m_1, m_2) satisfies (*) in all of the following cases:
(1) $m_1 = 1$, m_2 is arbitrary,
(2) $m_1 = 2$, $m_2 = 2$ or $2m + 1$,
(3) $m_1 = 4$, $m_2 = 1, 5, 4m + 3$,
(4) $m_1 = 8$, $m_2 = 1$,
(5) $k = 2$, $m_1 = 6$, $m_2 = 9$.

8.7.8. Corollary. *The set of multiplicities (m_1, m_2) of homogeneous, isoparametric, finite dimensional submanifolds with B_2 as its Coxeter groups is*

$$\{(1, m),\ (2, 2m + 1),\ (4, 4m + 3),\ (9, 6),\ (4, 5),\ (2, 2)\}.$$

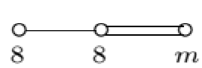
8.7.9. Open problems. If we compare Theorem 8.7.5 and 8.7.7, it is natural to pose the following problems:

(1) Is it possible to have an irreducible, rank 3, finite dimensional isoparametric submanifold, whose marked Dynkin diagram is the following?

D_4 

(It would be interesting if such an example does exist, however we expect that most likely it does not. Of course a negative answer to this problem would also imply the non-existence of marked Dynkin diagrams with uniform multiplicity 4 of D_k-type, $k > 5$ or E_k-type, $k = 6, 7, 8$.

(2) Is it possible to have an isoparametric submanifold whose marked Dynkin diagram is of the following type with $m > 1$:

B_3 

(3) Let $M^n \subseteq \boldsymbol{R}^{n+k}$ be an irreducible isoparametric submanifold with uniform multiplicities. Is it necessarily homogeneous ?

If the answer to problem 3 is affirmative and if the answers to problem 1 and 2 are both negative, then the remaining fundamental problem would be:

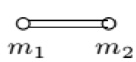
(4) Are there examples of non-homogeneous irreducible isoparametric submanifolds of rank $k > 3$?

It follows from section 6.4 that if $n = 2(m_1 + m_2 + 1)$, and $f : \boldsymbol{R}^n \to \boldsymbol{R}$ is a homogeneous polynomial of degree 4 such that

$$\triangle f(x) = 8(m_2 - m_1)\,\|x\|^2, \quad \|\nabla f(x)\|^2 = 16\,\|x\|^6, \tag{8.7.1}$$

then the polynomial map $x \mapsto (|x|^2, f(x))$ is isoparametric and its regular levels are isoparametric submanifolds of $\boldsymbol{R}^{n+2}$ with

B_2



as its marked Dynkin diagram, i.e., B_2 is the associated Weyl group with (m_1, m_2) as multiplicities. Solving (8.7.1), Ozeki and Takeuchi found the first two families of non-homogeneous rank 2 examples. In fact, they constructed the isoparametric polynomial explicitly as follows:

8.7.10. Examples. (Ozeki-Takeuchi [OT1,2]) Let $(m_1, m_2) = (3, 4r)$ or $(7, 8r)$, $F = H$ or Ca (the quarternions or Cayley numbers) for $m_1 = 3$ or 7 respectively, and let $n = 2(m_1 + m_2 + 2)$. Let $u \mapsto \bar{u}$ denote the canonical involution of F. Then

$$(u, v) = \frac{1}{2}(u\bar{v} + v\bar{u})$$

defines an inner product on F, that gives an inner product on F^m. We let

$$f_0 : \boldsymbol{R}^n = F^{2(r+1)} = F^{1+r} \times F^{1+r} \to \boldsymbol{R},$$

$$f_0(u, v) = 4(\|u\bar{v}^t\|^2 - (u, v)^2) + (\|u_1\|^2 - \|v_1\|^2 + 2(u_0, v_0))^2,$$

where $u = (u_0, u_1)$, $v = (v_0, v_1)$ and $u_0, v_0 \in F$, $u_1, v_1 \in F^r$. Then

$$f(u, v) = (\|u\|^2 + \|v\|^2)^2 - 2f_0(u, v)$$

satisfies (8.7.1). So the intersection of a regular level of f and $\boldsymbol{S}^{n-1}$ is isoparametric with B_2 as the associated Weyl group and $(3, 4r)$ or $(7, 8r)$ as multiplicities. These examples correspond to $(3, 4r)$ and $(7, 8r)$ are non-homogeneous. But there is also a homogeneous example with B_2 as its Weyl group and $(3, 4)$ as its multiplicities. So the marked Dynkin diagram does not characterize an isoparametric submanifold.

8.7.11. Examples. Another family of non-homogeneous rank 2 isoparametric examples is constructed from the representations of the Clifford algebra $C\ell^{m+1}$ by Ferus, Karcher and Münzner (see [FKM] for detail). It is known from representation theory that every irreducible representation space of $C\ell^{m+1}$ is of even dimension, and it is given by a "Clifford system" $(P_0, \dots, P_m)$ on $\boldsymbol{R}^{2r}$, i.e., the $P_i's$ are in $\boldsymbol{SO}(2r)$ and satisfy

$$P_iP_j + P_jP_i = 2\delta_{ij}Id.$$

Let $f : \boldsymbol{R}^{2r} \to \boldsymbol{R}$ be defined by

$$f(x) = \|x\|^4 - \sum_{i=0}^{m} \langle P_i(x), x\rangle^2.$$

Then f satisfies (8.7.1) with $m_1 = m$ and $m_2 = r - m_1 - 1$. If $m_1, m_2 > 0$, then the regular levels of the map $x \mapsto (\|x\|^2, f(x))$ are isoparametric with Coxeter group B_2 and multiplicities (m_1, m_2). Most of these examples are non-homogeneous.

8.7.12. Remark. The classification of isoparametric submanifolds is still far from being solved. For example we do not know

(1) what the set of the marked Dynkin diagrams for rank 2 finite dimensional isoparametric submanifolds is,

(ii) what the rank k homogeneous infinite dimensional isoparametric submanifolds are.