Proper Fredholm Submanifolds in Hilbert Spaces

In this chapter we generalize the submanifold theory of Euclidean space to Hilbert space. In order to use results the infinite dimensional differential topological we restrict ourself to the class of proper Fredholm immersions (defined below).

Proper Fredholm immersions

Let M be an immersed submanifold of the Hilbert space V (i.e., TM_x is a closed linear subspace of V), and let $\nu(M)_x = (TM_x)^\perp$ denote the normal plane of M at x in V. Using the same argument as in chapter 2, we conclude that, given a smooth normal field v on M and $u \in TM_{x_0}$, the orthogonal projection of $dv_{x_0}(u)$ onto TM_{x_0} depends only on $v(x_0)$ and not on the derivatives of v at x_0; it be denoted by $-A_{v(x_0)}(u)$ (the shape operator of M with respect to the normal vector $v(x_0)$). The first and second fundamentals forms I, II and the normal connection ∇^ν on M can be defined in the same (invariant) manner as in the finite dimensional case, i.e.,

$$I(x) = \langle\ ,\ \rangle | TM_x,$$

$$\langle II(x)(u_1, u_2), v\rangle = \langle A_v(u_1), u_2\rangle,$$

$$\nabla^\nu v = \text{the orthogonal projection of } dv \text{ onto } \nu(M).$$

Since all these local invariants for M are well-defined, the method of moving frame is valid here (because when we expand well-defined tensor fields in terms of local orthonormal frame field, then the infinite series are convergent). Arguing the same way as in the finite dimensional case, we can prove that I, II and the induced normal connection ∇^ν satisfy the Gauss, Codazzi and Ricci equations. Moreover, the Fundamental Theorem 2.3.1 is valid for immersed submanifolds of Hilbert space. As a consequence of the Ricci equation, we also have the analogue of Proposition 2.1.2:

7.1.1. Proposition. *Suppose M is an immersed submanifold of the Hilbert space V and the normal bundle $\nu(M)$ is flat. Then the family $\{A_v | v \in \nu(M)_x\}$ of shape operators is a commuting family of operators on TM_x.*

Although these elementary parts of the theory of submanifold geometry work just as in the finite dimensional case, many of the deeper results are not true in general. For example, the infinite dimensional differential topology developed by Smale and infinite dimensional Morse theory developed by Palais and Smale will not work for general submanifolds of Hilbert space without further restrictions. Recall also that the spectral theory of the shape operators and the Morse theory of the Euclidean distance functions of submanifolds of $\boldsymbol{R}^n$ are closely related and play essential roles in the study of the geometry and topology of submanifolds of $\boldsymbol{R}^n$. Here again, without some restrictions important aspects of these theories will not carry over to the infinite dimensional setting. One of the main goals of this section is to describe a class of submanifolds of Hilbert space for which the techniques of infinite dimensional geometry and topology can be applied to extend some of the deeper parts of the theory of submanifold geometry.

The end point map $Y : \nu(M) \to V$ for an immersed submanifold M of a Hilbert space V is defined just as in Definition 4.1.7; i.e., $Y : \nu(M) \to V$ is given by $Y(v) = x + v$ for $v \in \nu(M)_x$.

7.1.2. Definition. An immersed finite codimension submanifold M of V is *proper Fredholm* (PF), if

(i) the end point map Y is Fredholm,

(ii) the restriction of Y to each normal disk bundle of finite radius r is proper.

Since the basic theorems of differential calculus and local submanifold geometry work for PF submanifolds just as for submanifolds of $\boldsymbol{R}^n$, Proposition 4.1.8 is valid for PF submanifolds of Hilbert spaces. In particular, we have

$$dY_v = (I - A_v, id), \tag{7.1.1}$$

which implies that

7.1.3. Proposition. *The end point map Y of an immersed submanifold M of a Hilbert space V is Fredholm if and only if $I - A_v$ is Fredholm for all normal vector v of M.*

7.1.4. Remarks.

(i) An immersed submanifold M of $\boldsymbol{R}^n$ is PF if and only if the immersion is proper.

(ii) If M is a PF submanifold of V, and M is contained in the sphere of radius r with center x_0 in V, then $v(x) = x_0 - x$ is a normal field on M with length r, and $Y(x, v(x)) = x_0$. Since Y is proper on the r-disk normal bundle, M is compact. This implies that M must be finite dimensionional. It follows that PF submanifolds of an infinite dimensional Hilbert space V cannot lie on a hypersphere of V. In particular, the unit sphere of V is not PF.

7.1.5. Examples.

(1) A finite codimension linear subspace of V is PF.

(2) Let $\varphi : V \to V$ be a self-adjoint, injective, compact operator. Then the hypersurface

$$M = \{x \in V | \ \langle \varphi(x), x\rangle = 1\}$$

is PF. To see this we note that $v(x) = \varphi(x)/\|\varphi(x)\|$ is a unit normal field to M, and $A_{v(x)}(u) = -\varphi(u)^{TM_x}/\|\varphi(x)\|$ is a compact operator on TM_x, where $\varphi(u)^{TM_x}$ denote the orthogonal projection of $\varphi(u)$ onto TM_x. So it follows from Proposition 7.1.3 that the end point map of M is Fredholm. Next assume that $x_n \in M$, $\{\lambda_n\varphi(x_n)\}$ is bounded, and $Y(x_n, \lambda_n\varphi(x_n)) = x_n + \lambda_n\varphi(x_n) \to y$. Then x_n is bounded, and $\langle x_n+\lambda_n\varphi(x_n), x_n\rangle = \|x_n\|^2+\lambda_n$ is bounded, which implies that λ_n is bounded. Since φ is compact and $\{\lambda_n x_n\}$ is bounded, $\varphi(\lambda_n x_n)$ has a convergent subsequence, and so $\{x_n\}$ has a convergent subsequence.

7.1.6. Theorem. *Suppose G is an infinite dimensional Hilbert Lie group, G acts on the Hilbert space V isometrically, and the action is proper and Fredholm. Then every orbit Gx is an immersed PF submanifold of V.*

PROOF. First we prove that the end point map Y of $M = Gx$ is Fredholm. Because every isometry of V is an affine transformation, we have

$$(I - A_v)(\xi(x)) = (\xi(x+v))^{T_x},$$

where $\xi \in \mathcal{G}$, $v \in \nu(M)_x$, and u^{T_x} denotes the tangential component of u with respect to the decomposition $V = TM_x \oplus \nu(M)_x$. It follows from the definition of Fredholm action that the differential of the orbit map at e is Fredholm. So the two maps $\xi \mapsto \xi(x)$ and $\xi \mapsto \xi(x+v)$ are Fredholm maps from $\mathcal{G}$ to V. In particular, $T(Gx)_x$ and $T(G(x+v))_{x+v}$ are of finite codimension. So the map $P : T(G(x+v))_{x+v} \to T(Gx)_x$ defined by $P(u) = u^{T_x}$ is Fredholm. Hence $I - A_v$ is Fredholm, i.e., Y is Fredholm. Next we assume that $x_n \in M$, $v_n \in \nu(M)_{x_n}$, $\|v_n\| \le r$, and $Y(x_n, v_n) \to y$. Then there exist linear isometry φ_n of V and $c_n \in V$ such that $g_n = \varphi_n + c_n \in G$ and $x_n = g_n(x)$. Note that $dg_n = \varphi_n$, $u_n = \varphi_n^{-1}(v_n) \in \nu(M)_x$, and

$$Y(g_n x, v_n) = \varphi_n(x)+c_n+\varphi_n(u_n) = \varphi_n(x+u_n)+c_n = g_n(x+u_n) \to y.$$

Since $\{u_n\}$ is a bounded sequence in the finite dimensional Euclidean space $\nu(M)_x$, there exists a convergent subsequence $u_{n_i} \to u$. So we have $g_{n_i}(x + u_{n_i}) \to y$ and $x + u_{n_i} \to x + u$. It then follows from the definition of proper action that g_{n_i} has a convergent subsequence in G, which implies that x_{n_i} has a convergent subsequence in M. ∎

7.1.7. Proposition. *Let M be an immersed PF submanifold of V, $x \in M$, $v \in \nu(M)_x$, and A_v the shape operator of M with respect to v. Then:*

(1) A_v has no residual spectrum,

(2) the continuous spectrum of A_v is either $\{0\}$ or empty,

(3) the eigenspace corresponding to a non-zero eigenvalue of A_v is of finite dimension,

(4) A_v is compact.

PROOF. Since A_v is self-adjoint, it has no residual spectrum. Note that the eigenspace of A_v with respect to a non-zero eigenvalue λ is

$$\mathrm{Ker}(\lambda I - A_v) = \mathrm{Ker}(I - \frac{1}{\lambda} A_v) = \mathrm{Ker}(I - A_{\frac{v}{\lambda}}).$$

So (3) follows from Proposition 7.1.3. Now suppose $\lambda \neq 0, \mathrm{Ker}(A_v - \lambda I) = 0$, and $\mathrm{Im}(A_v - \lambda I)$ is dense in TM_x. Since $A_v - \lambda I$ is Fredholm, $\mathrm{Im}(A_v - \lambda I)$ is closed and equal to TM_x, i.e., $A_v - \lambda I$ is invertible, which proves (2). To prove (4) it suffices to show that if λ_i is a sequence of distinct real numbers in the discrete spectrum of A_v and $\lambda_i \to \lambda$ then $\lambda = 0$. But if $\lambda \neq 0$, then the self-adjoint Fredholm operator $P = I - A_{v/\lambda}$ induces an isomorphism $\tilde{P}$ on $V/\mathrm{Ker}(P)$, so $\tilde{P}$ is bounded. Let δ denote $\|\tilde{P}\|$. Then $|(1 - \lambda_i/\lambda)^{-1}| \leq \delta$, and hence $|\lambda - \lambda_i|/|\lambda| \geq 1/\delta > 0$, contradicting $\lambda_i \to \lambda$. ∎

It follows from (7.1.1) that $e \in \nu(M)_x$ is a regular point of Y if and only if $I - A_e$ is an isomorphism. Moreover, the dimension of $\mathrm{Ker}(I - A_e)$ and $\mathrm{Ker}(dY_e)$ are equal, which is finite by Proposition 7.1.3. Hence the Definition 4.2.1 of focal points and multiplicities makes sense for PF submanifolds.

7.1.8. Definition. Let $e \in \nu(M)_x$. The point $a = Y(e)$ in V is called a non-focal point for a PF submanifold M of V with respect to x if dY_e is an isomorphism. If $m = \dim(\mathrm{Ker}(dY_e)) > 0$ then a is called a focal point of multiplicity m for M with respect to x.

The set Γ of all the focal points of V is called the *focal set* of M in V, i.e., Γ is the set of all critical values of the normal bundle map Y. So applying the Sard-Smale Transversality theorem [Sm2] for Fredholm maps to the end point map Y of M, we have:

7.1.9. Proposition. *The set of non-focal points of a PF submanifold M of V is open and dense in V.*

By the same proof as in Proposition 4.1.5, we have:

7.1.10. Proposition. *Let M be an immersed PF submanifold of V, and $a \in V$. Let $f_a : M \to \boldsymbol{R}$ denote the map defined by $f_a(x) = \|x - a\|^2$. Then:*

(i) $\nabla f_a(x) = 2(x-a)^{T_x}$, the projection of $(x-a)$ onto TM_x, so in particular x_0 is a critical point of f_a if and only if $(x_0 - a) \in \nu(M)_{x_0}$,

(ii) $\frac{1}{2}\nabla^2 f_a(x_0) = I - A_{(a-x_0)}$ at the critical point x_0 of f_a,

(iii) f_a is non-degenerate if and only if a is a non-focal point of M in V,

It follows from Propositions 7.1.9 and 7.1.10 that:

7.1.11. Corollary. *If M is an immersed PF submanifold of V, then f_a is non-degenerate for all a in an open dense subset of V.*

As a consequence of Proposition 7.1.7 and 7.1.10:

7.1.12. Proposition. *Let M be an immersed PF submanifold of V. Suppose x_0 is a critical point of f_a and V_λ is the eigenspace of $A_{(a-x_0)}$ with respect to the eigenvalue $\lambda \neq 0$.*
Then:

(i) $\dim(V_\lambda)$ is finite,

(ii) $\mathrm{Index}(f_a, x_0) = \sum\{\dim(V_\lambda) | \lambda > 1\}$, which is finite.

Morse theory relates the homology of a smooth manifold to the critical point structure of certain smooth functions. This theory was extended to infinite dimensional Hilbert manifolds in the 1960's by Palais and Smale ([Pa2],[Sm1]) for the class of smooth functions satisfying Condition C (see Part II, chapter 1).

7.1.13. Theorem. *Let M be an immersed PF submanifold of a Hilbert space V, and $a \in V$. Then the map $f_a : M \to \boldsymbol{R}$ defined by $f_a(x) = \|x - a\|^2$ satisfies condition C.*

PROOF. We will write f for f_a. Suppose

$$|f(x_n)| \le c, \ \ \|\nabla f(x_n)\| \to 0.$$

Let u_n be the orthogonal projection of $(x_n - a)$ onto TM_{x_n}, and v_n the projection of $(x_n - a)$ onto $\nu(M)_{x_n}$. Since $\|x_n - a\|^2 \le c$ and $u_n \to 0$, $\{v_n\}$ is bounded (say by r). So $(x_n, -v_n)$ is a sequence in the r-disk normal bundle of M, and

$$Y(x_n, -v_n) = x_n - v_n = (x_n - a) - v_n + a = u_n + a \to a.$$

Since M is a PF submanifold, $(x_n, -v_n)$ has a convergent subsequence in $\nu(M)$, which implies that x_n has a convergent subsequence in M. ∎

7.1.14. Remark. Let M be an immersed submanifold of V (not necessarily PF). Then the condition that all f_a satisfy condition C is equivalent to the condition that the restriction of the end point map to the unit disk normal bundle is proper.

7.2. Isoparametric submanifolds in Hilbert spaces

In this section we will study the geometry of isoparametric submanifolds of Hilbert spaces. They are defined just as in $\boldsymbol{R}^n$.

7.2.1. Definition. An immersed PF submanifold M of a Hilbert space $(V, \langle\, ,\, \rangle)$ is called isoparametric if

(i) $\nu(M)$ is globally flat,

(ii) if v is a parallel normal field on M then the shape operators $A_{v(x)}$ and $A_{v(y)}$ are orthogonally equivalent for all $x, y \in M$.

7.2.2. Remark. Although Definition 5.7.2 seems weaker than Definition 7.2.1 (where we only assume that $\nu(M)$ is flat), if $V = \boldsymbol{R}^n$, we have proved in Theorem 6.4.4 that $\nu(M)$ *is* globally flat. So these two definitions agree when V is a finite dimensional Hilbert space.

7.2.3. Definition. An immersed submanifold $f : M \to V$ is *full*, if $f(M)$ is not included in any affine hyperplane of V. M is a rank k immersed isoparametric submanifold of V if M is a full, codimension k, isoparametric submanifold of V.

7.2.4. Remarks.

(i) Since PF submanifolds of V have finite codimension, an isoparametric submanifold of V is of finite codimension.

(ii) It follows from Remark 7.1.4 that if M is a full isoparametric submanifold of V and M is contained in the sphere of radius r centered at c_0, then both M and V must be of finite dimension.

Since compact operators have eigen-decompositons and the normal bundle of an isoparametric submanifold of V is flat, it follows from Proposition 7.1.1 and 7.1.7 that:

7.2.5. Proposition. *If M is an isoparametric PF submanifold of a Hilbert space V, then there exist E_0 and a family of finite rank smooth distributions $\{E_i|\ i \in I\}$ such that $TM = E_0 \bigoplus \{E_i | i \in I\}$ is the common eigen-decomposition for all the shape operators A_v of M and $A_v|E_0 = 0$.*

Since A_v is linear for $v \in V$, there exist smooth sections λ_i of $\nu(M)^*$ such that

$$A_v|E_i = \lambda_i(v) id_{E_i},$$

for all $i \in I$. Identifying $\nu(M)^*$ with $\nu(M)$ by the induced inner product from V, we obtain smooth normal fields v_i on M such that

$$A_v|E_i = \langle v, v_i \rangle id_{E_i}, \tag{7.2.1}$$

for all $i \in I$. These E_i's, λ_i's and v_i's are called the curvature distributions, principal curvatures, and curvature normals for M respectively. If v is a parallel normal field on an isoparametric submanifold M then A_v has constant eigenvalues. So it follows from (7.2.1) that each curvature normal field v_i is parallel.

7.2.6. Proposition. *If M is a rank k isoparametric PF submanifold of Hilbert space, and $\{v_i| \, i \in I\}$ are its curvature normals, then there is a positive constant c such that $\|v_i\| \leq c$ for all $i \in I$.*

PROOF. Let F denote the continuous function defined on the unit sphere $\boldsymbol{S}^{k-1}$ of the normal plane $\nu(M)_q$ by $F(v) = \|A_v\|$. Since $\boldsymbol{S}^{k-1}$ is compact, there is a constant $c > 0$ such that $F(v) \leq c$. Since the eigenvalues of A_v are $\langle v, v_i \rangle$, we have $|\langle v, v_i \rangle| \leq c$ for all $i \in I$ and all unit vector $v \in \nu(M)_q$. ∎

7.2.7. Proposition. *Let M be a rank k immersed isoparametric submanifold of Hilbert space, $\nu_q = q + \nu(M)_q$ the affine normal plane at q, and $\Gamma_q = \Gamma \cap \nu_q$ the set of focal points for M with respect to q. Then:*

(i) $\Gamma_q = \bigcup\{\ell_i(q)| \, i \in I\}$, where $\ell_i(q)$ is the hyperplane in ν_q defined by

$$\ell_i(q) = \{q + v| \, v \in \nu(M)_q, \ \langle v, v_i(q) \rangle = 1\}.$$

(ii) $\mathcal{H} = \{\ell_i(q)| \, i \in I\}$ is locally finite, i.e., given any point $p \in \nu_q$ there is an open neighborhood U of p in ν_q such that $\{i \in I | \ell_i(q) \cap U \neq \emptyset\}$ is finite.

PROOF. Let Y be the end point map of M. By (7.2.1), $x = q + e \in \Gamma_q$ if and only if 1 is an eigenvalue of A_e. So there exists $i_0 \in I$ such that $1 = \langle e, v_{i_0} \rangle$, i.e., $x \in \ell_{i_0}(q)$. This proves (i).

Let $J(x) = \{i \in I| \, x \in \ell_i(q)\}$ for $x = q + e \in \nu_q$. Then the eigenspace V_1 of A_e corresponding to eigenvalue 1 is $\bigoplus\{E_j | j \in J(x)\}$. Since A_e is compact and $\{\langle e, v_i \rangle| \, i \in I\}$ are the eigenvalues of A_e, the set $J(x)$ is finite and there exist $\delta > 0$ such that $|1 - \langle e, v_i \rangle| > \delta$ for all i not in $J(x)$. By analytic geometry, if i is not in $J(x)$ then

$$d(x, \ell_i(q)) = \frac{|1 - \langle e, v_i \rangle|}{\|v_i\|^2} > \frac{\delta}{c},$$

where c is the upper bound for $\|v_i\|$ as in Proposition 7.2.6. So we conclude that the ball $B(x, \delta/c)$ of radius δ/c and center x meets only finitely many $\ell_i(q)$ (in fact it intersects $\ell_i(q)$ only for $i \in J(x)$). ■

We next note the following:

(i) the Frobenius integrability theorem is valid for finite rank distributions on Banach manifolds,

(ii) the proof of the existence of a Coxeter group in Chapter 6 depended only on the facts that all the curvature distributions and $\nu(M)$ are of finite rank and the family of focal hyperplanes $\{\ell_i \mid i \in I\}$ is locally finite.

So it is not difficult to see that most of the results in sections 6.2 and 6.3 for isoparametric submanifolds of $\boldsymbol{R}^n$ can be generalized to the infinite dimensional case. In particular the statements from 6.2.3 to 6.2.9, from 6.3.1 to 6.3.5, and the Slice Theorem 6.5.9 are all valid if we replace M by a rank k isoparametric submanifold of a Hilbert space and the index set $1 \leq i \leq p$ of curvature normals by $\{i \mid i \in I\}$. In particular the analogues of Theorem 6.3.2 and 6.3.5 for infinite dimensional isoparametric submanifolds give:

7.2.8. Theorem. *Let φ_i be the involution associated to the curvature distribution E_i.*

(i) There exists a bijection $\sigma_i : I \to I$ such that $\sigma_i(i) = i$, $\varphi_i^*(E_j) = E_{\sigma_i(j)}$ and $m_j = m_{\sigma_i(j)}$.

(ii) Let R_i^q denote the reflection of ν_q in $\ell_i(q)$. Then

$$R_i^q(\ell_j(q)) = \ell_{\sigma_i(j)}(q),$$

i.e., R_i^q permutes $\mathcal{H} = \{\ell_i(q) | i \in I\}$.

Note that R_i^q permutes hyperplanes in $\mathcal{H}$ and $\mathcal{H}$ is locally finite, so by Theorem 5.3.6 the subgroup of isometries of $\nu_q = q + \nu(M)_q$ generated by $\{R_i^q \mid i \in I\}$ is a Coxeter group.

7.2.9. Theorem. *Let M be an immersed isoparametric submanifold in the Hilbert space V, E_i the curvature normals, and $\{v_i | i \in I\}$ the set of curvature normals. Let W^q be the subgroup of the group of isometries of the affine normal plane $\nu_q = q + \nu(M)_q$ generated by reflections φ_i in $\ell_i(q)$. Then W^q is a Coxeter group. Moreover, let $\pi_{q,q'} : \nu(M)_q \to \nu(M)_{q'}$ denote the parallel translation with respect to the induced normal connection, then the map $P_{q,q'} : \nu_q \to \nu_{q'}$, defined by $P_{q,q'}(q + u) = q' + \pi_{q,q'}(u)$, conjugates W^q to $W^{q'}$ for any q and q' in M.*

7.2.10. Corollary. *Let M be a rank k immersed isoparametric submanifold of the infinite dimensional Hilbert space V, $\{E_i | i \in I\}$ the curvature* *distributions, and $\{\ell_i(q)|i \in I\}$ the curvature normal vectors at $q \in M$. Then associated to M there is a Coxeter group W with $\{\ell_i(q)|i \in I\}$ as its root system.*

7.2.11. Corollary. *Let M be an isoparametric submanifold of the infinite dimensional Hilbert space V, $\{E_i|i \in I\}$ the curvature distributions, and $\{v_i|i \in I\}$ the curvature normals. Suppose $0 \in I$ and $v_0 = 0$.*

(1) If I is a finite set, then

(i) there exists a constant vector $c_0 \in V$ such that $\bigcap\{\ell_i(q)|i \in I\} = \{c_0\}$ for all $q \in M$,

(ii) the Coxeter group associated to M is a finite group,

(iii) the rank of E_0 is infinite,

(iv) $\tilde{E} = \bigoplus\{E_i|i \neq 0,\ i \in I\}$ is integrable,

(v) $M \simeq S \times E_0$, where S is an integral submanifold of $\tilde{E}$.

(2) If I is an infinite set, then the Coxeter group associated to M is an infinite group.

Let $\triangle_q$ be the connected component of $\nu_q \setminus \bigcap\{\ell_i \mid i \in I\}$ containing q. If I is an infinite set, then W is an affine Weyl group, the closure $\bar{\triangle}_q$ is a fundamental domain of W and its boundary has $k+1$ faces. If $\varphi \in W$ and $\varphi(\ell_i) = \ell_j$ then $m_i = m_j$. It follows that:

7.2.12. Corollary. *Let M be a rank k isoparametric submanifold of an infinite dimensional Hilbert space having infinitely many curvature distributions. Then there is associated to M a well-defined marked Dynkin diagram with $k+1$ vertices, namely the Dynkin diagram of the associated affine Weyl group with multiplicities m_i.*

7.2.13. Example. Let $\hat{G}$ be the H^1-loops on the compact simple Lie group G, V the Hilbert space of H^0-loops on the Lie algebra $\mathcal{G}$ of G, and let $\hat{G}$ act on V by gauge transformations as in Example 5.8.1. This action is polar, so the principal $\hat{G}$-orbits in V are isoparametric. In the following we calculate the basic local invariants of these orbits as submanifolds of V. Let $\triangle^+$ denote the set of positive roots of G. Then there exist x_α and y_α in $\mathcal{G}$ for all $\alpha \in \triangle^+$ such that

$$\mathcal{G} = \mathcal{T} \bigoplus \{\boldsymbol{R}x_a \oplus \boldsymbol{R}y_\alpha | \alpha \in \triangle^+\},$$

$$[h, x_\alpha] = \alpha(h) y_\alpha,\ [h, y_\alpha] = -\alpha(h) x_\alpha,$$

for all $h \in \mathcal{T}$. If rank$(G) = k$ and $\{t_1, \ldots, t_k\}$ is a bases of $\mathcal{T}$, then the union of the following sets

$$\{x_\alpha \cos n\theta,\ y_\alpha \cos n\theta |\ \alpha \in \triangle^+,\ n \geq 0 \text{ an integer}\},$$

$$\{x_\alpha \sin m\theta,\ y_\alpha \sin m\theta |\ \alpha \in \triangle^+, m > 0 \text{ an integer}\},$$

$$\{t_i \cos n\theta,\ t_i \sin\ m\theta |\ 1 \leq i \leq k,\ n \geq 0,\ m > 0, \text{ are integers}\}$$

is a separable basis for V. An orbit $M = \hat{G}\hat{t}_0$ is principal if and only if $\alpha(t_0) + n \neq 0$ for all $\alpha \in \triangle^+$ and $n \in \boldsymbol{Z}$. Let $\hat{t}_1 \in \hat{\mathcal{T}}$ be a regular point. Then the shape operator of M along the direction $\hat{t}_1$ is

$$A_{\hat{t}_1}(v' + [v, \hat{t}_0]) = [v, \hat{t}_1].$$

Using the above separable basis for V, it is easily seen that $A_{\hat{t}_1}$ is a compact operator, the eigenvalues are

$$\{\alpha(t_1)/\alpha(t_0) + n \mid \alpha \in \triangle^+,\ n \in \boldsymbol{Z}\},$$

and each has multiplicity 2. So the associated Coxeter group of M as an isoparametric submanifold is the affine Weyl group $W(\mathcal{T}^0)$ of the section $\mathcal{T}^0$, and all the multiplicities $m_i = 2$.