Isoparametric Submanifolds

In section 5.7, we defined a submanifold of a space form to be isoparametric if its normal bundle is flat and if the principal curvatures along any parallel normal vector field are constant (Definition 5.7.2). These submanifolds arise naturally in representation theory for, as we saw, an orbit of an orthogonal representation is isoparametric if and only if it is a principal orbit of a polar representations, so in particular principal coadjoint orbits are isoparametric. And because their local invariants are so simple,isoparametric manifolds are also natural models to use in the classification theory of submanifolds. Although the principal orbits of a polar action are isoparametric, not all isoparametric submanifolds in $\boldsymbol{R}^m$ and $\boldsymbol{S}^m$ are orbits. Nevertheless, as we will see in this chapter, every isoparametric submanifold of $\boldsymbol{R}^m$ or $\boldsymbol{S}^m$ has associated to it a singular, orbit-like foliation, and this foliation has many of the same remarkable properties of the orbit foliations of polar actions. Thus isoparametric submanifolds can be viewed as a geometric generalization of principal orbits of polar actions.

There is an interesting history of this subject, which explains the origin of the name "isoparametric". A hypersurface is always given locally as the level set of some smooth function f, and then $\|\nabla f\|^2$, $\triangle f$ are called the first and second differential parameters of the hypersurface. So it is natural to make the following definition: a smooth function $f : \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ is called *isoparametric* if $\|\nabla f\|^2$ and $\triangle f$ are functions of f. The family of the level hypersurfaces of f is then called an isoparametric family, since clearly the first and second differential parameters are constant on each hypersurface of the family. It is not difficult to show that an isoparametric family in $\boldsymbol{R}^n$ must be either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. This was proved by Levi-Civita [Lc] for $n = 2$, and by B. Segré [Se] for arbitrary n. Shortly after this work of Levi-Civita and Segré, É. Cartan ([Ca3]-[Ca5]) considered isoparametric functions f on space forms, and discovered many interesting examples for $\boldsymbol{S}^{n+1}$. Among other things Cartan showed that the level hypersurfaces of f have constant principal curvatures. And conversely, he showed that if M is a hypersurface of $N^{n+1}(c)$ with constant principal curvatures, then there is at least a local isoparametric function having M as a level set. Cartan called such hypersurfaces isoparametric. In the past dozen years, many people carried forward this research. Around mid 1970's, Münzner [Mü1,2] completed a beautiful structure theory of isoparametric hypersurfaces in spheres, reducing their classification to a difficult, but purely algebraic problem. Although many people have subsequently made significant contributions to this classification problem, including Abresch [Ab], Ferus, Karcher, Münzner [FKM], Ozeki and

Takeuchi [OT1,2], it is still far from being completely solved. There have also been applications of isoparametric hypersurface theory to harmonic maps [Ee] and minimal hypersurfaces ([No],[FK]). Recently, with the purpose in mind of constructing harmonic maps, Eells [Ee] gave a definition of isoparametric map that generalizes the concept of isoparametric function. Carter and West [CW2] also gave a definition of isoparametric maps $\boldsymbol{S}^{n+k} \to \boldsymbol{R}^k$; their purpose being to generalize Cartan's work to higher codimension. Using their definition, they showed that the regular level of an isoparametric map is an isoparametric submanifold. They also showed that there is a Coxeter group associated to each codimension two isoparametric submanifold of a sphere, but they did not obtain a similar result for higher codimension. This work led Terng [Te2] to the definition used in this section.

6.1. Isoparametric maps

6.1.1. Definition. A smooth map $f = (f_{n+1}, \ldots, f_{n+k}) : N^{n+k}(c) \to \boldsymbol{R}^k$ is called *isoparametric* if

(1) f has a regular value,

(2) $\langle \nabla f_\alpha, \nabla f_\beta \rangle$ and $\triangle f_\alpha$ are functions of f for all α, β,

(3) $[\nabla f_\alpha, \nabla f_\beta]$ is a linear combination of $\nabla f_{n+1}, \ldots, \nabla f_{n+k}$, with coefficients being functions of f, for all α and β.

This definition agrees with Cartan's when $k = 1$. In the following we will proceed to prove that regular level submanifolds of an isoparametric map are isoparametric.

Hereafter we will use the notation introduced in Chapter 2. Suppose $f : N^{n+k}(c) \to \boldsymbol{R}^k$ is isoparametric. Applying the Gram-Schmidt process to $\{\nabla f_\alpha\}$ we may assume that at any regular point of f, there is a local orthonormal frame field $e_1, \ldots, e_{n+k}$ with dual coframe $\omega_1, \ldots, \omega_{n+k}$ such that

$$df_\alpha = \sum_\beta c_{\alpha\beta}\, \omega_\beta, \tag{6.1.1}$$

with $\operatorname{rank}(c_{\alpha\beta}) = k$, and where the $c_{\alpha\beta}$ are functions of f. So

$$dc_{\alpha\beta} \equiv 0 \mod (\omega_{n+1}, \ldots, \omega_{n+k}). \tag{6.1.2}$$

It is obvious that $\omega_\alpha = 0$ defines the level submanifolds of f. Condition (3) implies that the *normal distribution* defined by $\omega_i = 0$ on the set of regular points of f is completely integrable.

6.1.2. Proposition. *Let $f : N^{n+k}(c) \to \boldsymbol{R}^k$ be isoparametric, $b = f(q)$ a regular value, $M = f^{-1}(b)$, and F the leaf of the normal distribution through q. Then*

(i) F is totally geodesic,

(ii) $\nu(M)$ is flat and has trivial holonomy group.

PROOF. Take the exterior differential of (6.1.1), and using the structure equations, we obtain

$$\sum_{\beta} dc_{\alpha\beta} \wedge \omega_\beta + \sum_{\beta i} c_{\alpha\beta}\omega_{\beta i} \wedge \omega_i + \sum_{\beta\gamma} c_{\alpha\beta}\omega_{\beta\gamma} \wedge \omega_\gamma = 0. \qquad (6.1.3)$$

From (6.1.2), since the coefficient of $\omega_i \wedge \omega_\gamma$ in (6.1.3) is zero, we obtain

$$\sum_{\beta} c_{\alpha\beta}(-\omega_{\beta i}(e_\gamma) + \omega_{\beta\gamma}(e_i)) = 0. \qquad (6.1.4)$$

But $\operatorname{rank}(c_{\alpha\beta}) = k$, hence:

$$\omega_{\beta i}(e_\gamma) = \omega_{\beta\gamma}(e_i).$$

From condition (3) of Definition 6.1.1, we have

$$\begin{aligned}
[e_\alpha, e_\beta] &= \sum_{\gamma} u_{\alpha\beta\gamma} e_\gamma = \nabla_{e_\alpha} e_\beta - \nabla_{e_\beta} e_\alpha \\
&= \sum_i (\omega_{\beta i}(e_\alpha) - \omega_{\alpha i}(e_\beta)) e_i + \sum_\gamma (\omega_{\beta\gamma}(e_\alpha) - \omega_{\alpha\gamma}(e_\beta)) e_\gamma.
\end{aligned}$$

Hence

$$\begin{aligned}
\omega_{\beta i}(e_\alpha) &= \omega_{\alpha i}(e_\beta), \\
\omega_{\beta\gamma}(e_\alpha) - \omega_{\alpha\gamma}(e_\beta) &= u_{\alpha\beta\gamma},
\end{aligned}$$

where $u_{\alpha\beta\gamma}$ is a function of f. In particular, we have

$$\omega_{\beta\alpha}(e_\alpha) = u_{\alpha\beta\alpha},$$

is a function of f. Using (6.1.4), we have

$$\begin{aligned}
\omega_{\beta i}(e_\alpha) &= \omega_{\beta\alpha}(e_i) \\
&= \omega_{\alpha i}(e_\beta) = \omega_{\alpha\beta}(e_i) = -\omega_{\beta\alpha}(e_i).
\end{aligned}$$

So $\omega_{\alpha\beta}(e_i) = 0$ and $\omega_{\alpha i}(e_\beta) = 0$, i.e., $\omega_{\alpha\beta} = 0$ on M, and $\omega_{\alpha i} = 0$ on F. This implies that the e_α are parallel normal fields on M and that F is totally geodesic.

Note that e_α on M can be obtained by applying the Gram-Schmidt process to $\nabla f_{n+1}, \ldots, \nabla f_{n+k}$, so e_α is a global parallel normal frame on M, hence the holonomy of $\nu(M)$ is trivial. ∎

6.1.3. Corollary. *With the same assumption as in Proposition 6.1.2,*
(i) $\nabla f_\alpha | M$ is a parallel normal field on M for all $n+1 \leq \alpha \leq n+k$,
(ii) if v is a parallel normal field on M, then there exists $t_0 > 0$ such that $\{exp_x(tv(x)) |\ x \in M\}$ is a regular level submanifold of f for $|t| < t_0$.

In order to prove that a regular level submanifold of an isoparametric map is isoparametric we need the following simple and direct generalization of Theorem 3.4.2 on Bonnet transformations.

6.1.4. Proposition. *Suppose $X : M^n \to N^{n+k}(c)$ is an isometric immersion with flat normal bundle, and v is a unit parallel normal field. Then $X^* = aX + bv$ is an immersion if and only if $(aI - bA_v)$ is non-degenerate on M. Here A_v is the shape operator of M in the direction v, and $(a, b) = (1, t)$ for $c = 0$, is $(\cos t, \sin t)$ for $c = 1$, and is $(\cosh t, \sinh t)$ for $c = -1$. Moreover:*
(i) $\nu(M^*)$ is flat,
(ii) $TM_q = TM^*_{q^*}$, $\nu(M)_q = \nu(M^*)_{q^*}$, where $q^* = X^*(q)$,
(iii) $v^* = -cbX + av$ is a parallel normal field on M^*,
(iv) $A^*_{v^*} = (cbI + aA_v)(aI - bA_v)^{-1}$.

PROOF. We will prove only the case $c = 0$, the other cases being similar. Let e_A be an adapted frame on M. Taking the differential of X^* we obtain

$$dX^* = I - tA_v + t(\nabla^\nu v).$$

Since v is parallel, we have $dX^* = I - tA_v$. Hence X^* is an immersion if and only if $(I - tA_v)$ is invertible. So e_A is also an adapted frame for M^*, and the dual coframe is $\omega^*_i = \sum_j (\delta_{ij} - t(A_v)_{ij})\omega_j$. Moreover, $\omega^*_{i\alpha} = \omega_{i\alpha}$, so we have $A^*_{v^*} = A_v(I - tA_v)^{-1}$. ∎

6.1.5. Proposition. *With the same assumptions as in Proposition 6.1.2:*
(i) the mean curvature vector of M is parallel,
(ii) the principal curvatures of M along a parallel normal field are constant.

PROOF. We choose a local orthonormal frame e_A as in the proof of Proposition 6.1.2. Let

$$d(f_\alpha) = \sum_A (f_\alpha)_A \omega_A,$$

$$\nabla^2 f_\alpha = \sum_{AB} (f_\alpha)_{AB}\, \omega_A \otimes \omega_B.$$

Using (6.1.1) we have

$$(f_\alpha)_i = 0, \quad (f_\alpha)_\beta = c_{\alpha\beta}.$$

Now (1.3.6) gives

$$(f_\alpha)_{ii} = -\sum_\beta c_{\alpha\beta} h_{i\beta i},$$

$$(f_\alpha)_{\beta\beta} = dc_{\alpha\beta}(e_\beta) + \sum_\gamma c_{\alpha\gamma}\omega_{\gamma\beta}(e_\beta),$$

so we have

$$\triangle f_\alpha = \sum_\beta dc_{\alpha\beta}(e_\beta) - \sum_\beta c_{\alpha\beta} H_\beta + \sum_{\beta\gamma} c_{\alpha\gamma}\omega_{\gamma\beta}(e_\beta),$$

where $H_\beta = \sum_i h_{i\beta i}$ is the mean curvature of level submanifolds of f in the direction of e_β. Since $\triangle f_\alpha$, $c_{\alpha\beta}$ and $\omega_{\gamma\beta}(e_\beta)$ are all functions of f, $\sum_\beta c_{\alpha\beta} H_\beta$ is a function of f. However rank$(c_{\alpha\beta}) = k$, and hence the H_α are functions of f, i.e., each H_α's is constant on M. But the e_α are parallel normal fields on M, so (i) is proved.

To prove (ii) we use the method used by Nomizu [No] in codimension one. Let X be the position function of M in $\boldsymbol{R}^{n+k}$. By Corollary 6.1.3, there exists $t_0 > 0$ such that $X^* = X + te_\alpha$ is an immersion if $|t| < t_0$, and $X^*(M)$ is a regular level of f. Then, by (i), the mean curvature H^*_α of X^* in the direction of $e^*_\alpha = e_\alpha$ is constant. Using Proposition 6.1.4 (iv) and the identity:

$$A(I - tA)^{-1} = A\sum_{m=0}^{\infty} t^m A^m = \sum_{m=0}^{\infty} A^{m+1} t^m,$$

we have

$$H^*_\alpha = \sum_{m=0}^{\infty} (\mathrm{tr}(A^{m+1}_{e_\alpha}))t^m. \tag{6.1.5}$$

Note that H^*_α is independent of $x \in M$, so the right hand side of (6.1.5) is a function of t alone. Hence $\mathrm{tr}(A^{m+1}_{e_\alpha})$ is a function of t for all m and this implies that the eigenvalues of A_{e_α} are constant on M. ∎

As a consequence of Propositions 6.1.2 and 6.1.5, we have

6.1.6. Theorem. *Let $f : N^{n+k}(c) \to \boldsymbol{R}^k$ be isoparametric, b a regular value, and $M = f^{-1}(b)$. Then M is isoparametric.*

6.2. Curvature distributions

In this section we assume that M^n is an immersed isoparametric submanifold of $\boldsymbol{R}^{n+k}$. Since $\nu(M)$ is flat, by Proposition 2.1.2, $\{A_v | \, v \in \nu(M)_q\}$ is a family of commuting self-adjoint operators on TM_q, so there exists a common eigendecomposition $TM_q = \bigoplus_{i=1}^{p} E_i(q)$. Let $\{e_\alpha\}$ be a local orthonormal parallel normal frame. By definition of isoparametric, $A_{e_\alpha(x)}$ and $A_{e_\alpha(q)}$ have same eigenvalues. So E_i's are smooth distributions and $TM = \bigoplus E_i$. The E_i's are characterized by the equation

$$A_{e_\alpha} | E_i = n_{i\alpha} id_{E_i},$$

together with the conditions that if $i \neq j$ then there exists α_0 with $n_{i\alpha_0} \neq n_{j\alpha_0}$. Note that the $E_i(q)$ are the common eigenspaces of all the shape operators at q, so they are independent of the choice of the e_α, and are uniquely determined up to a permutations of their indices. These distributions E_i are called the *curvature distributions* of M.

We will make the following standing assumptions:

(1) M has p curvature distributions $E_1, \ldots, E_p$, and $m_i = \operatorname{rank}(E_i)$.

(2) Let $\{e_i\}$ be a local orthonormal tangent frame for M such that E_i is spanned by $\{e_j | \mu_{i-1} < j \leq \mu_i\}$, where $\mu_i = \sum_{s=1}^{i} m_s$. So we have

$$\omega_{\alpha\beta} = 0, \tag{6.2.1}$$

$$\omega_{i\alpha} = \lambda_{i\alpha} \omega_i, \tag{6.2.2}$$

where $\lambda_{i\alpha}$ are constant. In fact, $\lambda_{i\alpha} = n_{j\alpha}$ if $\mu_{j-1} < i \leq \mu_j$.

(3) Let $v_i = \sum_\alpha n_{i\alpha} e_\alpha$. Then

$$A_v | E_i = \langle v, v_i \rangle id_{E_i}, \tag{6.2.3}$$

for any normal field v. Clearly (6.2.3) characterizes the v_i, so in particular v_i is independent of the choice of e_α, i.e., each v_i is a well-defined normal field associated to E_i. In fact, if $\bar{e}_\alpha$ is another local parallel normal frame on M and $\bar{n}_{i\alpha}$ the eigenvalues of $A_{\bar{e}_\alpha}$ then

$$v_i = \sum_\alpha \bar{n}_{i\alpha} \bar{e}_\alpha = \sum_\alpha n_{i\alpha} e_{i\alpha}.$$

We call v_i the *curvature normal* of M associated to E_i.

(4) Let $n_i = (n_{in+1}, \ldots, n_{in+k})$.

If M is isoparametric in $\boldsymbol{R}^{n+k}$ then M is also isoparametric in $\boldsymbol{R}^{n+k+1}$. To avoid this redundancy, we make the following definition:

6.2.1. Definition. A submanifold M of $\boldsymbol{R}^{n+k}$ is *full* if M is not included in any affine hyperplane of $\boldsymbol{R}^{n+k}$.

6.2.2. Definition. An immersed, full, isoparametric submanifold M^n of $\boldsymbol{R}^{n+k}$ is called a rank k isoparametric submanifold in $\boldsymbol{R}^{n+k}$.

6.2.3. Proposition. *An immersed isoparametric submanifold M^n of $\boldsymbol{R}^{n+k}$ is full if and only if the curvature normals $v_1, \ldots, v_p$ spans $\nu(M)$. In particular, if M^n is full and isoparametric in $\boldsymbol{R}^{n+k}$ then $k \leq n$.*

PROOF. Note that $v_1, \ldots, v_p$ span $\nu(M)$ if and only if the rank of the $k \times p$ matrix $N = (n_{i\alpha})$ is k. Suppose M is contained in a hyperplane normal to a constant unit vector $u_0 \in \boldsymbol{R}^{n+k}$. Then we can choose $e_{n+1} = u_0$, so $n_{in+1} = 0$ for all i, and $\text{rank}(N) < k$. Conversely, if $\text{rank}(N) < k$ then there exists a unit vector $c = (c_\alpha) \in \boldsymbol{R}^k$ such that $\langle c, n_i \rangle = 0$ for all $1 \leq i \leq p$. We claim that $v = \sum_\alpha c_\alpha e_\alpha$ is a constant vector b in $\boldsymbol{R}^{n+k}$. To see this, we note that the eigenvalues of A_v are $\langle v, v_i \rangle = \langle c, n_i \rangle = 0$, i.e., $A_v = 0$. But $dv = -A_v$, so v is constant on M. Then it follows that

$$d(\langle X, b \rangle) = \langle dX, b \rangle = \sum_i \omega_i \langle e_i, b \rangle = 0.$$

Hence $\langle X, b \rangle = c_0$ a constant, i.e., M is contained in a hyperplane. ∎

Recall that the endpoint map $Y : \nu(M) \to \boldsymbol{R}^{n+k}$ is defined by $Y(v) = x + v$ for $v \in \nu(M)_x$. Using the frame e_A, we can write

$$Y = Y(x, z) = x + \sum_\alpha z_\alpha e_\alpha(x).$$

The differential of Y is

$$\begin{aligned} dY &= dX + \sum_\alpha z_\alpha de_\alpha + \sum_\alpha dz_\alpha e_\alpha \\ &= \sum_{i=1}^{p} (1 - \langle z, n_i \rangle) id_{E_i} + \sum_\alpha dz_\alpha e_\alpha. \end{aligned}$$

Now recall also that a point y of $\boldsymbol{R}^{n+k}$ is called a focal point of M if it is a singular value of Y, that is if it is of the form $y = Y(v)$ where dY_v has rank less than $n + k$. The set Γ of all focal points of M is called the focal set of M

6.2.4. Proposition. *Let M be an immersed isoparametric submanifold M^n of $\boldsymbol{R}^{n+k}$ and Γ its focal set. For each $q \in M$ let Γ_q denote the intersection of Γ with the normal plane $q + \nu(M)_q$ to M at q. Then Γ, is the union of the Γ_q, and each*

Γ_q is the union of the p hyperplanes $\ell_i(q) = \{q+v \mid v \in \nu(M)_q,\ \langle v, v_i\rangle = 1\}$ in $q+\nu(M)_q$. These $\ell_i(q)$ are called the focal hyperplane associated to E_i at q.

6.2.5. Corollary.
(1) The curvature normal $v_i(q)$ is normal to the focal hyperplane $\ell_i(q)$ in $q + \nu(M)_q$.
(2) The distance $d(q, \ell_i(q))$ from q to $\ell_i(q)$ is $1/\|v_i\|$.

6.2.6. Proposition. *Let $X : M^n \to \boldsymbol{R}^{n+k}$ be an immersed isoparametric submanifold, and v a parallel normal field. Then $X + v$ is an immersion if and only if $\langle v_i, v\rangle \neq 1$ for all $1 \leq i \leq p$. Moreover,*
(i) the parallel set M_v defined by v, i.e., the image of $X + v$, is an immersed isoparametric submanifold,
(ii) let $q^* = q + v(q)$, then $TM_q = T(M_v)_{q^*}$, $\nu(M)_q = \nu(M_v)_{q^*}$, and $q + \nu(M)_q = q^* + \nu(M_v)_{q^*}$
(iii) if $\{e_\alpha\}$ is a local parallel normal frame on M then $\{\bar{e}_\alpha\}$ is a local parallel normal frame on M_v, where $\bar{e}_\alpha(q^*) = e_\alpha(q)$,
(iv) $E_i^*(q^*) = E_i(q)$ are the curvature distributions of M_v, and the corresponding curvature normals are given by

$$v_i^*(q^*) = v_i(q)/(1 - \langle v, v_i\rangle),$$

(v) the focal hyperplane $\ell_i^*(q^*)$ of M_v associated to E_i^* is the same as the focal hyperplane $\ell_i(q)$ of M associated to E_i.

PROOF. Since v is parallel, there exist constants z_α such that $v = \sum_\alpha z_\alpha e_\alpha$. The differential of $X + v$ is

$$\begin{aligned} d(X+v) &= dX + \sum_\alpha z_\alpha de_\alpha \\ &= \sum_i \omega_i e_i - \sum_{i,\alpha} z_\alpha \omega_{i\alpha} e_i \\ &= \sum_i (1 - \sum_\alpha z_\alpha \lambda_{i\alpha})\omega_i e_i. \end{aligned}$$

So we may choose the following local frame on M_v:

$$e_A^* = e_A,\ \ \omega_i^* = (1 - \sum_\alpha z_\alpha \lambda_{i\alpha})\omega_i.$$

Then $\omega^*_{AB} = \langle de^*_A, e^*_B \rangle = \omega_{AB}$. In particular, we have

$$\omega^*_{\alpha\beta} = 0,$$

$$\omega^*_{i\alpha} = \lambda_{i\alpha}\omega_i = \frac{\lambda_{i\alpha}}{1 - \sum_\beta z_\beta \lambda_{i\beta}} \omega^*_i,$$

which proves the proposition. ∎

Next we will prove that the curvature distributions are integrable. First we need some formulas for the Levi-Civita connection of M in terms of E_i. Using (6.2.1), (6.2.2) and the structure equations, we have

$$\begin{aligned} d\omega_{i\alpha} &= d(\lambda_{i\alpha}\omega_i) = \lambda_{i\alpha} d\omega_i = \lambda_{i\alpha} \sum_j \omega_{ij} \wedge \omega_j \\ &= \sum_j \omega_{ij} \wedge \omega_{j\alpha} = \sum_j \lambda_{j\alpha} \omega_{ij} \wedge \omega_j, \end{aligned}$$

so

$$\sum_j (\lambda_{i\alpha} - \lambda_{j\alpha}) \omega_{ij} \wedge \omega_j = 0.$$

Suppose $\omega_{ij} = \sum_m \gamma_{ijm}\omega_m$, then we have

$$\sum_{j,m} (\lambda_{i\alpha} - \lambda_{j\alpha}) \gamma_{ijm} \omega_m \wedge \omega_j = 0.$$

This implies that

6.2.7. Proposition. *Let $\omega_{ij} = \sum_m \gamma_{ijm}\omega_m$. Then*

$$(\lambda_{i\alpha} - \lambda_{j\alpha})\gamma_{ijm} = (\lambda_{i\alpha} - \lambda_{m\alpha})\gamma_{imj}, \ \text{ if } j \neq m.$$

In particular, if $e_i,\ e_m \in E_{i_1}$, $e_j \in E_{i_2}$, and $i_1 \neq i_2$, then $\gamma_{ijm} = 0$.

6.2.8. Theorem. *Let M^n be an immersed isoparametric submanifold of $\boldsymbol{R}^{n+k}$. Then each curvature distribution E_i is integrable.*

PROOF. For simplicity, we assume $i = 1$ and $m = m_1$. E_1 is defined by the following 1-form equations on M:

$$\omega_i = 0, \ \ m < i \leq n.$$

Using the structure equation, we have

$$d\omega_i = \sum_{j=1}^{m} \omega_{ij} \wedge \omega_j = \sum_{j,s=1}^{m} \gamma_{ijs}\omega_s \wedge \omega_j.$$

Since $\omega_{ij} = -\omega_{ji}$, $\gamma_{ijs} = -\gamma_{jis}$, which is zero by Proposition 6.2.7. So E_1 is integrable. ∎

6.2.9. Theorem. *Let M^n be a complete, immersed, isoparametric submanifold of $\boldsymbol{R}^{n+k}$, E_i the curvature distributions, v_i the corresponding curvature normals, and $\ell_i(q)$ the focal hyperplane associated to E_i at $q \in M$. Let $S_i(q)$ denote the leaf of E_i through q.*

(1) If $v_i \neq 0$ then

(i) $E_i(x) \oplus Rv_i(x)$ is a fixed (m_i+1)-plane ξ_i in $\boldsymbol{R}^{n+k}$ for all $x \in S_i(q)$,

(ii) $x + (v_i(x)/\|v_i(x)\|^2)$ is a constant $c_0 \in \xi_i$ for all $x \in S_i(q)$,

(iii) $S_i(q)$ is the standard sphere of $c_0 + \xi_i$ with radius $1/\|v_i\|$ and center at c_0,

(iv) $E_i(x) \oplus \nu(M)_x$ is a fixed (m_i+k)-plane η_i in $\boldsymbol{R}^{n+k}$ for all $x \in S_i(q)$,

(v) $\ell_i(x) = \ell_i(q)$ for all $x \in S_i(q)$, which is the $(k-1)$-plane perpendicular to $c_0 + \xi_i$ in $c_0 + \eta_i$ at c_0,

(vi) given $y \in \ell_i(q)$ we have $\|x - y\| = \|q - y\|$ for all $x \in S_i(q)$.

(2) If $v_i = 0$ then $E_i(x) = E_i(q)$ is a fixed m_i-plane for all $x \in S_i(q)$ and $S_i(q)$ is the plane parallel to $E_i(q)$ passes through q.

PROOF. It suffices to prove this theorem for E_1. Let $m = m_1$. To obtain (1), we compute the differential of the map $f = e_1 \wedge \ldots \wedge e_m \wedge v_1$ from $S_1(q)$ to the Grassman manifold $\boldsymbol{Gr}(m+1, n+k)$. Since

$$e_1 \wedge \ldots \wedge e_m \wedge dv_1 = 0$$

on $S_1(q)$, we have

$$d(e_1 \wedge \ldots \wedge e_m \wedge v_1) =$$

$$\sum_{i \leq m} e_1 \wedge \ldots \wedge (\sum_{j>m} \omega_{ij} e_j + \sum_{\alpha} \omega_{i\alpha} e_\alpha) \wedge e_{i+1} \wedge \ldots \wedge e_m \wedge v_1.$$

Using Proposition 6.2.7, we have $\omega_{ij} = \sum_{s \leq m} \gamma_{ijs}\omega_s = 0$ if $i \leq m$ and $j > m$. So we have

$$\begin{aligned} df &= \sum_{i \leq m, \alpha} e_1 \wedge \ldots \wedge \omega_{i\alpha} e_\alpha \wedge e_{i+1} \wedge \ldots \wedge e_m \wedge v_1 \\ &= \sum_{i \leq m, \alpha, \beta} e_1 \wedge \ldots \wedge n_{1\alpha}\omega_i e_\alpha \wedge e_{i+1} \wedge \ldots \wedge e_m \wedge n_{1\beta} e_\beta \\ &= \sum_{i \leq m, \alpha, \beta} n_{1\alpha} n_{1\beta} \omega_i e_1 \wedge \ldots e_\alpha \wedge e_{i+1} \ldots e_m \wedge e_\beta = 0, \end{aligned}$$

which proves (1)(i). Similarly one can prove (1)(iv) by showing that

$$d(e_1 \wedge \ldots \wedge e_m \wedge e_{n+1} \ldots \wedge e_{n+k}) = 0$$

on $S_1(q)$. Next we calculate the differential of $X + (v_1/\|v_1\|^2)$ on $S_1(q)$:

$$d\left(X + \frac{v_1}{\|v_1\|^2}\right) = Id_{E_1} - \frac{1}{\|v_1\|^2} A_{v_1}|E_1.$$

Since $A_v|E_i = \langle v, v_i \rangle id_{E_i}$, (1)(ii) follows, and (1)(iii) is a direct consequence. Note that $v_1(x)$ is normal to $\ell_1(x)$ in $c_0 + \xi_1$, so it follows from (1)(i) and (1)(iv) that $\ell_1(x)$ is perpendicular to $c_0 + \xi_1$ in $c_0 + \eta_1$ at c_0 for all $x \in S_1(q)$. Hence (1)(v) and (vi) follow.

If $v_1 = 0$ then $\omega_{i\alpha} = 0$ for $i \leq m$. By Proposition 6.2.7, $\omega_{ij} = 0$ on $S_1(q)$ if $i \leq m$ and $j > m$. So $d(e_1 \wedge \ldots \wedge e_m) = 0$ on $S_1(q)$, which proves (2). ∎

Because an m_0-plane is not compact, we have

6.2.10. Corollary. *If M^n is a compact, immersed, full isoparametric submanifold of $\boldsymbol{R}^{n+k}$, then all the curvature normals of M are non-zero.*

6.2.11. Proposition. *Let $\omega_{ij} = \sum_m \gamma_{ijm}\omega_m$. Then*
(i) $(\lambda_{i\alpha} - \lambda_{j\alpha})\gamma_{ijm} = h_{i\alpha jm}$,
(ii) if $e_i \in E_{i_1}$, $e_j \in E_{i_2}$ and $i_1 \neq i_2$, then $\gamma_{ijj} = 0$.

PROOF. Using (2.1.19), we obtain

$$0 = \sum_m h_{j\alpha jm}\omega_m,$$

$$(\lambda_{i\alpha} - \lambda_{j\alpha})\omega_{ij} = \sum_m h_{i\alpha jm}\omega_m. \quad ∎$$

6.3. Coxeter groups associated to isoparametric submanifolds

In this section we assume that $X : M^n \to \boldsymbol{R}^{n+k}$ is an immersed full isoparametric submanifold. Let $E_0, E_1, \ldots, E_p$ be the curvature distributions, v_i the corresponding curvature normals, and $\ell_i(q)$ the focal hyperplane in $q + \nu(M)_q$ associated to E_i. We may assume $v_0 = 0$, so $v_i \neq 0$ for all $i > 0$. We will use the following standing notations:

(1) $\nu_q = q + \nu(M)_q$.
(2) R_i^q denotes the reflection of ν_q across the hyperplane $\ell_i(q)$.
(3) T_i^q denotes the linear reflection of $\nu(M)_q$ along $v_i(q)$, i.e.,

$$T_i^q(v) = v - 2\frac{\langle v, v_i(q)\rangle}{\|v_i\|^2} v_i(q).$$

(3) Let φ_i be the diffeomorphism of M defined by $\varphi_i(q)$ = the antipodal point of q in the leaf sphere $S_i(q)$ of E_i for $i > 0$. Note that φ_i^2 is clearly the identity map of M; we call it the involution associated to E_i.
(4) S_p will denote the group of permutations of $\{1, \ldots, p\}$.

It follows from (1) of Theorem 6.2.9 that

$$\varphi_i = X + 2\frac{v_i}{\|v_i\|^2},$$

$$\varphi_i(q) = R_i^q(q).$$

Since φ_i is a diffeomorphism it follows from Proposition 6.2.6 that:

6.3.1. Proposition. *If $v_i \neq 0$ then $1 - 2(\langle v_i, v_j\rangle/\|v_i\|^2)$ never vanishes for $0 \leq j \leq p$.*

6.3.2. Theorem. *There exist permutations $\sigma_1, \ldots, \sigma_p$ in S_p such that*
(1) $E_j(\varphi_i(q)) = E_{\sigma_i(j)}(q)$, i.e., $\varphi_i^*(E_j) = E_{\sigma_i(j)}$, in particular we have $m_j = m_{\sigma_i(j)}$,
(2) $v_{\sigma_i(j)}(q) = \left(1 - 2\,\frac{\langle v_i, v_{\sigma_i(j)}\rangle}{\|v_i\|^2}\right) v_j(\varphi_i(q))$,
(3) $T_i^q(v_j(q)) = \left(1 - 2\,\frac{\langle v_i, v_{\sigma_i(j)}\rangle}{\|v_i\|^2}\right)^{-1} v_{\sigma_i(j)}(q)$.

PROOF. It suffices to prove the theorem for E_1. Note that $\varphi_1 = X + v$ and $M_v = \varphi_1(M) = M$, where $v = 2v_1/\|v_1\|^2$ is parallel. So by Proposition 6.2.6 (iv), there exists $\sigma \in S_p$ such that (1) is true.

By Proposition 6.2.6 (iii), $\bar{e}_\alpha(x) = e_\alpha(\varphi_1(x))$ gives a parallel normal frame on M. So the two parallel normal frames $\bar{e}_\alpha$ and e_α differ by a constant matrix C in $\boldsymbol{O}(k)$. To determine C, we parallel translate $e_\alpha(q)$ with respect to the induced normal connection of M in $\boldsymbol{R}^{n+k}$ to $q^* = \varphi_1(q)$. Let ξ_1, η_1, c_0 be as in Theorem 6.2.9. Then the leaf $S_1(q)$ of E_1 at q is the standard sphere in the (m_1+1)-plane $c_0 + \xi_1$, which is contained in the (m_1+k)-plane $c_0 + \eta_1$, and $e_\alpha|S_1(q)$ is a parallel normal frame of $S_1(q)$ in $c_0 + \eta_1$. In particular, the normal parallel translation of $e_\alpha(q)$ to q^* on $S_1(q)$ in $c_0 + \eta_1$ is the same as the normal parallel translation on M in $\boldsymbol{R}^{n+k}$. Note that the normal planes of

$S_1(q)$ at q and q^* in $c_0 + \eta_1$ are the same. Let π denote the parallel translation in the normal bundle of $S_1(q)$ in $c_0 + \eta$ from q to q^*. Then it is easy to see that $\pi(v_1(q)) = -v_1(q)$ and $\pi(u) = u$ if u is a normal vector at q perpendicular to $v_1(q)$, i.e., π is the linear reflection R_1^q of $\nu(M)_q$ along $v_1(q)$. So

$$e_\alpha(q^*) = T_1^q(e_\alpha(q)) = T_1^q(\bar{e}_\alpha(q^*)).$$

Since $(T_1^q)^{-1} = T_1^q$,

$$\bar{e}_\alpha(q^*) = T_1^q(e_\alpha(q^*)) = e_\alpha(q^*) - 2\frac{\langle v_1(q), e_\alpha(q^*)\rangle}{\|v_1\|^2} v_1(q).$$

But $v_1(q^*) = -v_1(q)$, so we have

$$\begin{aligned} \bar{e}_\alpha &= e_\alpha - 2\frac{\langle e_\alpha, v_1\rangle}{\|v_1\|^2} v_1 \\ &= \sum_{\alpha,\beta} (\delta_{\alpha\beta} - 2\frac{n_{1\alpha} n_{1\beta}}{\|n_1\|^2}) e_\beta. \end{aligned} \tag{6.3.1}$$

Let $\lambda_{i\alpha}$ and $\bar{\lambda}_{i\alpha}$ be the eigenvalues of A_{e_α} and $A_{\bar{e}_\alpha}$ on E_i respectively. Then (6.3.1) implies that

$$\bar{\lambda}_{i\alpha} = \sum_\beta (\delta_{\alpha\beta} - 2\frac{n_{1\alpha} n_{1\beta}}{\|n_1\|^2}) \lambda_{i\beta}.$$

We have proved that $E_i(q^*) = E_{\sigma(i)}(q)$, so using Proposition 6.2.6 (iv) we have

$$\lambda_{i\alpha} = \frac{\lambda_{\sigma(i)\alpha}}{1 - 2\frac{\langle v_1, v_{\sigma(i)}\rangle}{\|v_1\|^2}}. \tag{6.3.2}$$

Note that

$$\begin{aligned} v_i(\varphi_1(q)) &= T_1^q(v_i(q)), \quad \text{since } v_i \text{ is parallel} \\ &= (v_i - 2\frac{\langle v_1, v_i\rangle}{\|v_1\|^2} v_1)(q) \\ &= \sum_\alpha \bar{\lambda}_{i\alpha} \bar{e}_\alpha(\varphi_1(q)) = \sum_\alpha \bar{\lambda}_{i\alpha} e_\alpha(q), \text{ by (6.3.2)} \\ &= \sum_\alpha \frac{\lambda_{\sigma(i)\alpha}}{1 - 2\frac{\langle v_1, v_{\sigma(i)}\rangle}{\|v_1\|^2}} e_\alpha(q), \\ &= \left(1 - 2\frac{\langle v_1, v_{\sigma(i)}\rangle}{\|v_1\|^2}\right)^{-1} v_{\sigma(i)}(q). \quad \blacksquare \end{aligned}$$

As a consequence of Theorem 6.3.2 (3) and Corollary 5.3.7, we have

6.3.3. Corollary. *The subgroup W^q of $\boldsymbol{O}(\nu(M)_q)$ generated by the linear reflections $T_1^q, \ldots, T_p^q$ is a finite Coxeter group.*

From the fact that the curvature normals are parallel we have:

6.3.4. Proposition. *Let $\pi_{q,q'} : \nu(M)_q \to \nu(M)_{q'}$ denote the parallel translation map. Then $\pi_{q,q'}$ conjugates the group W^q to $W^{q'}$. In particular, we have associated to M a well-defined Coxeter group W.*

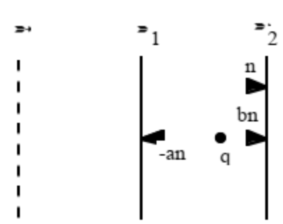
6.3.5. Theorem. $R_i^q(\ell_j(q)) = \ell_{\sigma_i(j)}(q)$.

PROOF. It suffices to prove the theorem for $i = 1, j = 2$. We may assume that $\sigma_1(2) = 3$. In our proof q is a fixed point of M, so we will drop the reference to q whenever there is no possibility of confusion. Let $\ell = R_1(\ell_2)$. Since v_i is normal to ℓ_i, it follows from Theorem 6.3.2(3) that ℓ is parallel to ℓ_3. Choose $q' \in \ell_3$ and $Q \in \ell$ such that $\|q - q'\| = d(q, \ell_3) = c$ and $\|q - Q\| = d(q, \ell)$ respectively. Let $1/a = \|v_1\|$ and $1/b = \|v_2\|$. By Theorem 6.2.9, $\overrightarrow{qq'} = v_3/\|v_3\|^2$. Note that 6.3.2(3) gives

$$T_1(v_2) = \left(1 - 2\frac{\langle v_1, v_3\rangle}{\|v_1\|^2}\right)^{-1} v_3. \tag{6.3.3}$$

We claim that $\overrightarrow{qq'} = \overrightarrow{qQ}$, which will prove that $\ell = \ell_3$. It is easily seen that $\overrightarrow{qq'}$ and $\overrightarrow{qQ}$ are parallel. We divide the proof of the claim into four cases:

(Case i) $\ell_1 \parallel \ell_2$ and $v_1,\ v_2$ are in the opposition directions.



Let n be the unit direction of v_2. Then $\overrightarrow{qQ} = -(2a + b)n$. Note that v_3 is equal to $(\epsilon/c)n$ for $\epsilon = 1$ or -1. Using (6.3.3), we have

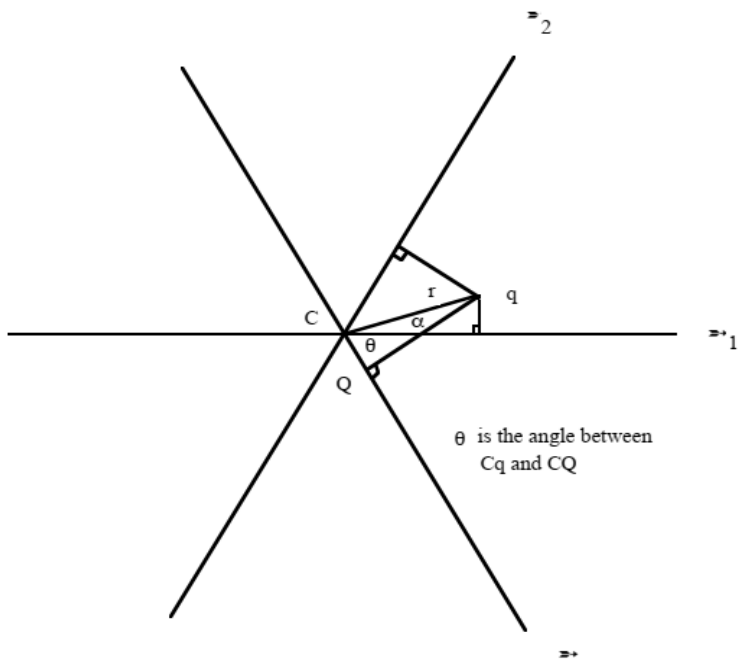
$$\begin{aligned} T_1(v_2) &= T_1\left(\frac{1}{b}n\right) = -\frac{1}{b}n \\ &= \left(1 - 2\frac{\langle v_1, v_3\rangle}{\|v_1\|^2}\right)^{-1} v_3 = \left(1 + 2\frac{\epsilon a}{c}\right)^{-1} \frac{\epsilon}{c} n. \end{aligned}$$

So

$$-1/b = \frac{\epsilon}{c}(1 + 2a\epsilon/c)^{-1}. \tag{6.3.4}$$

If $\epsilon = 1$ then the right hand side of (6.3.4) is positive, a contradiction. So $\epsilon = -1$. Then (6.3.4) implies $c = 2a + b$, which proves the claim.

(Case ii) $\ell_1 \cap \ell_2 \neq \emptyset$, and $\langle v_1, v_2 \rangle < 0$.



Note that $\langle T_1(v_2), v_1 \rangle = \langle v_2, T_1(v_1) \rangle = \langle v_2, -v_1 \rangle > 0$ and $T_1(v_2)$ and $\overrightarrow{qQ}$ are in the same direction. We claim that v_3 and $T_1(v_2)$ are in the same direction. If not then it follows from $\langle T_1(v_2), v_1 \rangle > 0$ that $\langle v_3, v_1 \rangle < 0$. By (6.3.3), $T_1(v_2)$ and v_3 are in the same direction, a contradiction. So $\langle v_1, v_3 \rangle > 0$. Let θ denote the angle between v_1 and v_3, which is also the angle between v_1 and $T_1(v_2)$. Let $\|v_3\| = 1/c$; computing the length of both sides of (6.3.3) gives

$$1/b = \frac{1}{c}(1 - 2a\cos\theta/c)^{-1},$$

i.e., $c = b + 2a\cos\theta$. Let α and γ be the angles shown in the diagram. Then

$$\begin{aligned}\overrightarrow{qQ} &= r\sin(\theta + \alpha)\\ &= r(\sin(\theta - \alpha) + 2\cos\theta\sin\alpha)\\ &= b + 2a\ \cos\theta.\end{aligned}$$

This proves $\overrightarrow{qQ} = \overrightarrow{qq'}$.

The proofs of the claim for the following two cases are similar to those for (i) and (ii) respectively and are left to the reader.
(iii) $\ell_1 \parallel \ell_2$ and $v_1,\ v_2$ are in the same direction.
(iv) $\ell_1 \cap \ell_2 \neq \emptyset$, and $\langle v_1, v_2\rangle \geq 0$. ∎

As a consequence of Corollary 5.3.7, we have:

6.3.6. Corollary. *If M^n is a rank k isoparametric submanifold of $\boldsymbol{R}^{n+k}$, then:*
(i) the subgroup of isometries of $\nu_q = q+\nu(M)_q$ generated by the reflections R_i^q in the focal hyperplanes $\ell_i(q)$ is a finite rank k Coxeter group, which is isomorphic to the Coxeter group W associated to M,
(ii) $\bigcap\{\ell_i(q)|1 \leq i \leq p\}$ consists of one point.

Let $\triangle_q$ be the connected component of $\nu_q - \bigcap\{\ell_i|1 \leq i \leq p\}$ containing q. Then the closure $\bar{\triangle}_q$ is a simplicial cone and a fundamental domain of W and $\{v_i|\ \ell_i(q)$ contains a $(k-1)-$simplex of $\bar{\triangle}_q\}$ is a simple root system for W. If $\varphi \in W$ and $\varphi(\ell_i) = \ell_j$ then by Theorem 6.3.2 we have $m_i = m_j$. So we have

6.3.7. Corollary. *We associate to each rank k isoparametric submanifold M^n of $\boldsymbol{R}^{n+k}$ a well-defined marked Dynkin diagram with k vertices, namely the Dynkin diagram of the associated Coxeter group with multiplicities m_i.*

6.3.8. Examples. Let G be a compact, rank k simple Lie group, and $\mathcal{G}$ its Lie algebra with inner product $\langle\ ,\ \rangle$, where $-\langle\ ,\ \rangle$ is the Killing form of G. Let $\mathcal{T}$ be a maximal abelian subalgebra of $\mathcal{G}$, $a \in \mathcal{T}$ a regular point, and $M = Ga$ the principal orbit through a. Since this orthogonal action is polar ($\mathcal{T}$ is a section), M is isoparametric in $\mathcal{G}$ of codimension k. Note that

$$TM_x = \{[\xi, x]|\ \xi \in \mathcal{G}\},$$
$$\nu(M)_{gag^{-1}} = g\mathcal{T}g^{-1}.$$

Given $b \in \mathcal{T}$, $\hat{b}(gag^{-1}) = gbg^{-1}$ is a well-defined normal field on M. Since $d\hat{b}_a([\xi, a]) = [\xi, b]$ and

$$\langle[\xi, b], t\rangle = \langle-[b, \xi], t\rangle = \langle\xi, [b, t]\rangle = \langle\xi, 0\rangle = 0$$

for all $t \in \nu(M)_a$, $\hat{b}$ is parallel and the shape operator is

$$A_{\hat{b}}([\xi, a]) = -[\xi, b]. \tag{6.3.5}$$

To obtain the common eigendecompositions of $\{A_{\hat{b}}\}$, we recall that if $\triangle^+$ is a set of positive roots of $\mathcal{G}$ then there exist x_α, y_α in $\mathcal{G}$ for each $\alpha \in \triangle^+$ such that

$$\mathcal{G} = \mathcal{T} \oplus \{Rx_\alpha \oplus Ry_\alpha|\ \alpha \in \triangle^+\},$$

$$[t, x_\alpha] = \alpha(t) y_\alpha, \;\; [t, y_\alpha] = -\alpha(t) x_\alpha, \tag{6.3.6}$$

where $\alpha(t) = \langle \alpha, t \rangle$, and $t \in \mathcal{T}$. Using (6.3.5) and (6.3.6), we have

$$A_b(x_\alpha) = -\frac{\alpha(b)}{\alpha(a)} x_\alpha, \;\; A_b(y_\alpha) = -\frac{\alpha(b)}{\alpha(a)} y_\alpha.$$

This implies that the curvature distributions of M are given by $E_\alpha = Rx_\alpha \oplus Ry_\alpha$ for $\alpha \in \triangle^+$ and the curvature normals are given by $v_\alpha = -\alpha/\langle \alpha, a \rangle$. So the Coxeter group associated to M as an isoparametric submanifold is the Weyl group of G, and all the multiplicities are equal to 2.

If $v \in \nu(M)_q$, then $q + v \in \ell_i(q)$ if and only if $\langle v, v_i \rangle = 1$, so as a consequence of Corollary 6.3.6 (ii), we have:

6.3.9. Corollary. *If M^n is a rank k isoparametric submanifold of $\boldsymbol{R}^{n+k}$, then there exists $a \in \boldsymbol{R}^k$ such that $\langle a, n_i \rangle = 1$ for all $1 \le i \le p$.*

6.3.10. Corollary. *Suppose $X : M^n \to \boldsymbol{R}^{n+k}$ is a rank k immersed isoparametric submanifold and all the curvature normals are non-zero. Then there exist vectors $a \in \boldsymbol{R}^k$ and $c_0 \in \boldsymbol{R}^{n+k}$ such that M is contained in the sphere of radius $\|a\|$ centered at c_0 in $\boldsymbol{R}^{n+k}$ so that*

$$X + \sum_\alpha a_\alpha e_\alpha = c_0.$$

In particular, we have

$$\bigcap \{\ell_i(q) |\; q \in M, \; 1 \le i \le p\} = \{c_0\}.$$

PROOF. By Corollary 6.3.9, there exists $a \in \boldsymbol{R}^k$ such that $\langle a, n_i \rangle = 1$. We claim that the map $X + \sum_\alpha a_\alpha e_\alpha$ is a constant vector $c_0 \in \boldsymbol{R}^{n+k}$ on M, because

$$d\left(X + \sum_\alpha a_\alpha e_\alpha \right) = \sum_{i=1}^p (1 - \langle a, n_i \rangle) \, id_{E_i} = 0.$$

So we have

$$\|X - c_0\|^2 = \| \sum_\alpha a_\alpha e_\alpha \|^2 = \|a\|^2. \quad \blacksquare$$

6.3.11. Corollary. *The following statements are equivalent for an immersed isoparametric submanifold M^n of $\boldsymbol{R}^{n+k}$:*

(i) M is compact,
(ii) all the curvature normals of M are non-zero.
(iii) M is contained in a standard sphere in $\boldsymbol{R}^{n+k}$.

6.3.12. Corollary. *If M^n is a rank k isoparametric submanifold of $\boldsymbol{R}^{n+k}$ and zero is one of the curvature normals for M corresponding to the curvature distribution E_0, then there exists a compact rank k isoparametric submanifold M_1 of $\boldsymbol{R}^{n+k-m_0}$ such that $M = E_0 \times M_1$.*

PROOF. By Corollary 6.3.9, there exists $a \in \boldsymbol{R}^k$ such that $\langle a, n_i \rangle = 1$ for all $1 \le i \le p$. Consider the map $X^* = X + \sum_\alpha a_\alpha e_a : M \to \boldsymbol{R}^{n+k}$. Then

$$dX^* = \sum_{i=0}^{p} (1 - \langle a, n_i \rangle)\, id_{E_i} = id_{E_0},$$

and $M^* = X^*(M)$ is a flat m_0-plane of $\boldsymbol{R}^{n+k}$. So $X^* : M \to M^*$ is a submersion, and in particular the fiber is a smooth submanifold of M. But the tangent plane of the fiber is $\bigoplus_{i=1}^p E_i$, so it is integrable. On the other hand $\bigoplus_{i=1}^p E_i$ is defined by

$$\omega_i = 0, \ \ i \le m_0,$$

so we have

$$0 = d\omega_i = \sum_{j > m_0} \omega_{ij} \wedge \omega_j = \sum_{j,m > m_0} \gamma_{ijm}\, \omega_m \wedge \omega_j.$$

Hence

$$\gamma_{ijm} = \gamma_{imj}, \ \ \text{for } i \le m_0, \ j \ne m > m_0. \tag{6.3.7}$$

By Proposition 6.2.7, we have

$$\lambda_{j\alpha}\gamma_{ijm} = \lambda_{m\alpha}\gamma_{imj}, \ \text{for } i \le m_0. \tag{6.3.8}$$

If $e_j, e_m \in E_s$ for some $s > 0$, then Proposition 6.2.7 imply that $\gamma_{ijm} = 0$. If e_j and e_m belong to different curvature distributions, then there exists α_0 such that $\lambda_{j\alpha_0} \ne \lambda_{m\alpha_0}$. So (6.3.7) and (6.3.8) implies that $\gamma_{ijm} = 0$. Therefore we have proved that

$$\omega_{ij} = 0, \ \ \omega_{i\alpha} = 0, \ \ i \le m_0, \ j > m_0,$$

on M. Let $M_1 = (X^*)^{-1}(q^*)$. Then both M and $M_1 \times E_0$ have flat normal bundles and the same first, second fundamental forms. So the fundamental theorem of submanifolds (Corollary 2.3.2) implies that $M = M_1 \times E_0$. ∎

Next we discuss the irreduciblity of the associated Coxeter group of an isoparametric submanifold, which leads to a decomposition theorem for isoparametric submanifolds.

If $M_i^{n_i}$ is isoparametric in $\boldsymbol{R}^{n_i+k_i}$ with Coxeter group W_i on $\boldsymbol{R}^{k_i}$ for $i = 1, 2$, then $M_1 \times M_2$ is isoparametric in $\boldsymbol{R}^{n_1+n_2+k_1+k_2}$ with Coxeter group $W_1 \times W_2$ on $\boldsymbol{R}^{k_1} \times \boldsymbol{R}^{k_2}$. The converse is also true.

6.3.13. Theorem. *Let M^n be a compact rank k isoparametric submanifold of $\boldsymbol{R}^{n+k}$, and W its associated Coxeter group. Suppose $\boldsymbol{R}^k = \boldsymbol{R}^{k_1} \times \boldsymbol{R}^{k_2}$ and $W = W_1 \times W_2$, where W_i is a Coxeter group on $\boldsymbol{R}^{k_i}$. Then there exist two isoparametric submanifolds M_1, M_2 with Coxeter groups W_1, W_2 respectively such that $M = M_1 \times M_2$.*

PROOF. We may assume that $n_i \in \boldsymbol{R}^{k_1} \times 0,\ R_i \in W_1$ for $i \leq p_1$, and $n_j \in 0 \times \boldsymbol{R}^{k_2},\ R_j \in W_2$ for $j > p_1$. Since W_1 is a finite Coxeter group, there exists a constant vector $a \in \boldsymbol{R}^{k_1} \times 0$ such that $\langle a, n_i \rangle = 1$ for all $i \leq p_1$. Consider $X^* = X + \sum_\alpha a_\alpha e_\alpha$. Since $\langle a, n_j \rangle = 0$ for all $j > p_1$, we have

$$dX^* = \sum_{j>p_1}^{p} id_{E_j}.$$

So an argument similar to that in Corollary 6.3.12 implies that $V = \bigoplus_{i \leq p_1} E_i$ and $H = \bigoplus_{j>p_1}^{p} E_j$ are integrable, and that M is the product of a leaf of V and a leaf of H. ■

6.3.14. Definition. An isoparametric submanifold M^n of $\boldsymbol{R}^{n+k}$ is called *irreducible*, if M is not the product of two lower dimensional isoparametric submanifolds.

As a consequence of Theorem 6.3.13, we have:

6.3.15. Proposition. *An isoparametric submanifold of Euclidean space is irreducible if and only if its associated Coxeter group is irreducible.*

Since every Coxeter group can be written uniquely as the product of irreducible Coxeter groups uniquely up to permutation, we have:

6.3.16. Theorem. *Every isoparametric submanifold of Euclidean space can be written as the product of irreducible ones, and such decomposition is unique up to permutation.*

As a consequence of Corollary 6.3.11 and the following proposition we see that the set of compact, isoparametric submanifolds of Euclidean space coincides with the set of compact isoparametric submanifolds of standard spheres.

6.3.17. Proposition. *If M^n is an isoparametric submanifold of $\boldsymbol{S}^{n+k}$ then M is an isoparametric submanifold of $\boldsymbol{R}^{n+k+1}$.*

PROOF. Let $X : M \to \boldsymbol{S}^{n+k}$ be the immersion, and $\{e_A\}$ the adapted frame for X, ω_i the dual coframe, and ω_{AB} the Levi-Civita connection 1-form. We may assume that e_α's are parallel, i.e., $\omega_{\alpha\beta} = 0$ for $n < \alpha,\ \beta \leq n + k$. Set $e_{n+k+1} = X$, then $\{e_1, \ldots, e_{n+k+1}\}$ is an adapted frame for M as an immersed submanifold of $\boldsymbol{R}^{n+k+1}$. Since

$$de_{n+k+1} = dX = \sum \omega_i e_i,$$

we have $\omega_{n+k+1,\alpha} = 0$ and $A_{e_{n+k+1}} = -id$. This implies that M is isoparametric in $\boldsymbol{R}^{n+k+1}$. ∎

6.4. Existence of isoparametric polynomial maps

In this section, given an isoparametric submanifold M^n of $\boldsymbol{R}^{n+k}$, we will construct a polynomial isoparametric map on $\boldsymbol{R}^{n+k}$ which has M as a level submanifold. This construction is a generalization of the Chevalley Restriction Theorem in Example 5.6.16.

By Corollary 6.3.11 and 6.3.12, we may assume that M^n is a compact, rank k isoparametric submanifold of $\boldsymbol{R}^{n+k}$, and $M \subseteq \boldsymbol{S}^{n+k-1}$. Let W be the Coxeter group associated to M, and p the number of reflection hyperplanes of W, i.e., M has p curvature normals. In the following we use the same notation as in section 6.2.

Given $q \in M$, there is a simply connected neighborhood U of q in M such that U is embedded in $\boldsymbol{R}^{n+k}$. Let e_α be a parallel normal frame, v_i the curvature normals $\sum_\alpha n_{i\alpha} e_\alpha$, and $n_i = (n_{in+1}, \ldots, n_{in+k})$. Let $Y : U \times \boldsymbol{R}^k \to \boldsymbol{R}^{n+k}$ be the endpoint map, i.e., $Y(x, z) = x + \sum_\alpha z_\alpha e_\alpha(x)$. Then there is a small ball B centered at the origin in $\boldsymbol{R}^k$ such that $Y|U \times B$ is a local coordinate system for $\boldsymbol{R}^{n+k}$. In particular, $z \cdot n_i < 1$ for all $z \in B$ and $1 \leq i \leq p$. We denote $Y(U \times B)$ by $\mathcal{O}$. In fact, $\mathcal{O}$ is a tubular neighborhood of M in $\boldsymbol{R}^{n+k}$. Since $M \subseteq \boldsymbol{S}^{n+k-1}$, by Corollary 6.3.10 there exists a vector $a \in \boldsymbol{R}^k$ such that

$$X = \sum_\alpha a_\alpha e_\alpha.$$

Then

$$Y = X + \sum_\alpha z_\alpha e_\alpha = \sum_\alpha (z_\alpha - a_\alpha) e_\alpha.$$

Let $y = z - a$ (note that $y = 0$ corresponds to the origin of $\boldsymbol{R}^{n+k}$ and the W-action on $q + \nu(M)_q$ induces an action on $\boldsymbol{R}^k$ which is linear in y). Then y_α is a smooth function defined on the tubular neighborhood $\mathcal{O}$ of M in $\boldsymbol{R}^{n+k}$. It is easily seen that any W-invariant smooth function u on $\boldsymbol{R}^k$ can be extended uniquely to a smooth function f on $\mathcal{O}$ that is constant on all the parallel submanifolds of the form M_v, where v is a parallel normal field on M with $v(q) \in B$. That is, we extend f by the formula $f(Y(x, z)) = u(z - a) = u(y)$. We will call this f simply the *extension* of u.

In order to construct a global isoparametric map for M, we need the following two lemmas.

6.4.1. Lemma. *If $u : \boldsymbol{R}^k \to \boldsymbol{R}$ is a W-invariant homogeneous polynomial of degree k, then the function*

$$\varphi(y) = \sum_{i=1}^{p} m_i \frac{\nabla u(y) \cdot n_i}{y \cdot n_i}$$

is a W-invariant homogeneous polynomial of degree $k - 2$.

PROOF. Let R_i denote the reflection of $\boldsymbol{R}^k$ along the vector n_i. Since $u(R_i y) = u(y)$, $\nabla u(R_i(y)) = R_i(\nabla u(y))$. We claim that $\nabla u(y) \cdot n_i = 0$ if $y \cdot n_i = 0$. For if $y \cdot n_i = 0$ then $R_i(y) = y$, so $\nabla u(y) = R_i(\nabla u(y))$, i.e., $\nabla u(y) \cdot n_i = 0$. Therefore $\varphi(y)$ is a homogeneous polynomial of degree $k - 2$. To check that φ is W-invariant, we note that

$$\begin{aligned}
\varphi(R_i(y)) &= \sum_j m_j \frac{\nabla u(R_i(y)) \cdot n_j}{R_i(y) \cdot n_j} \\
&= \sum_j m_j \frac{R_i(\nabla u(y)) \cdot n_j}{R_i(y) \cdot n_j} \\
&= \sum_j m_j \frac{\nabla u(y) \cdot R_i(n_j)}{y \cdot R_i(n_j)}.
\end{aligned}$$

Then the lemma follows from Theorem 6.3.2 (3). ∎

6.4.2. Lemma. *Let $u : \boldsymbol{R}^k \to \boldsymbol{R}$ be a W-invariant homogeneous polynomial of degree k, and $f : \mathcal{O} \to \boldsymbol{R}$ its extension. Then*

(i) $\triangle f$ is the extension of a W-invariant homogeneous polynomial of degree $(k - 2)$ on $\boldsymbol{R}^k$,

(ii) $\|\nabla f\|^2$ is the extension of a W-invariant homogeneous polynomial of degree $2(k-1)$ on $\boldsymbol{R}^k$

PROOF. Since

$$dY = \sum_i (1 - z \cdot n_i)\, id_{E_i} + \sum_\alpha dz_\alpha e_\alpha$$
$$= \sum_i y \cdot z_i\, id_{E_i} + \sum_\alpha dy_\alpha e_\alpha,$$

we may choose a local frame field $e_A^* = e_A$ on $\mathcal{O} \subset \boldsymbol{R}^{n+k}$, and the dual coframe is

$$\omega_j^* = (y \cdot n_i)\omega_j, \text{ if } \sum_{r=1}^{i-1} m_r < j \le \sum_{r=1}^{i} m_r,$$
$$\omega_a^* = dy_\alpha.$$

The Levi-Civita connection 1-form on $\mathcal{O}$ is $\omega_{AB}^* = \omega_{AB}$. Then by (1.3.6) we have

$$\triangle y_\alpha = -\sum_{i=1}^{p} \frac{m_i n_{i\alpha}}{y \cdot n_i}.$$

Since $f(x, y) = u(y)$, we have

$$df = \sum_\alpha u_\alpha \omega_\alpha^*, \quad \|\nabla f\|^2 = \|\bar{\nabla} u\|^2,$$

$$\triangle f = \bar{\triangle} u + \sum_i m_i \frac{\bar{\nabla} u \cdot n_i}{y \cdot n_i},$$

where $\bar{\triangle}$, $\bar{\nabla}$ are the standard Laplacian and gradient on $\boldsymbol{R}^k$. Then (i) follows from Lemma 6.4.1. To prove (ii), we note that $\bar{\nabla} u(R_i(y)) = R_i(\bar{\nabla} u(y))$, so $\|\bar{\nabla} u\|^2$ is a W-invariant polynomial of degree $2(k-1)$ on $\boldsymbol{R}^k$. ∎

6.4.3. Theorem. *Let M^n be a rank k isoparametric submanifold in $\boldsymbol{R}^{n+k}$, W the associated Coxeter group, q a point on M, and $\nu_q = q + \nu(M)$ the affine normal plane at q. If $u : \nu_q \to \boldsymbol{R}$ is a W-invariant homogeneous polynomial of degree m, then u can be extended uniquely to a homogeneous degree m polynomial f on $\boldsymbol{R}^{n+k}$ such that f is constant on M.*

PROOF. We may assume $\nu_q = \boldsymbol{R}^k$. We prove this theorem on $\mathcal{O}$ by using induction on the degree k of u. The theorem is obvious for $m = 0$. Suppose it is true for all $\ell < m$. Given a degree m W-invariant homogeneous polynomial u on $\boldsymbol{R}^k$, by Lemma 6.4.2, $\|df\|^2$ is again the extension of a W-invariant homogeneous polynomial of degree $2k-2$ on $\boldsymbol{R}^k$. Applying Lemma 6.4.2 repeatedly, we have $\triangle^{m-1}(\|df\|^2)$ is the extension of a degree zero W-invariant polynomial, hence it is a constant. Therefore

$$\begin{aligned} 0 &= \triangle^m(\|df\|^2) \\ &= \sum_{r=0}^{m} \sum_{\substack{s+s'=m-r \\ i,i_1,\ldots,i_r}} c_{r,s}(\triangle^s f)_{i,i_1,\ldots,i_r}(\triangle^{s'} f)_{i,i_1,\ldots,i_r}, \end{aligned}$$

where $c_{r,s}$ are constants depending on r and s. We claim that

$$(\triangle^s f)_{i,i_1,\ldots,i_r}(\triangle^{s'} f)_{i,i_1,\ldots,i_r}, \quad s' = m-r-s,$$

is zero if $r < m$. For we may assume that $s \geq m-r-s$, i.e., $s \geq s'$, so $s \geq 1$. By Lemma 6.4.2, $\triangle^s f$ is the extension of a degree $m-2s$ W-invariant polynomial on $\boldsymbol{R}^k$. By the induction hypothesis, $\triangle^s f$ is a homogeneous polynomial on $\mathcal{O} \subset \boldsymbol{R}^{n+k}$ of degree $m-2s$, hence all the partial derivatives of order bigger than $m-2s$ will be zero. We have $r+1 > r \geq m-2s$ by assumption, so we obtain

$$0 = \sum_{i,i_1,\ldots,i_m} f^2_{i,i_1,\ldots,i_m},$$

i.e., $D^\alpha f = 0$ in $\mathcal{O}$ for $|\alpha| = k+1$. This proves that f is a homogeneous polynomial of degree k in $\mathcal{O}$. There is a unique polynomial extension on $\boldsymbol{R}^{n+k}$, which we still denote by f. ∎

By Theorem 5.3.18 there exist k homogeneous W-invariant polynomials $u_1, \ldots, u_k$ on $\boldsymbol{R}^k$ such that the ring of W-invariant polynomials on $\boldsymbol{R}^k$ is the polynomial ring $R[u_1, \ldots, u_k]$.

6.4.4. Theorem. *Let M, W, q, ν_q be as in Theorem 6.4.3, and let $u_1, \ldots, u_k$ be a set of generators of the W-invariant polynomials on ν_q. Then $u = (u_1, \ldots, u_k)$ extends uniquely to an isoparametric polynomial map $f : \boldsymbol{R}^{n+k} \to \boldsymbol{R}^k$ having M as a regular level set. Moreover,*

(1) each regular set is connected,
(2) the focal set of M is the set of critical points of f,
(3) $\nu_q \cap M = W \cdot q$,
(4) $f(\boldsymbol{R}^{n+k}) = u(\nu_q)$,
(5) for $x \in \nu_q$, $f(x)$ is a regular value if and only if x is W-regular,
(6) $\nu(M)$ is globally flat.

PROOF. Let $f_1, \ldots, f_k$ be the extended polynomials on $\boldsymbol{R}^{n+k}$. Because $u_1, \ldots, u_k$ are generators, $f = (f_1, \ldots, f_k)$ will automatically satisfies condition (1) and (2) of Definition 6.1. Since y_α are part of local coordinates,

$[y_\alpha, y_\beta] = 0$. But f is a function of y, so condition (3) of Definition 6.1.1 is satisfied. Then (1)-(5) follow from the fact that $u_1, \ldots, u_k$ separate the orbits of W and that regular points of the map $u = (u_1, \ldots, u_k)$ are just the W-regular points. Finally, since $\{\nabla f_1, \ldots, \nabla f_k\}$ is a global, parallel, normal frame for M, $\nu(M)$ is globally flat. ■

6.4.5. Corollary. *Let M^n be an immersed isoparametric submanifold of $\boldsymbol{R}^{n+k}$. Then*
(i) M is embedded,
(ii) $\nu(M)$ is globally flat.

The above proof also gives a constructive method for finding all compact irreducible isoparametric submanifolds of Euclidean space. To be more specific, given an irreducible Coxeter group W on $\boldsymbol{R}^k$ with multiplicity m_i for each reflection hyperplane ℓ_i of W such that $m_i = m_j$ if $g(\ell_i) = \ell_j$ for some $g \in W$, i.e., given a marked Dynkin diagram. Suppose W has p reflection hyperplanes $\ell_1, \ldots, \ell_p$. Let a_i be a unit normal vector to ℓ_i. Set $n = \sum_{i=1}^p m_i$. Let $u_1, \ldots, u_k$ be a fixed set of generators for the ring of W-invariant polynomials on $\boldsymbol{R}^k$, which can be chosen to be homogeneous of degree k_i. Then there are polynomials V_i, Φ_i, U_{ij}, and Ψ_{ijm} on $\boldsymbol{R}^k$ such that

$$\triangle u_i = V_i(u), \quad \nabla u_i \cdot \nabla u_j = U_{ij}(u),$$

$$\sum_j m_j \frac{\nabla u_i \cdot a_j}{y \cdot a_j} = \Phi_i(u), \quad [\nabla u_i, \nabla u_j] = \sum_m \Psi_{ijm}(u) \nabla u_m.$$

Then any polynomial solution $f = (f_1, \ldots, f_k) : \boldsymbol{R}^{n+k} \to \boldsymbol{R}^k$, with f_i being homogeneous of degree k_i, of the following system is an isoparametric map:

$$\begin{aligned} \triangle f_i &= V_i(f) + \Phi_i(f), \\ \nabla f_i \cdot \nabla f_j &= U_{ij}(f), \\ [\nabla f_i, \nabla f_j] &= \sum_m \Psi_{ijm}(f) \nabla f_m. \end{aligned} \tag{6.4.1}$$

Moreover, if M is any regular level submanifold of such an f, then the associated Coxeter group and multiplicities of M are W and m_i respectively.

Since $u_1 : \boldsymbol{R}^k \to \boldsymbol{R}$ can be chosen to be $\sum_{i=1}^k x_i^2$, the extension f_1 on $\boldsymbol{R}^{n+k}$ is $\sum_{i=1}^{n+k} x_i^2$. So (6.4.1) is a system of equations for $(k-1)$ functions. Because both the coefficients and the admissible solutions for (6.4.1) are homogeneous polynomials, the problem of classifying isoparametric submanifolds becomes a purely algebraic one.

6.4.6. Remark. Theorem 6.4.4 was first proved by Münzner in [Mü1,2] for the case of isoparametric hypersurfaces of spheres, i.e., for rank 2 isoparametric submanifolds of Euclidean space. Suppose W is the dihedral group of $2p$ elements on $\boldsymbol{R}^2$. Then W has p reflection lines in $\boldsymbol{R}^2$, and we may choose $a_j = (\cos(j\pi/p), \sin(j\pi/p))$ for $0 \leq j < p$. By Theorem 6.3.2, all m_i's are equal to some integer m if p is odd, and $m_1 = m_3 = \cdots,\ m_2 = m_4 = \cdots$ if p is even. So we have $n = pm$ if p is odd, and $n = p(m_1 + m_2)/2$ if p is even. It is easily seen that we can choose

$$u_1(x, y) = x^2 + y^2,\ \ u_2(x, y) = Re((x + iy)^p).$$

Let $f_i : \boldsymbol{R}^{n+2} \to \boldsymbol{R}$ be the extensions. Then $f_1(x) = \|x\|^2$. Let $F = f_2$. Then it follows from a direct computation that (6.4.1) becomes the equations given by Münzner in [Mü1,2]:

$$\begin{aligned} \triangle F(x) &= c\|x\|^{p-2} \\ \|\nabla F(x)\|^2 &= p^2\|x\|^{2p-2}, \end{aligned}$$

where $c = 0$ if p is odd and $c = (m_2 - m_1)p^2/2$ if p is even.