Transformation Groups

The theory of Lie groups of transformations of finite dimensional manifolds is a complex, rich, and beautiful one, with many applications to different branches of mathematics. For a systematic introduction to this subject we refer the reader to [Br] and [Dv]. Because our interest is in the Riemannian geometry of Hilbert manifolds, we will concentrate on *isometric* actions on such manifolds. In studying the action of a Lie group G on a finite dimensional manifold M, it is well known that without the assumption that the group G is compact or, more generally, that the action is proper (cf. definition below) all sorts of comparatively pathological behavior can occur. For example, orbits need not be regularly embedded closed submanifolds, the action may not admit slices, and invariant Riemannian metrics need not exist. In fact, in the finite dimensional case, properness is both necessary and sufficient for G to be a closed subgroup of the group of isometries of M with respect to some Riemannian metric. In infinite dimensions properness is no longer necessary for the latter, but it is sufficient when coupled with one other condition. This other condition on an action, defined below as "Fredholm", is automatically satisfied in finite dimensions. As we shall see, much of the richness of the classical theory of compact transformation groups carries over to proper, Fredholm actions on Hilbert manifolds.

G-manifolds

A Hilbert manifold M is a differentiable manifold locally modeled on a separable Hilbert space $(V, \langle\ ,\ \rangle)$. The foundational work on Hilbert (and Banach) manifolds was carried out in the 1960's. The standard theorems of differential calculus (e.g., the inverse function theorem and the local existence and uniqueness theorem for ordinary differential equations) remain valid ([La]), and in [Sm2] Smale showed that one of the basic tools of finite dimensional differential topology, Sard's Theorem, could be recovered in infinite dimensions if one restricted the morphisms to be smooth Fredholm maps.

A Riemannian metric on M is a smooth section g of $S^2(T^*M)$ such that $g(x)$ is an inner product for TM_x equivalent to the inner product $\langle\ ,\ \rangle$ on V for all $x \in M$. Such an (M, g) is called a Riemannian Hilbert manifold. For fixed vector fields X and Z the right hand side of (1.2.2) defines a continuous linear functional of TM_x. Since TM_x^* is isomorphic to TM_x, (1.2.2) defines a unique element $(\nabla_Z X)(x)$ in TM_x, and the argument for a unique compatible, torsion free connection for g is valid for infinite dimensional Riemannian manifolds, so geodesics and the exponential map $\exp : TM \to M$ can be defined just as in finite dimensions. A diffeomorphism $\varphi : M \to M$ is an isometry if $d\varphi_x : TM_x \to TM_{\varphi(x)}$ is a linear isometry for all $x \in M$.

5.1.1. Definition. Let M and N be Hilbert manifolds. A smooth map $\varphi : M \to N$ is called an *immersion* if $d\varphi_x$ is injective and $d\varphi_x(TM_x)$ is a closed linear subspace of $TN_{\varphi(x)}$ for all $x \in M$.

If the dimension of N is finite, then $d\varphi_x(TM_x)$ is always a closed linear subspace of $TN_{\varphi(x)}$. So this definition agrees with the finite dimensional case.

5.1.2. Definition. A Hilbert Lie group G is a Hilbert manifold with a group structure such that the map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ from $G \times G \to G$ is smooth.

In this chapter we will always assume that manifolds are Hilbert manifolds and that Lie groups are Hilbert Lie groups. They can be either of finite or infinite dimension.

Let G be a Lie group, and M a smooth manifold. A smooth G-action on M is a smooth map $\rho : G \times M \to M$ such that

$$ex = x, \quad (g_1 g_2)x = g_1(g_2 x),$$

for all $x \in M$ and $g_1, g_2 \in G$. Here e is the identity element of G and $gx = \rho(g, x)$. This defines a group homomorphism, again denoted by ρ, from G to the group $\mathrm{Diff}(M)$ of diffeomorphisms of M; namely $\rho(g)(x) = gx$. Given a fixed such G-action, we say that G acts on M, or that M is a G-manifold.

5.1.3. Definition. A G-manifold M with action ρ is

(i) *linear*, if M is a vector space V and $\rho(G) \subseteq \boldsymbol{GL}(V)$, i.e., ρ is a linear representation of G,

(2) *affine*, if M is an affine space V and $\rho(G)$ is a subgroup of the affine group of V,

(3) *orthogonal*, if M is a Hilbert space V and $\rho(G)$ is a subgroup of the group of linear isometries of V, i.e., ρ is an orthogonal representation of G,

(4) *Riemannian* or *isometric*, if M is a Riemannian manifold and $\rho(G)$ is included in the group of isometries of M.

5.1.4. Examples.

(1) The natural orthogonal action of $\boldsymbol{SO}(n)$ on $\boldsymbol{R}^n$ given by taking ρ to be the inclusion of $\boldsymbol{SO}(n)$ into $\mathrm{Diff}(\boldsymbol{R}^n)$.

(2) The Adjoint action of G on G, defined by by $Ad(g)(h) = ghg^{-1}$.

(3)The adjoint action of G on its Lie algebra $\mathcal{G}$ given by $g \to d(Ad(g))_e$, the differential of $Ad(g)$ at the identity e. If G is compact and semi-simple then the Killing form b is negative definite, and the adjoint action is orthogonal with respect to the inner product $-b$.

(4) $\boldsymbol{SO}(n)$ acts on the linear space $\mathcal{S}$ of trace zero symmetric $n \times n$ matrices by conjugation, i.e., $g \cdot x = gxg^{-1}$. This action is orthogonal with respect to the inner product $\langle x, y \rangle = \mathrm{tr}(xy)$.

(5) $\boldsymbol{SU}(n)$ acts on the linear space $\mathcal{M}$ of Hermitian $n \times n$ trace zero matrices by conjugation. This action is orthogonal with respect to the inner product $\langle x, y \rangle = \mathrm{tr}(x\bar{y})$.

The differential of the group homomorphism $\rho : G \to \mathrm{Diff}(M)$ at the identity e gives a Lie algebra homomorphism from the Lie algebra $\mathcal{G}$ to the Lie algebra $C^\infty(TM)$ of $\mathrm{Diff}(M)$. We will denote the vector field $d\rho_e(\xi)$ by ξ again, and identify $\mathcal{G}$ as a Lie subalgebra of $C^\infty(TM)$. In fact, if g_t is the one-parameter subgroup in G generated by ξ then $\xi(x) = \frac{d}{dt}|_{t=0}\, g_t x$. If M is a Riemannian G-manifold, then the vector field ξ is a Killing vector field.

5.1.5. Definition. If M is a G-manifold and $x \in M$ then Gx, the *G-orbit through x*, and G_x, *the isotropy subgroup at x* are defined respectively by:

$$Gx = \{gx \mid g \in G\},$$

$$G_x = \{g \in G \mid gx = x\}.$$

The *orbit map* $\omega_x : G \to M$ is the map $g \mapsto gx$. It is constant on G_x cosets and hence defines a map $\varpi_x : G/G_x \to M$ that is clearly injective, with image Gx. Since G/G_x has a smooth quotient manifold structure, this means that we can (and will) regard each orbit as a smooth manifold by carrying over the differentiable structure from G/G_x. Since the action is smooth the orbits are even smoothly "immersed" in M, but it is important to note that without additional assumptions the orbits will *not* be regularly embedded in M, i.e., the manifold topology that Gx inherits from G/G_x will not in general be the topology induced from M. Moreover Gx will not in general be closed in M, and even the tangent space of Gx at x need not be closed in TM_x. The assumptions of properness and Fredholm, defined below, are required to avoid these unpleasant possibilities.

To prepare for the definition of Fredholm actions we recall the definition of a Fredholm map between Hilbert manifolds. If V, W are Hilbert spaces, then a bounded linear map $T : V \to W$ is *Fredholm* if $\mathrm{Ker}\, T$ and $\mathrm{Coker}\, T$ are of finite dimension. It is then a well-known, easy consequence of the closed graph theorem that $T(V)$ is closed in W. If M and N are Hilbert manifolds, then a differentiable map $f : M \to N$ is *Fredholm* if df_x is Fredholm for all x in M.

5.1.6. Definition. The G-action on M is called *Fredholm* if for each $x \in M$ the orbit map $\omega_x : G \to M$ is Fredholm. In this case we also say that M is a *Fredholm G-manifold*.

5.1.7. Remark. Clearly any smooth map between finite dimensional manifolds is Fredholm, so if G is a finite dimensional Lie group and M is a finite dimensional G-manifold then the action of G on M is automatically Fredholm.

5.1.8. Proposition. *If M is any G manifold, then*
(i) $G_{gx} = gG_x g^{-1}$,
(ii) if $Gx \cap Gy \neq \emptyset$, *then* $Gx = Gy$,
(iii) $T(Gx)_x = \{\xi(x) \mid \xi \in \mathcal{G}\}$.
(iv) If the action is Fredholm then each isotropy group G_x *has finite dimension and each orbit* Gx *has finite codimension in* M.

Let M/G denote the set of all orbits, and $\pi : M \to M/G$ the orbit map defined by $x \mapsto Gx$. The set M/G equipped with the quotient topology is called the *orbit space* of the G-manifold M and will also be denoted by $\widetilde{M}$. The conjugacy class of a closed subgroup H of G will be denoted by (H) and is called a G-isotropy type. If Gx is any orbit of a G-manifold M, then the set of isotropy groups $G_{gx} = gG_x g^{-1}$ at points of Gx is an isotropy type, called the isotropy type of the orbit, and two orbits (of possibly different G-manifolds) are said to be of the same type if they have the same isotropy types.

5.1.9. Definition. Let M and N be G-manifolds. A mapping $F : M \to N$ is *equivariant* if $F(gx) = gF(x)$ for all $(g, x) \in G \times M$. A function $f : M \to \boldsymbol{R}$ is invariant if $f(gx) = f(x)$ for all $(g, x) \in G \times M$.

If $F : M \to N$ is equivariant, then it is easily seen that $F(Gx) = G(F(x))$, and $G_x \subseteq G_{F(x)}$ with equality if and only if F maps Gx one-to-one onto $G(F(x))$. It follows that two orbits have the same type if and only if they are equivariantly diffeomorphic.

5.1.10. Definition. Let M be a G-manifold. An orbit Gx is a *principal orbit* if there is a neighborhood U of x such that for all $y \in U$ there exists a G-equivariant map from Gx to Gy (or equivalently there exists $g \in G$ such that $G_x \subseteq gG_y g^{-1}$). (G_x) is a principal isotropy type of M if Gx is a principal orbit.

A point x is called a *regular point* if Gx is a principal orbit, and x is called a *singular point* if Gx is not a principal orbit. The set of all regular points, and the set of all singular points of M will be denoted by M_r and M_s respectively.

5.1.11. Definition. Let M be a G-manifold. A submanifold S of M is called a *slice* at x if there is a G-invariant open neighborhood U of Gx and a smooth equivariant retraction $r : U \to Gx$, such that $S = r^{-1}(x)$.

5.1.12. Proposition. *If M is a G-manifold and S is a slice at x, then*
(i) $x \in S$ and $G_x S \subseteq S$,
(ii) $gS \cap S \neq \emptyset$ implies that $g \in G_x$,
(iii) $GS = \{gs \mid (g, s) \in G \times S\}$ is open in M.

PROOF. Let $r : U \to Gx$ be an equivariant retraction and $S = r^{-1}(x)$. Then $G_y \subseteq G_x$ for all $y \in S$, hence $r|Gy$ is a submersion. This implies that x is a regular value of r, so S is a submanifold of M. If $y \in S$ and $gy \in S$, then $r(gy) = x = gr(y) = gx$, i.e., $g \in G_x$. If $g \in G_x$ and $s \in S$, then $r(gs) = gr(s) = gx = x$. So we have $G_x S \subseteq S$. ∎

5.1.13. Corollary. *If S is a slice at x, then*
(1) S is a G_x-manifold,
(2) if $y \in S$, then $G_y \subseteq G_x$,
(3) if Gx is a principal orbit and G_x is compact, then $G_y = G_x$ for all $y \in S$, i.e., all nearby orbits of Gx are principal of the same type.
(4) two G_x-orbits $G_x s_1$ and $G_x s_2$ of S are of the same type if and only if the two G-orbits Gs_1 and Gs_2 of M are of the same type,
(5) $S/G_x = GS/G$, which is an open neighborhood of the orbit space M/G near Gx.

PROOF. (1) and (2) follow from the definition of slice. If $y \in S$ then G_y is a closed subgroup of G_x, hence if G_x is compact so is G_y. If Gx is principal then, by definition, for y near x we also have that G_x is conjugate to a subgroup of G_y. But if two compact Lie groups are each isomorphic to a subgroup of the other then they clearly have the same dimension and the same number of components. It then follows that for y in S we must have $G_y = G_x$. Let $K = G_x$ and $s \in S$. Using the condition (ii) of the Proposition we see that $K_s = G_s$, and (4) and (5) follow. ∎

Exercises.

1. What are the orbits of the actions in (1), (4) and (5) of Example 5.1.4 ?
2. Find all orbit types of the actions in (1), (4) and (5) of Example 5.1.4.
3. Describe the orbit space of the actions in (1), (4) and (5) of Example 5.1.4.
4. Let $\mathcal{S}$ be the $\boldsymbol{SO}(n)$-space in Example 5.1.4 (4), and Σ the set of all trace zero $n \times n$ real diagonal matrices. Show that:
   (i) Σ meets every orbit of $\mathcal{S}$,
   (ii) if $x \in \Sigma$, then Gx is perpendicular to Σ,
   (iii) let $\Sigma^0 = \{\mathrm{diag}(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \text{ are distinct}\}$, and S a connected component of Σ^0. Then S is a slice at x for all $x \in S$.

5. Let $\mathcal{M}$ be the $\boldsymbol{SU}(n)$-space in Example 5.1.4 (5), and Σ the set of all trace zero $n \times n$ real diagonal matrices. Show that
   (i) Σ meets every orbit of $\mathcal{M}$,
   (ii) if $x \in \Sigma$, then Gx is orthogonal to Σ,
   (iii) let $\Sigma^0 = \{\text{diag}(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \text{ are distinct}\}$, and S a connected component of Σ^0, then S is a slice at x for all $x \in S$.
6. Describe the orbit space of the action of Example 5.1.4 (3) for $G = \boldsymbol{SU}(n)$ and $G = \boldsymbol{SO}(n)$.

5.2. Proper actions

5.2.1. Definition. A G-action on M is called *proper* if $g_n x_n \to y$ and $x_n \to x$ imply that g_n has a convergent subsequence.

5.2.2. Remark. Either of the following two conditions is necessary and sufficient for a G-action on M to be proper:
(i) the map from $G \times M$ to $M \times M$ defined by $(g, x) \mapsto (gx, x)$ is proper,
(ii) given compact subsets K and L of M, the set $\{g \in G | \, gK \cap L \neq \emptyset\}$ is compact.

5.2.3. Remark. If G is compact then clearly any G-action is proper. Also, if G acts properly on M, then all the isotropy subgroups G_x are compact.

Next we discuss the relation between proper actions and Riemannian actions.

5.2.4. Proposition. *Let M be a finite dimensional Riemannian G-manifold. If G is closed in the group of all isometries of M then the action of G on M is proper.*

PROOF. Suppose $g_n x_n \to y$ and $x_n \to x$ in M. Since M is of finite dimension, there exist compact neighborhoods K and L of x and y such that $x_n \in K$ and $g_n K \subseteq L$. Because the $g_n : M \to M$ are isometries, $\{g_n\}$ is equicontinuous, and it then follows from Ascoli's theorem that a subsequence of $\{g_n\}$ converges uniformly to some isometry g of M. Thus if G is closed in the group of isometries of M, g_n has a convergent subsequence in G. ∎

The above proposition is not true for infinite dimensional Riemannian G-manifolds. A simple counterexample is the standard orthogonal action on an infinite dimensional Hilbert space V (the isotropy subgroup at the origin is the group $\boldsymbol{O}(V)$, which is not compact). However, if M is a proper Fredholm (PF) G-manifold, then there exists a G-invariant metric on M, i.e., G acts on M

isometrically with respect to this metric. In order to prove this fact, we need the following two theorems. (Although these two theorems were proved in [Pa1] for proper actions on finite dimensional G-manifolds, they generalize without difficulty to infinite dimensional PF G-manifolds):

5.2.5. Theorem. *If M is a PF G-manifold and $\{U_\alpha\}$ is a locally finite open cover consisting of G-invariant open sets, then there exists a smooth partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$ such that each f_α is G-invariant.*

Such $\{f_\alpha\}$ is called a *G-invariant partition of unity*. Roughly speaking, it is a partition of unity subordinate to the open cover $\{\tilde{U}_\alpha\}$ of the orbit space $\tilde{M}$.

5.2.6. Theorem. *If M is a PF G-manifold, then given any $x \in M$ there exists a slice at x.*

5.2.7. Theorem. *If M is a PF G-manifold, then there exists a G-invariant metric on M, i.e., the G-action on M is isometric with respect to this metric.*

PROOF. Using Theorem 5.2.6, given any $x \in M$ there exists a slice S_x at x. Then $\{U_x = GS_x \mid x \in M\}$ is a G-invariant open cover of M. So we may assume that there exists a locally finite G-invariant open cover $\{U_\alpha\}$ such that $U_\alpha = GS_\alpha$ and S_α is the slice at x_α. Let $\{f_\alpha\}$ be a G-invariant partition of unity subordinate to $\{U_\alpha\}$.

Since G_{x_α} is compact, by the averaging method we can obtain an orthogonal structure b_α on $TM|S_\alpha$, which is G_{x_α}-invariant. Extend b_α to $TM|U_\alpha$ by requiring that $b_\alpha(gs)(dg_s(u_1), dg_s(u_2)) = b_\alpha(s)(u_1, u_2)$ for $g \in G$ and $s \in S_\alpha$. This is well-defined because b_α is G_{x_α}- invariant. Then $b = \sum f_\alpha b_\alpha$ is a G-invariant metric on M. ∎

As a consequence of Proposition 5.2.4 and Theorem 5.2.7 we see that a finite dimensional G-manifold M is proper if and only if there exists a Riemannian metric on M such that G is a closed subgroup of $\mathrm{Iso}(M)$.

5.3. Coxeter groups

In this sections we will review some of the standard results concerning Coxeter groups. For details see [BG] and [Bu].

Coxeter groups can be defined either algebraically, in terms of generators and relations, or else geometrically. We will use the geometric definition. In the following we will use the term hyperplane to mean a translate ℓ of a linear subspace of codimension one in some $\boldsymbol{R}^k$, and we let R_ℓ denote the reflection in the hyperplane ℓ. Given a constant vector $v \in \boldsymbol{R}^k$, we let $T_v : \boldsymbol{R}^k \to \boldsymbol{R}^k$ denote the translation given by v, i.e., $T_v(x) = x + v$. Recall that any isometry φ of $\boldsymbol{R}^k$ is of the form $\varphi(x) = g(x) + v_0$ (i.e., the composition of T_{v_0} and g) for some $g \in \boldsymbol{O}(k)$ and $v_0 \in \boldsymbol{R}^k$.

5.3.1. Definition. Let $\{\ell_i \mid i \in I\}$ be a family of hyperplanes in $\boldsymbol{R}^k$. A subgroup W of Iso$(\boldsymbol{R}^k)$ generated by reflections $\{R_{\ell_i} | \ i \in I\}$ is a *Coxeter group* if the topology induced on W from Iso$(\boldsymbol{R}^n)$ is discrete and the W-action on $\boldsymbol{R}^k$ is proper. An infinite Coxeter group is also called an *affine Weyl group*.

5.3.2. Definition. Let W be a subgroup of Iso$(\boldsymbol{R}^k)$ generated by reflections. A hyperplane ℓ of $\boldsymbol{R}^k$ is called a *reflection hyperplane* of W if the reflection R_ℓ is an element of W. A unit normal vector to a reflection hyperplane of W is called a *root* of W.

5.3.3. Definition. A family $\mathfrak{H}$ of hyperplanes in $\boldsymbol{R}^k$ is *locally finite* if given any $x \in \boldsymbol{R}^k$ there exists a neighborhood U of x such that $\{\ell \mid \ell \cap U \neq \emptyset, \ \ell \in \mathfrak{H}\}$ is finite.

5.3.4. Definition. Let $\mathfrak{H} = \{\ell_i \mid i \in I\}$ be a family of hyperplanes in $\boldsymbol{R}^k$, and v_i a unit vector normal to ℓ_i. The rank of $\mathfrak{H}$ is defined to be the maximal number of independent vectors in $\{v_i \mid i \in I\}$. If W is the Coxeter group generated by $\{R_\ell \mid \ell \in \mathfrak{H}\}$, then the rank of W is defined to be the rank of $\mathfrak{H}$.

5.3.5. Proposition. *Suppose $\mathfrak{H}$ is a locally finite family of hyperplanes in $\boldsymbol{R}^k$ with rank $m < k$. Then there exists an m-dimensional plane E in $\boldsymbol{R}^k$ such that the subgroup of Iso$(\boldsymbol{R}^k)$ generated by $\{R_\ell \mid \ell \in \mathfrak{H}\}$ is isomorphic to the subgroup of Iso(E) generated by reflections of E in the hyperplanes $\{\ell \cap E | \ell \in \mathfrak{H}\}$, the isomorphism being given by $g \mapsto g|E$.*

Thus, without loss of generality, we may assume that a rank k Coxeter group is a subgroup of Iso$(\boldsymbol{R}^k)$.

5.3.6. Theorem [Te5]. *Let W the subgroup of Iso$(\boldsymbol{R}^k)$ generated by a set of reflections $\{R_i | \ i \in I\}$, and let $\mathfrak{H}$ denote the set of all reflection hyperplanes of W. Then W is a Coxeter group if and only if $\mathfrak{H}$ is locally finite.*

5.3.7. Corollary. *Let W be a subgroup of Iso$(\boldsymbol{R}^k)$ generated by a set of reflections $\{R_i | \ i \in I\}$, and $\mathfrak{H}$ the set of reflection hyperplanes of W. Suppose that $\mathfrak{H}$ is locally finite and* $\mathrm{rank}(\mathfrak{H}) = k$. *Then*

(i) W is a Coxeter group of rank k,

(ii) W permutes the hyperplanes in $\mathfrak{H}$,

(iii) if $\mathfrak{H}$ has finitely many hyperplanes, then W is a finite group and $\bigcap\{\ell | \ \ell \in \mathfrak{H}\} = \{x_o\}$ is a point,

(iv) if $\mathfrak{H}$ has infinitely many hyperplanes, then W is an infinite group.

5.3.8. Theorem. *Let W be a rank k Coxeter group on $\boldsymbol{R}^k$, and $\mathfrak{H}$ the set of reflection hyperplanes of W. Let U be a connected component of $\boldsymbol{R}^k \setminus \bigcup\{\ell_i \mid i \in I\}$, and $\bar{U}$ the closure of U. Then*

(i) $\bar{U}$ is a fundamental domain of W, i.e., each W-orbit meets $\bar{U}$ at exactly one point, and $\bar{U}$ is called a Weyl chamber of W,

(ii) $\bar{U}$ is a simplicial cone if W is finite, and $\bar{U}$ is a simplex if W is infinite.

5.3.9. Theorem. *Let W be a rank k finite Coxeter group on $\boldsymbol{R}^k$, and $\bar{U}$ a Weyl chamber of W. Then there are k reflection hyperplanes $\ell_1, \ldots, \ell_k$ of W such that*

(i) $\bigcap\{\ell_i \mid 1 \le i \le k\} = \{x_0\}$, is one point,

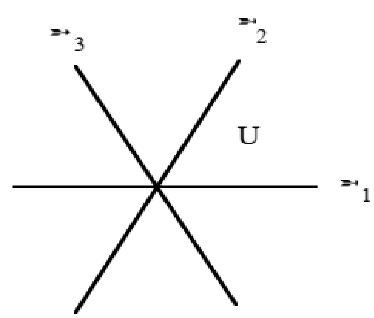
(ii) the boundary of $\bar{U}$ is contained in $\bigcup\{\ell_i \mid 1 \le i \le k\}$,

(iii) W is generated by reflections $\{R_{\ell_i} \mid 1 \le i \le k\}$,

(iv) there exist unit vectors v_i normal to ℓ_i such that

$$\bar{U} = \{x \in \boldsymbol{R}^k \mid \langle x, v_i \rangle \ \ge 0 \text{ for all } 1 \le i \le k\},$$

and $\{v_1, \ldots, v_k\}$ is called a simple root system of W.



5.3.10. Theorem. *Let W be a rank k infinite Coxeter group on $\boldsymbol{R}^k$, and $\bar{U}$ a Weyl chamber of W. Then there are $k+1$ reflection hyperplanes $\ell_1, \ldots, \ell_{k+1}$ of W such that*

(i) $\bigcap\{\ell_i \mid 1 \le i \le k+1\} = \emptyset$,

(ii) the boundary of $\bar{U}$ is contained in $\bigcup\{\ell_i \mid 1 \le i \le k+1\}$,

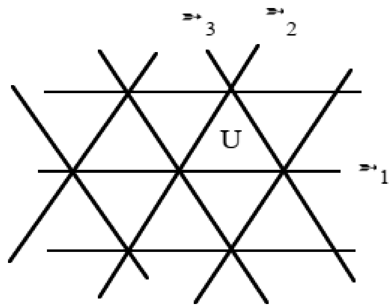
(iii) W is generated by reflections $\{R_{\ell_i} \mid 1 \le i \le k+1\}$,

(iv) there exist unit vectors v_i normal to ℓ_i such that

$$\bar{U} = \{x \in \boldsymbol{R}^k \mid \langle x, v_i \rangle \ \ge 0 \text{ for all } 1 \le i \le k+1\},$$

and $\{v_1, \ldots, v_{k+1}\}$ *is called a simple root system for* W,

(v) if $Q = \{v \in \boldsymbol{R}^k \mid T_v \in W\}$ *is the subgroup of translations in* W, *then* W *is the semi-direct product of* W_p *and* Q, *where* p *is a vertex of* $\bar{U}$ *and* W_p *is the isotropy subgroup of* W *at* p.



5.3.11. Definition. A rank k Coxeter group W on $\boldsymbol{R}^k$ is called *crystallographic*, if there is a rank k integer lattice Γ which is invariant under W. A finite crystallographic group is also called a *Weyl group*.

5.3.12. Theorem. *Let* W *be a Coxeter group generated by reflections in affine hyperplanes* $\{\ell_i \mid i \in I_0\}$. *Then* W *is crystallographic if and only if the angles between any* ℓ_i *and* ℓ_j *is* π/p, *for some* $p \in \{1, 2, 3, 4, 6\}$, *or equivalently, if and only if the order* m_{ij} *of* $R_{\ell_i} \circ R_{\ell_j}$, $i \neq j$ *is either infinite or is equal to* $2, 3, 4$ *or* 6.

Note that if, for $i = 1, 2$, W_i is a Coxeter group on $\boldsymbol{R}^{k_i}$, then $W_1 \times W_2$ is a Coxeter group on $\boldsymbol{R}^{k_1+k_2}$.

5.3.13. Definition. A Coxeter group W on $\boldsymbol{R}^k$ is *irreducible* if it cannot be written as a product two Coxeter groups.

5.3.14. Theorem. *Every Coxeter group can be written as the direct product of finitely many irreducible Coxeter groups.*

Let W be a finite Coxeter group of rank k, $\{v_1, \ldots, v_k\}$ a system of simple roots, R_i the reflection along v_i, and m_{ij} the order of $R_i \circ R_j$. The *Coxeter graph* associated to W is a graph with k vertices with the i^{th} and j^{th} vertices joined by a line (called a branch) with a mark m_{ij} if $m_{ij} > 2$ and is not joined by a branch if $m_{ij} = 2$. As a matter of convenience we shall usually suppress the label on any branch for which $m_{ij} = 3$. The Dynkin diagram is a Coxeter graph with the further restriction that $m_{ij} = 2, 3, 4, 6$ or ∞, in which branches marked with 4 are replaced by double branches and branches marked with 6 are replaced by triple branches. Similarly, we associate to an infinite Coxeter group of rank k a graph of $k + 1$ vertices.

5.3.15. Theorem.

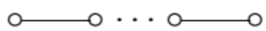
(1) A Coxeter group is irreducible if and only if its Coxeter graph is connected.

(2) If the Coxeter graph of W_1 and W_2 are the same then W_1 is isomorphic to W_2.

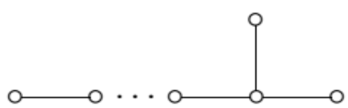
(3) If W is isomorphic to the product of irreducible Coxeter groups $W_1 \times \ldots \times W_r$ and D_i is the Coxeter graph for W_i, then the Coxeter graph of W is the disjoint union of $D_1, \ldots, D_r$.

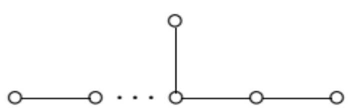
Therefore the classification of Coxeter graphs gives the classifications of Coxeter groups.

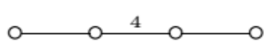
5.3.16. Theorem. *If W is an irreducible finite Coxeter group of rank k, then its Coxeter graph must be one of the following:*

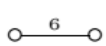
A_k 

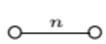
B_k 

D_k 

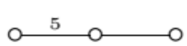
E_k  k=6,7,8

F_4 

G_2 

H_2^n 

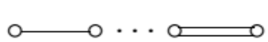
(n=5 or n>6)

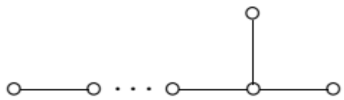
I_3 

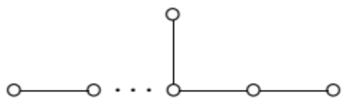
I_4 

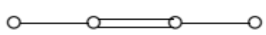
5.3.17. Corollary. *If W is a rank k finite Weyl group, then its Dynkin diagram must be one of the following:*

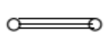
A_k 

B_k 

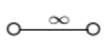
D_k 

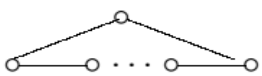
E_k  k=6,7,8

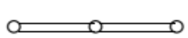
F_4 

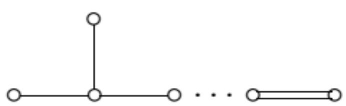
G_2 

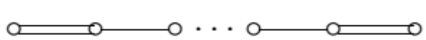
5.3.18. Theorem. *If W is an irreducible infinite Coxeter group of rank k, then its Dynkin diagram must be one of the following:*

$\tilde{A}_1$ 

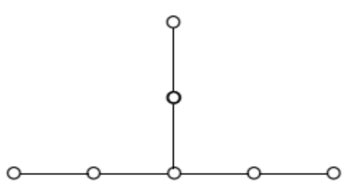
$\tilde{A}_k$ 

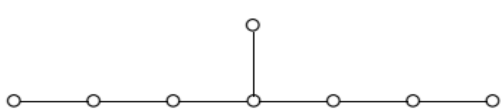
$\tilde{B}_2$ 

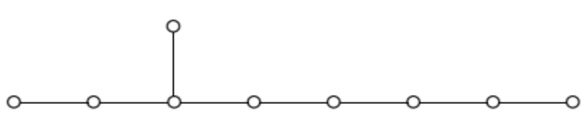
$\tilde{B}_k$ 

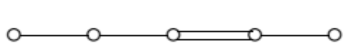
$\tilde{C}_k$ 

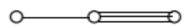
$\tilde{D}_k$ 

$\tilde{E}_6$ 

$\tilde{E}_7$ 

$\tilde{E}_8$ 

$\tilde{F}_4$ 

$\tilde{G}_2$ 

5.3.19. Chevalley Theorem. *Let W be a finite Coxeter group of rank k on $\boldsymbol{R}^k$. Then there exist k W-invariant polynomials $u_1, \ldots, u_k$ such that the ring of W-invariant polynomials on $\boldsymbol{R}^k$ is the polynomial ring $R[u_1, \ldots, u_k]$.*

Exercises.

1. Classify rank 1 and 2 Coxeter groups directly by analytic geometry and standard group theory.
2. Suppose W is a rank 3 finite Coxeter group on $\boldsymbol{R}^3$.
   (i) Show that W leaves $\boldsymbol{S}^2$ invariant,
   (ii) Describe the fundamental domain of W on $\boldsymbol{S}^2$ for $W = A_3,\ B_3$.

5.4. Riemannian G-manifolds

Let M be a Riemannian Hilbert manifold. Recall that a smooth curve α is a geodesic if $\nabla_{\alpha'}\alpha' = 0$. Let $\exp_p : TM_p \to M$ denote the exponential map at p. That is, $\exp_p(v) = \alpha(1)$, where α is the unique geodesic with $\alpha(0) = p$ and $\alpha'(0) = v$. Then $\exp_p(0) = p$ and $d(\exp_p)_0 = id$. It follows that for $r > 0$ sufficiently small the restriction φ of $\exp_p$ to the ball $B_r(0)$ of radius r about the origin of TM_p is a diffeomorphism of $B_r(0)$ onto a neighborhood of p in M. Then φ is called a geodesic coordinate system for M at p. The supremum of all such r is called the injectivity radius of M at p. If $\varphi : M \to M$ is an isometry and σ is a geodesic, then $\varphi(\sigma)$ is also a geodesic. In particular we have

5.4.1. Proposition. *If M is a Riemannian Hilbert manifold and $\varphi : M \to M$ is an isometry, then*

$$\varphi(\exp_p(tv)) = \exp_{\varphi(p)}(t\, d\varphi_p(v)),$$

for $p \in M$, and $v \in TM_p$. In particular, if $\varphi(p_0) = p_0$ then in geodesic coordinates near p_0, φ is linear.

5.4.2. Corollary. *If M is a Riemannian Hilbert manifold and $\varphi : M \to M$ is an isometry, then the fixed point set of φ:*

$$F = \{x \in M \mid \varphi(x) = x\}$$

is a totally geodesic submanifold of M.

PROOF. This follows from the fact that TF_x is the eigenspace of the linear map $d\varphi_x$ with respect to the eigenvalue 1. ∎

In section 5.2 we used the existence of slices for PF G-manifolds to prove the existence of G-invariant metrics. We will now see that conversely the existence of slices for PF Riemannian actions is easy.

Let N be an embedded closed submanifold of a Riemannian manifold M. For $r > 0$ we let $S_r(x) = \{\exp_x(u) | \ x \in N, \ u \in \nu(N)_x, \ \|u\| < r\}$, and $\nu_r(N) = \{u \in \nu(N)_x \mid x \in N, \ \|u\| < r\}$. If exp maps $\nu_r(N)$ diffeomorphically onto the open subset $U_r = \exp(\nu_r(N))$, then U_r is called a *tubular neighborhood* of N. Suppose M is a PF Riemannian G-manifold and $N = Gp$. Then there exists an $r > 0$ such that $\exp_p$ is diffeomorphic on the r-ball B_r of TM_p and $\exp_p(B_r) \cap N$ has only one component (or, equivalently, $d_M(p, N \setminus \exp_p(B_r)) \geq r$). Then $U_{r/2}$ is a tubular neighborhood of $N = Gp$ in M.

5.4.3. Proposition. *Let M be a Riemannian PF G-manifold. Let $r > 0$ be small enough that $U_r = \exp(\nu_r(Gx))$ is a tubular neighborhood of Gx in M. Let S_x denote $\exp_x(\nu_r(Gx)_x)$. Then*
(1) $S_{gx} = gS_x$,
(2) S_x is a slice at x, which will be called the normal slice at x.

PROOF. (1) is a consequence of Proposition 5.4.1. Since $\nu_r(Gx)$ is a tubular neighborhood, S_x and S_y are disjoint if $x \neq y$. So if $gS_x \cap S_x \neq \emptyset$, then $S_{gx} = S_x$ and $gx = x$.

Let M be a G-manifold. The differential of the action G_x defines a linear representation ι of G_x on TM_x called the *isotropy representation* at x. Now suppose that M is a Riemannian G-manifold. Then ι is an orthogonal representation, and the tangent space $T(Gx)_x$ to the orbit of x is an invariant linear subspace. So the orthogonal complement $\nu(Gx)_x$, i.e., the normal plane of Gx in M at x, is also an invariant linear subspace, and the restriction of the isotropy representation of G_x to $\nu(Gx)_x$ is called the *slice representation* at x.

5.4.4. Example. Let $M = G$ be a compact Lie group with a bi-invariant metric. Let $G \times G$ act on G by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Then M is a Riemannian $G \times G$-manifold (in fact a symmetric space), G_e is the diagonal subgroup $\{(g, g) | \ g \in G\}$, and the isotropy representation of $G_e \simeq G$ on $TG_e = \mathcal{G}$ is just the adjoint action as in Example 5.1.4 (3).

5.4.5. Example. Let $M = G/K$ be a compact symmetric space, and $\mathcal{G} = \mathcal{K} + \mathcal{P}$ is the orthogonal decomposition with respect to $-b$, where b is the Killing form on $\mathcal{G}$. Then $TM_{eK} = \mathcal{P}$ and $G_{eK} = K$. Let ad denote the adjoint representation of G on $\mathcal{G}$. Then $ad(K)(\mathcal{P}) \subseteq \mathcal{P}$. So it gives a representation of K on $\mathcal{P}$, which is the isotropy representation of M at eK. For example, $M = (G \times G)/G$ gives Example 5.4.4.

5.4.6. Remark. The set of all isotropy representations for non-compact symmetric spaces is the same as the set of all isotropy representations for compact symmetric spaces.

5.4.7. Proposition. *Let M be a Riemannian PF G-manifold, and $x \in M$. Then Gx is a principal orbit if and only if the slice representation at x is trivial.*

PROOF. Let S denote the normal slice at x. Then $G_y \subseteq G_x$ for all $y \in S$. So Gx is a principal orbit if and only $G_y = G_x$ for all $y \in S$, i.e., G_x fixes S. Then the result follows from Proposition 5.4.1. ∎

5.4.8. Corollary. *Let M be a Riemannian G-manifold, x a regular point, and S_x the normal slice at x as in Proposition 5.4.3. Then $G_y = G_x$ for all $y \in S_x$.*

5.4.9. Corollary. *Let M be a Riemannian G-manifold, Gx a principal orbit, and $v \in \nu(Gx)_x$. Then $\hat{v}(gx) = dg_x(v)$ is a well-defined smooth normal vector field of Gx in M.*

PROOF. If $gx = hx$, then $g^{-1}h \in G_x$. By Proposition 5.4.7, $d(g^{-1}h)_x(v)$ v, which implies that $dg_x(v) = dh_x(v)$. ∎

5.4.10. Definition. Let M be a Riemannian G-manifold, and N an orbit of M. A section u of $\nu(N)$ is called an *equivariant normal field* if $dg_x(u(x)) = u(gx)$ for all $g \in G$ and $x \in N$.

5.4.11. Corollary. *Let M be a Riemannian G-manifold, Gx a principal orbit, and $\{v_\alpha\}$ an orthonormal basis for $\nu(Gx)_x$. Let $\hat{v}_\alpha$ be the equivariant normal field defined by v_α as in Corollary 5.4.9. Then $\{\hat{v}_\alpha\}$ is a global smooth orthonormal frame field on Gx. In particular, the normal bundle of Gx in M is trivial.*

5.4.12. Proposition. *Let M be a Riemannian G-manifold, N an orbit in M, and v an equivariant normal field on N. Then*

(1) $A_{v(gx)} = dg_x \circ A_{v(x)} \circ dg_x^{-1}$ for all $x \in N$, where A_v is the shape operator of N with respect to the normal vector v,

(2) the principal curvatures of N along v are constant,

(3) $\{\exp(v(x)) | \ x \in N\}$ is again a G-orbit.

PROOF. Since $dg_x(TN_x) = TN_{gx}$ and g is an isometry, (1) follows. (2) is a consequence of (1). Since $v(gx) = dg_x(v(x))$, (3) follows from Proposition 5.4.1. ∎

5.4.13. Corollary. *Let $N^n(c)$ be the simply connected space form with constant sectional curvature c, G a subgroup of Iso$(N^n(c))$, M a G-orbit, and v an equivariant normal field on M. Then $\{Y(v(x)) \mid x \in M\}$ is again a G-orbit, where Y is the endpoint map of M in $N^n(c)$.*

We will now consider the orbit types of PF actions.

5.4.14. Proposition. *If M is a PF G-manifold, then there exists a principal orbit type.*

PROOF. By Remark 5.2.3 all of the isotropy subgroups of M are compact. It follows that there exists an isotropy subgroup, G_x, having minimal dimension and, for that dimension, the smallest number of components. By Theorem 5.2.6, there exists a slice S at x. Then GS is an open subset, and $G_s \subseteq G_x$ for all $s \in S$. By the choice of x it follows that in fact $G_s = G_x$ for all $s \in S$. But then $G_{gs} = gG_sg^{-1} = gG_xg^{-1}$, so (G_x) is a principal orbit type. ∎

5.4.15. Theorem. *If M is a PF G-manifold, then the set M_r of regular points is open and dense.*

PROOF. Openness follows from the existence of slice. To prove denseness, we proceed as follows: Let U be an open subset of M, $x \in U$, and S a slice at x. Choose $y \in GS \cap U$ so that G_y has smallest dimension and, for that dimension, the smallest number of components. Let S_0 be a slice at y, and $z \in GS_0 \cap U \cap GS$. It follows form Corollary 5.1.13 (2) that there exists $g \in G$ such that $G_z \subseteq gG_yg^{-1}$. Since the dimension of G_y is less than or equal to the dimension of G_z, we conclude that G_z and G_y in fact have the same dimension, and then since the number of components of G_y is less than or equal to the number of components of G_z, $G_z = gG_yg^{-1}$. This proves that Gy is a principal orbit. ∎

5.4.16. Theorem. *If M is a PF G-manifold then given a point $p \in M$ there exists a G-invariant open neighborhood U containing p such that U has only finitely many G-orbit types.*

PROOF. By Theorem 5.2.7 we may assume that M is a PF Riemannian G-manifold. Let S be the normal slice at p. Then S is of finite dimension, and G_p is a compact group acting isometrically on S so, by 5.1.13(4), it will suffice to prove this theorem for Riemannian G-manifolds of finite dimension n. We prove this by induction. For $n = 0$ the theorem is trivial. Suppose it is true for all proper G-manifolds of dimension less than n and let M be a proper Riemannian G-manifold of dimension n, $p \in M$, and S the normal slice at p. By 5.1.13(4) again, it will suffice to prove that locally S has only finitely many orbit types. If $\dim(S) < n$, then this follows from the induction hypothesis, so assume that $\dim(S) = n$. Then by Proposition 5.4.1 the G_p-action ρ on S is an orthogonal action on $TM_p = \boldsymbol{R}^n$ with respect to geodesic coordinates. Now G_p leaves $\boldsymbol{S}^{n-1}$ invariant and, by the induction hypothesis, locally $\boldsymbol{S}^{n-1}$ has only finitely many orbit types. But then because $\boldsymbol{S}^{n-1}$ is compact, it has finitely many orbit types altogether. Now note that, in a linear representation, the isotropy group (and hence the type of an orbit) is constant on any line through the origin, except at the origin itself. So ρ has at most one more orbit type on S than on $\boldsymbol{S}^{n-1}$, and hence only finitely many orbit types. ∎

5.4.17. Theorem. *If M is a PF G-manifold, then the set $\tilde{M}_s = M_s/G$ of singular orbits does not locally disconnect the orbit space $\tilde{M} = M/G$.*

PROOF. Using the slice representation as in the previous theorem, it suffices to prove this theorem for linear orthogonal G-action on $\boldsymbol{R}^n$. We proceed by induction. If $n = 1$, then we may assume that $G = \boldsymbol{O}(1) = \boldsymbol{Z}_2$. It is easily seen that $\boldsymbol{R}/G$ is the half line $\{x | \ x \geq 0\}$ with 0 as the only singular orbit. So $\{0\}$ does not locally disconnect $\boldsymbol{R}/G$. Suppose $G \subseteq \boldsymbol{O}(n)$. Applying the induction hypothesis to the slice representation of $\boldsymbol{S}^{n-1}$, we conclude that the set of singular orbits of $\boldsymbol{S}^{n-1}$ does not locally disconnect $\boldsymbol{S}^{n-1}/G$. But $\boldsymbol{R}^n/G$ is the cone over $\boldsymbol{S}^{n-1}/G$. So the set of singular orbits of $\boldsymbol{R}^n$ does not locally disconnect $\boldsymbol{R}^n/G$. ∎

5.4.18. Corollary. *If M is a connected PF G-manifold, then*
(1) M/G is connected,
(2) M has a unique principal orbit type.

5.4.19. Corollary. *Suppose M is a connected PF G-manifold. Let $m = \inf\{\dim(G_x) \mid x \in M\}$, and k the smallest number of components of all the dimension m isotropy subgroups. Then an orbit Gx_0 is principal if and only if G_{x_0} has dimension m and k components.*

5.5. Riemannian submersions

A smooth map $\pi : E \to B$ is a *submersion* if B is a finite dimension manifold and the rank of $d\pi_x$ is equal to the dimension of B. Then $V = \ker(d\pi)$ is a smooth subbundle of TE called the tangent bundle along the fiber (or the vertical subbundle). In case E and B are Riemannian manifolds we define the horizontal subbundle $\mathcal{H}$ of TE to be the orthogonal complement $V^\perp$ of the vertical bundle.

5.5.1. Definition. Let E and B be Riemannian manifolds. A submersion $\pi : E \to B$ is called a *Riemannian submersion* if $d\pi_x$ maps $\mathcal{H}_x$ isometrically onto $TB_{\pi(x)}$ for all $x \in E$.

The theory of Riemannian submersions, first systematically studied by O'Neil [On], plays an important role in the study of isometric actions, as we will see in the following.

5.5.2. Remark. Let M be a PF Riemannian G-manifold. Suppose M has a single orbit type (H), and $H = G_x$. Then the orbit map $p : M \to \tilde{M}$ is a smooth fiber bundle. If S is a slice at x, then we get a local trivialization of p on the neighborhood GS of the orbit Gx using the diffeomorphism $G/H \times S \approx GS$ defined by $(gH, s) \mapsto gs$. There is a unique metric on $\tilde{M}$ such that p is a Riemannian submersion. To see this, we define the inner product on $T\tilde{M}_{p(x)}$ by requiring that $dp_x : \nu(Gx)_x \to T\tilde{M}_{p(x)}$ is an isometry. Since $dg_x(T(Gx)_x) = T(Gx)_{gx}$ and dg_x is an isometry, dg_x maps the inner product space $\nu(Gx)_x$ isometrically onto $\nu(Gx)_{gx}$. This shows that the metric on $\tilde{M}$ is well-defined, and it is easily seen to be smooth. Actually, in this case M is a smooth fiber bundle in a completely different but important way. First, it is clear that M is partitioned into the closed, totally geodesic submanifolds $F(gHg^{-1})$, where the latter denotes the fixed point set of the subgroup gHg^{-1}. Clearly $F(g_1Hg_1^{-1}) = F(g_2Hg_2^{-1})$ if and only if $g_1N(H) = g_2N(H)$, where $N(H)$ denotes the normalizer of H in G. Thus we get a smooth map $\Pi : M \to G/N(H)$ having the $F(gHg^{-1})$ as fibers. Note that $N(H)$ acts on $F(H)$, and it is easily seen that the fibration $\Pi : M \to G/N(H)$ is the bundle with fiber $F(H)$ associated to the principal $N(H)$-bundle $G \to G/N(H)$.

What is most important about this *second* realization of M as the total space of a differentiable fiber bundle is that it points the way to generalize the first when M has more than one orbit type. In this case let (H) be a fixed orbit type of M, say $H = G_x$. Then $F = F(H)$ is again a closed, totally geodesic submanifold of M, and $F^* = F^*(H) = \{x \in M \mid G_x = H\}$ is an open submanifold of F. Just as above, we see that $M_{(H)}$ is a smooth fiber bundle over $G/N(H)$ with fiber F^*. In particular each orbit type $M_{(H)}$ is a smooth G-invariant submanifold of M. But of course $M_{(H)}$ has a single orbit type, so as above its orbit space $\tilde{M}_{(H)}$ has a natural differentiable structure making the orbit map $p : M_{(H)} \to \tilde{M}_{(H)}$ a smooth fiber bundle, and a smooth Riemannian structure making p a Riemannian submersion. Now we have already seen that the decompositions of M and $\tilde{M}$ into the orbit types $M_{(H)}$ and $\tilde{M}_{(H)}$ are locally finite. In fact they have all the best properties one can hope for in such a situation. To be technical, they are stratifications of M and $\tilde{M}$ respectively and, by what we have just noted, the orbit map $p : M \to \tilde{M}$ is a stratified Riemannian submersion.

5.5.3. Definition. Let $\pi : E \to B$ be a Riemannian submersion, V the vertical subbundle, and $\mathcal{H}$ the horizontal subbundle. Then a vector field ξ on E is

(1) *vertical*, if $\xi(x)$ is in V_x for all $x \in E$,
(2) *horizontal*, if $\xi(x)$ is in $\mathcal{H}_x$ for all $x \in E$,
(3) *projectable*, if there exists a vector field η on B such that $d\pi(\xi) = \eta$,
(4) *basic*, if it is both horizontal and projectable.

5.5.4. Proposition. *Let $\pi : E \to B$ be a Riemannian submersion.*

(1) If τ is a smooth curve on B then given $p_0 \in \pi^{-1}(\tau(t_0))$ there exists a unique smooth curve $\tilde{\tau}$ on E such that $\tilde{\tau}'(t)$ is horizontal for all t, $\pi(\tilde{\tau}) = \tau$, and $\tilde{\tau}(t_0) = p_0$. $\tilde{\tau}$ is called the horizontal lifting of τ at p_0.

(2) If η is a vector field on B, then there exists a unique basic field $\tilde{\eta}$ on E such that $d\pi(\tilde{\eta}) = \eta$, which is called the horizontal lift of η. In fact, this gives a one to one correspondence between $C^\infty(TB)$ and the space of basic vector fields on E.

5.5.5. Proposition. *If X is vertical and Y is projectable then $[X, Y]$ is vertical.*

PROOF. This follows from the fact that

$$d\pi([X, Y]) = [d\pi(X), d\pi(Y)]. \quad \blacksquare$$

5.5.6. Proposition. *Let $\pi : E \to B$ be a Riemannian submersion, τ a geodesic in B, and $\tilde{\tau}$ its horizontal lifting in E. Let $L(\alpha)$ denote the arc length of the smooth curve α, and $E_b = \pi^{-1}(b)$. Then*

(1) $L(\tilde{\tau}) = L(\tau)$,

(2) $\tilde{\tau}$ perpendicular to the fiber $E_{\tau(t)}$ for all t,

(3) if τ is a minimizing geodesic joining p to q in B, then $L(\tilde{\tau}) = d(E_p, E_q)$, the distance between the fibers E_p and E_q,

(4) $\tilde{\tau}$ is a geodesic of E.

PROOF. (1) and (2) are obvious. (4) is a consequence of (3). It remains to prove (3). Suppose τ is a minimizing geodesic joining p and q in B. If α is a smooth curve in E joining a point in E_p to a point in E_q, then $\pi \circ \alpha$ is a curve on B joining p, q. So $L(\pi \circ \alpha) \geq L(\tau)$. Let $\alpha' = u + v$, where u is the horizontal component and v is the vertical component of α'. Since $\|d\pi(u)\| = \|u\|$ and $d\pi(v) = 0$, $\|d\pi(\alpha')\| \leq \|\alpha'\|$. So we have

$$L(\alpha) \geq L(\pi(\alpha)) \geq L(\tau) = L(\tilde{\tau}),$$

which implies that $\tilde{\tau}$ is a geodesic, $d(E_p, E_q) = L(\tau)$. ∎

5.5.7. Corollary. *Let $\pi : E \to B$ be a Riemannian submersion. If σ is a geodesic in E such that $\sigma'(t_0)$ is horizontal then $\sigma'(t)$ is horizontal for all t (or equivalently, if a geodesic σ of E is perpendicular to $E_{\sigma(t_0)}$ then it is perpendicular to all fibers $E_{\sigma(t)}$).*

PROOF. Let $p_0 = \sigma(t_0)$, τ the geodesic of B such that $\tau(t_0) = \pi(p_0)$ and $\tau'(t_0) = d\pi(\sigma'(t_0))$. Let $\tilde{\tau}$ be the horizontal lifting of τ at p_0. Then both σ and $\tilde{\tau}$ are geodesics of E passing through p_0 with the same tangent vector at p_0. So $\sigma = \tilde{\tau}$. ∎

5.5.8. Corollary. *Let $\pi : E \to B$ be a Riemannian submersion, and $\mathcal{H}$ the horizontal subbundle (or distribution).*

(1) If $\mathcal{H}$ is integrable then the leaves are totally geodesic.

(2) If $\mathcal{H}$ is integrable and S is a leaf of $\mathcal{H}$ then $\pi|S$ is a local isometry.

5.5.9. Remark. If $F = \pi^{-1}(b)$ is a fiber of π then $\mathcal{H}|F$ is just the normal bundle of F in E. There exists a canonical global parallelism on the normal bundle $\nu(F)$: a section $\tilde{v}$ of $\nu(F)$ is called *π-parallel* if $d\pi(\tilde{v}(x))$ is a fixed vector $v \in TB_b$ independent of x in F. Clearly $\tilde{v} \mapsto v$ is a bijective correspondence between π-parallel fields and TB_b. There is another parallelism on $\nu(F)$ given by the induced normal connection ∇^ν as in the submanifold geometry, i.e., a normal field ξ is parallel if $\nabla^\nu \xi = 0$. It is important to note that in general the π-parallelism in $\nu(F)$ is *not* the same as the parallel translation defined by the normal connection ∇^ν. (The latter is in general not flat, while the former is always both flat and without holonomy.) Nevertheless we shall see later that if $\mathcal{H}$ is integrable then these two parallelisms *do* coincide.

5.5.10. Remark. Let M be a Riemannian G-manifold, (H) the principal orbit type, and $\pi : M_{(H)} \to \tilde{M}_{(H)}$ the Riemannian submersion given by the orbit map. Then a normal field ξ of a principal orbit Gx is G-equivariant if and only if ξ is π-parallel.

5.5.11. Definition. A Riemannian submersion $\pi : E \to B$ is called *integrable* if the horizontal distribution $\mathcal{H}$ is integrable.

We will first discuss the local theory of Riemannian submersions. Let $\pi : E \to B$ be a Riemannian submersion. Then there is a local orthonormal frame field e_A on E such that $e_1, \ldots, e_n$ are vertical and $e_{n+1}, \ldots, e_{n+k}$ are basic. Then $\{e^*_\alpha = d\pi(e_\alpha)\}$ is a local orthonormal frame field on B. We use the same index convention as in section 2.1, i.e.,

$$1 \le i, j, k \le n, \ n+1 \le \alpha, \beta, \gamma \le n+k, \ 1 \le A, B, C \le n+k.$$

Let ω_A and ω^*_α be the dual coframe, and ω_{AB}, $\omega^*_{\alpha\beta}$ the Levi-Civita connections on E and B respectively. Then $\pi^*(\omega^*_\alpha) = \omega_\alpha$. Assume that

$$\omega_{i\alpha} = \sum_\beta a_{i\alpha\beta}\omega_\beta + \sum_j r_{i\alpha j}\omega_j \ , \tag{5.5.1}$$

$$\omega_{\alpha\beta} = \pi^*(\omega^*_{\alpha\beta}) + \sum_i b_{\alpha\beta i}\omega_i \ . \tag{5.5.2}$$

Note that

$$\begin{aligned} d\omega_\alpha &= d(\pi^*\omega^*_\alpha) = \pi^*(d\omega^*_\alpha) \\ &= \pi^*\left(\sum_\beta \omega^*_{\alpha\beta} \wedge \omega^*_\beta\right) \\ &= \sum_\beta \pi^*(\omega^*_{\alpha\beta}) \wedge \omega_\beta, \end{aligned}$$

which does not have $\omega_i \wedge \omega_\beta$ and $\omega_i \wedge \omega_j$ terms. But the structure equation gives

$$d\omega_\alpha = \sum_j \omega_{\alpha\beta} \wedge \omega_\beta + \sum_i \omega_{\alpha i} \wedge \omega_i. \tag{5.5.3}$$

So the coefficients of $\omega_i \wedge \omega_\beta$ and $\omega_i\omega_j$ in (5.5.3) are zero, i.e.,

$$b_{\alpha\beta i} = a_{i\alpha\beta}, \tag{5.5.4}$$

$$r_{i\alpha j} = r_{j\alpha i}. \tag{5.5.5}$$

Note that the restriction of $\omega_{i\alpha}$ and $\omega_{\alpha\beta}$ to the fiber F are the second fundamental forms and the normal connection of F in E. In fact, $\sum r_{i\alpha j}\omega_i \otimes \omega_j \otimes e_\alpha$ is the second fundamental form of F and $\omega_{\alpha\beta} = \sum_i b_{\alpha\beta i}\omega_i = \sum_i a_{i\alpha\beta}\omega_i$ is the induced normal connection of the normal bundle $\nu(F)$ in E.

Next we describe in our notation the two fundamental tensors A and T associated to Riemannian submersions by O'Neil in [On]. Let u^h and u^v denote the horizontal and vertical components of $u \in TE_p$. Then it is easy to check that

$$T(X,Y) = (\nabla_{X^v} Y^v)^h + (\nabla_{X^v} Y^h)^v,$$

$$A(X,Y) = (\nabla_{X^h} Y^h)^v + (\nabla_{X^h} Y^v)^h,$$

define two tensor fields on E. Using (5.5.1) and (5.5.2), these two tensors are

$$T = \sum r_{j\alpha i}(\omega_i \otimes \omega_j \otimes e_\alpha - \omega_i \otimes \omega_\alpha \otimes e_j),$$

$$A = \sum a_{j\beta\alpha}(\omega_\alpha \otimes \omega_i \otimes e_\beta - \omega_\alpha \otimes \omega_\beta \otimes e_i).$$

If $\mathcal{H}$ is integrable then, by Corollary 5.5.8, each leaf S of $\mathcal{H}$ is totally geodesic and $e_\alpha|S$ is a local frame field on S. Thus the second fundamental form on S is zero, i.e., $\nabla_{e_\alpha} e_j$ is vertical, or $a_{i\alpha\beta} = 0$. Note that $e_i|F$ form a tangent frame field for the fiber F, and $e_\alpha|F$ is a normal vector field of F. By Proposition 5.5.5, $[e_j, e_\alpha] = \nabla_{e_j} e_\alpha - \nabla_{e_\alpha} e_j$ is vertical, so we have $\nabla_{e_j} e_\alpha$ is vertical, i.e., $e_\alpha|F$ is parallel with respect to the induced normal connection of F in E.

Conversely, suppose $e_\alpha|F$ is parallel for every fiber F of π, i.e., $\nabla_{e_i} e_\alpha$ is vertical, or $\omega_{\alpha\beta}(e_i) = 0$. By (5.5.1) (5.5.2) and (5.5.4), we have

$$0 = \omega_{\alpha\beta}(e_i) = b_{\alpha\beta i} = a_{i\alpha\beta} = \omega_{i\alpha}(e_\beta).$$

The torsion equation implies that

$$[e_\alpha, e_\beta] = \nabla_{e_\alpha} e_\beta - \nabla_{e_\beta} e_\alpha = \sum (\omega_{\beta A}(e_\alpha) - \omega_{\alpha A}(e_\beta)) e_A.$$

Hence $[e_\alpha, e_\beta]$ is horizontal, i.e., $\mathcal{H}$ is integrable. So we have proved:

5.5.12. Theorem. *Let $\pi : E \to B$ be a Riemannian submersion. Then the following statements are equivalent:*

(i) π is integrable,

(ii) every π-parallel normal field on the fiber $F = \pi^{-1}(b)$ is parallel with respect to the induced normal connection of F in E,

(iii) the O'Neil tensor A is zero.

5.6. Sections

Henceforth M will denote a connected, complete Riemannian G-manifold, and M_r is the set of regular points of M. As noted above, we have a Riemannian submersion $\pi : M_r \to \tilde{M}_r$. We assume all the previous notational conventions. In particular we identify the Lie algebra $\mathcal{G}$ of G with the Killing fields on M generating the action of G.

5.6.1. Proposition. *If $\xi \in \mathcal{G}$ and σ is a geodesic on M, then the quantity $\langle \sigma'(t), \xi(\sigma(t)) \rangle$ is a constant independent of t.*

PROOF. If ξ is a Killing field and $\nabla \xi = \sum \xi_{ij} e_i \otimes \omega_j$, then $\xi_{ij} + \xi_{ji} = 0$. So $\langle \nabla_{\sigma'} \xi, \sigma' \rangle = 0$. Since σ is a geodesic, $\nabla_{\sigma'} \sigma' = 0$, which implies that

$$\frac{d}{dt} \langle \xi(\sigma), \sigma' \rangle = \langle \nabla_{\sigma'} \xi, \sigma' \rangle + \langle \xi(\sigma), \nabla_{\sigma'} \sigma' \rangle = 0. \quad \blacksquare$$

It will be convenient to introduce for each regular point x the set $\mathcal{T}(x)$, defined as the image of $\nu(Gx)_x$ under the exponential map of M, and also $\mathcal{T}_r(x) = \mathcal{T}(x) \cap M_r$ for the set of regular points of $\mathcal{T}(x)$. Note that $\mathcal{T}(x)$ may have singularities.

5.6.2. Proposition. *For each regular point x of M:*
(1) $g\mathcal{T}(x) = \mathcal{T}(gx)$ and $g\mathcal{T}_r(x) = \mathcal{T}_r(gx)$,
(2) for $v \in \nu(Gx)_x$ the geodesic $\sigma(t) = \exp_x(tv)$ is orthogonal to each orbit it meets,
(3) if G is compact then $\mathcal{T}(x)$ meets every orbit of M.

PROOF. (1) follows from Proposition 5.4.1, and (2) follows from Proposition 5.6.1. Finally suppose G is compact and given any $y \in M$, since Gy is compact, we can choose $g \in G$ so that gy minimizes the distance from x to Gy. Let $\sigma(t) = \exp(tv_0)$ be a minimizing geodesic from $x = \sigma(0)$ to $gy = \sigma(s)$. Then σ is perpendicular to Gy. By (2), σ is also orthogonal to Gx. In particular $v_0 = \sigma'(0) \in \nu(Gx)_x$ so the arbitrary orbit Gy meets $\mathcal{T}(x) = \exp(\nu(Gx)_x)$ at $\exp(sv_0) = gy$. $\blacksquare$

Let x be a regular point and S a normal slice at x. If S is orthogonal to each orbit it meets then so is gS. This implies that the Riemannian submersion $\pi : M_r \to \tilde{M}_r$ is integrable. Since for most Riemannian G-manifold M the submersion $\pi : M_r \to \tilde{M}_r$ is not integrable, a normal slice is in general *not* orthogonal to each orbit it meets.

5.6.3. Example. Let $\boldsymbol{S}^1$ act on $\boldsymbol{R}^2 \times \boldsymbol{R}^2$ by $e^{it}(z_1, z_2) = (e^{it} z_1, e^{it} z_2)$. Then $p = (1, 0)$ is a regular point. It is easy to check that $y = (1, 1) \in \mathcal{T}(p)$ and $\mathcal{T}(p)$ is not orthogonal to the orbit Sy.

5.6.4. Definition. A connected, closed, regularly embedded smooth submanifold Σ of M is called a *section* for M if it meets all orbits orthogonally.

The conditions on Σ are, more precisely, that $G\Sigma = M$ and that for each $x \in \Sigma, T\Sigma_x \subseteq \nu(Gx)_x$. But since $T(Gx)_x$ is just the set of $\xi(x)$ where $\xi \in \mathcal{G}$, this second condition has the more explicit form

(*) For each $x \in \Sigma$ and $\xi \in \mathcal{G}$, $\xi(x)$ is orthogonal to $T\Sigma_x$.

In the following we will discuss some basic properties for G-manifolds that admit sections. For more detail, we refer the reader to [PT2].

It is trivial that if Σ is a section for M then so is $g\Sigma$ for each $g \in G$. Since $G\Sigma = M$, it follows that if one section Σ exists then in fact there is a section through each point of M, and we shall say that M *admits sections*.

5.6.5. Example. All the examples in 5.1.4 admit sections. In fact, for (1), $\{ru \mid r \in R\}$ is a section, where u is any unit vector in $\boldsymbol{R}^n$; for (2) a maximal torus is a section; for (3) a maximal abelian (Cartan) subalgebra is a section; for (4) and (5), the space of all trace zero real diagonal matrices is a section.

5.6.6. Definition. The *principal horizontal distribution* of a Riemannian G-manifold M is the horizontal distribution of the Riemannian submersion on the principal stratum $\pi : M_r \to \tilde{M}_r$.

If Σ is a section of M then the set $\Sigma_r = \Sigma \cap M_r$ of regular points of Σ is an integral submanifold of the principal horizontal distribution $\mathcal{H}$ of the G-action. Since $\tilde{M}_r$ is always connected, it follows from Corollary 5.5.8, Remark 5.5.10 and Theorem 5.5.12 that we have:

5.6.7. Theorem. *If M admits sections, and Σ is a section, then*

(1) the principal horizontal distribution $\mathcal{H}$ is integrable;

(2) each connected component of $\Sigma_r = \Sigma \cap M_r$ is a leaf of $\mathcal{H}$;

(3) if F is the leaf of $\mathcal{H}$ through a regular point x then $\pi|F$ is a covering isometry onto $\tilde{M}_r$;

(4) Σ is totally geodesic;

(5) there is a unique section through each regular point x of M, namely $\mathcal{T}(x) = \exp(\nu(Gx)_x)$.

(6) an equivariant normal field on a principal orbit is parallel with respect to the induced normal connection.

5.6.8. Remark. One might naively hope that, conversely to Theorem 5.6.7(1), if $\mathcal{H}$ is integrable then M admits sections. To give a counterexample take $M = \boldsymbol{S}^1 \times \boldsymbol{S}^1$ and let $G = \boldsymbol{S}^1 \times \{e\}$ acting by translation. Let ξ denote the vector field on M generating the action of G and let η denote an element of the

Lie algebra of $\boldsymbol{S}^1 \times \boldsymbol{S}^1$ generating a nonclosed one parameter group γ. If we choose the invariant Riemannian structure for M making ξ and η orthonormal then a section for M would have to be a coset of γ, which is impossible since γ is not closed in M. This also gives a counter example to the weaker conjecture that if a compact G-manifold M has codimension 1 principal orbits then any normal geodesic to the principal orbit is a section. It *is* probably true that if $\mathcal{H}$ is integrable, then a leaf of $\mathcal{H}$ can be extended to be a complete immersed totally geodesic submanifold of M, which meets every orbit orthogonally. However we can prove this only in the real analytic case.

5.6.9. Proposition. *Suppose G is a compact Lie group, and M a Riemannian G-manifold. Let x_0 be a regular point of M, and $\mathcal{T} = \exp(\nu(Gx_0)_{x_0})$. If $\mathcal{H}$ is integrable and $\mathcal{T}$ is a closed properly embedded submanifold of M, then $\mathcal{T}$ is a section.*

PROOF. By Proposition 5.6.2(3), it suffices to show that $\mathcal{T}$ is orthogonal to Gx for all $x \in \mathcal{T}$. Let F denote the leaf of $\mathcal{H}$ through x_0. By Corollary 5.5.8, F is totally geodesic. So F is open in $\mathcal{T}$ and $\mathcal{T}$ is orthogonal to Gy for all $y \in F$. Now suppose $x \in \mathcal{T} \setminus F$. Since $\exp_x : T\mathcal{T}_x \to \mathcal{T}$ is regular almost everywhere, there is an open neighborhood U of the unit sphere of $T\mathcal{T}_x$ such that for all $v \in U$ there is an $r > 0$ such that $\sigma_v(r) = \exp_x(rv)$ is in F. Then by Proposition 5.6.2(2) $\sigma_v'(0) = v$ is normal to Gx. ∎

It is known that any connected totally geodesic submanifold of a simply connected, complete symmetric space can be extended uniquely to one that is complete and properly embedded (cf. [KN] Chapter 9, Theorem 4.3). So we have

5.6.10. Corollary. *Let $M = G/K$ be a simply connected complete symmetric space, and H a subgroup of G. Then the action of H on M admits sections if and only if the principal horizontal distribution of this action is integrable. In particular if the principal H-orbit is of codimension one then the H-action on M has a section.*

It follows from Theorem 5.5.12 that

5.6.11. Theorem. *The following statements are equivalent for a Riemannian G-manifold M:*

(1) the principal horizontal distribution $\mathcal{H}$ is integrable,

(2) every G-equivariant (i.e., π-parallel) normal vector field on a principal orbit is parallel with respect to the induced normal connection for the normal bundle $\nu(Gx)$ in M,

(3) for each regular point x of M, if S is the normal slice at x then for all

$\xi \in \mathcal{G}$ and $s \in S$, $\xi(s)$ is normal to S.

5.6.12. Proposition. *Let V be an orthogonal representation of G, x a regular point of V, and Σ the linear subspace of V orthogonal to the orbit Gx at x. Then the following are equivalent:*
(i) V admits sections,
(ii) Σ is a section for V,
(iii) for each v in Σ and ξ in $\mathcal{G}$, $\xi(v)$ is normal to Σ.

In the following, M is a Riemannian G-manifold that admits sections. Let x be a regular point of M, and Σ the section of M through x. Recall that a small enough neighborhood U of x in Σ is a slice at x and so intersects each orbit near Gx in a unique point. Also recall that G_x acts trivially on Σ.

In general given a closed subset S of M we let $N(S)$ denote the closed subgroup $\{g \in G | \; gS = S\}$ of G, the largest subgroup of G which induces an action on S, and we let $Z(S)$ denote the kernel of this induced action, i.e., $Z(S) = \{g \in G | \; gs = s, \; \forall \; s \in S\}$ is the intersection of the isotropy subgroups G_s, $s \in S$. Thus $N(S)/Z(S)$ is a Lie group acting effectively on S. In particular when S is a section Σ then we denote $N(\Sigma)/Z(\Sigma)$ by $W = W(\Sigma)$ and call it the *generalized Weyl group* of Σ.

5.6.13. Remark. If M is the compact Lie group G with the Adjoint action, then for a subgroup H of G, $N(H)$ and $Z(H)$ are respectively the normalizer and centralizer of H. If for H we take a maximal torus T of G (which is in fact a section of the Adjoint action) then $Z(T) = T$ and $W(T) = N(T)/T$ is the usual Weyl group of G.

5.6.14. Remark. Let x be a regular point, S a normal slice at x, and Σ a section at x. As remarked above $G_x \subseteq Z(S) \subseteq Z(\Sigma)$, and conversely from the definition of $Z(\Sigma)$ it follows that $Z(\Sigma) \subseteq G_x$, so $Z(\Sigma) = G_x$. Moreover if $g\Sigma = \Sigma$ then $g\Sigma$ is the section at the regular point gx. So $G_x = Z(\Sigma) = Z(g\Sigma) = G_{gx}$. Then it follows from $G_{gx} = gG_xg^{-1}$ that we have $N(\Sigma) \subseteq N(G_x)$ and $W(\Sigma) \subseteq N(G_x)/G_x$.

5.6.15. Proposition. *The generalized Weyl group $W(\Sigma)$ of a section Σ is a discrete group. Moreover if Σ' is a second section for M then $W(\Sigma')$ is isomorphic to $W(\Sigma)$ by an isomorphism which is well determined up to inner automorphism.*

Proof. Let Σ be the section and S the normal slice at the regular point x. Then S is an open subset of Σ. If $g \in N(\Sigma)$ is near the identity then $gx \in S$. Since S meets every orbit near x at a unique point, $gx = x$, i.e., $g \in G_x = Z(\Sigma)$, so $Z(\Sigma)$ is open in $N(\Sigma)$ and hence $W(\Sigma)$ is discrete. If Σ'

is a section then $\Sigma' = g_0\Sigma$ and so $g \mapsto g_0 g g_0^{-1}$ clearly induces an isomorphism of $W(\Sigma)$ onto $W(\Sigma')$. ∎

5.6.16. Example. The isotropy representation of the symmetric space $M = G/K$ at eK admits sections. In fact, let $\mathcal{G} = \mathcal{K} + \mathcal{P}$ be the orthogonal decomposition of the Lie algebra $\mathcal{G}$ of G as in Example 5.4.5 and $\mathfrak{A}$ a maximal abelian subalgebra in $\mathcal{P}$. Then $\mathfrak{A}$ is a section and the generalized Weyl group W is the standard Weyl group associated to the symmetric space G/K. These representations have the following remarkable properties:
(i) Given $p \in \mathcal{P}$, the slice representations of $\mathcal{P}$ again admits sections.
(ii) $\mathfrak{A}/W \simeq \mathcal{P}/K$.
(iii) Chevalley Restriction Theorem ([He],[Wa]): Let $\boldsymbol{R}[\mathcal{P}]^G$ be the algebra of G-invariant polynomials on $\mathcal{P}$, and $\boldsymbol{R}[\mathfrak{A}]^W$ the algebra of W-invariant polynomials on $\mathfrak{A}$. Then the restriction map $\boldsymbol{R}[\mathcal{P}]^G \to \boldsymbol{R}[\mathfrak{A}]^W$ defined by $f \mapsto f|\mathfrak{A}$ is an algebra isomorphism.

Following J. Dadok we shall say that an orthogonal representation space is *polar* if it admits sections. The following theorem of Dadok [Da] says that the isotropy representations of symmetric spaces are "essentially" the only polar representations.

5.6.17. Theorem. *Let $\rho : H \to \boldsymbol{O}(n)$ be a polar representation of a compact connected Lie group. Then there exists an n-dimensional symmetric space $M = G/K$ and a linear isometry $A : \boldsymbol{R}^n \to TM_{eK}$ mapping H-orbits onto K-orbits.*

5.6.18. Corollary. *If ρ is a finite dimensional polar representation, then the corresponding generalized Weyl group is a classical Weyl group.*

5.6.19. Definition. A G-manifold M is called *polar* if the G-action is proper, Fredholm, isometric, and admits sections.

5.6.20. Remark. The generalized Weyl group of a polar G-manifold is not a Weyl group in general. In fact we will now construct examples with an arbitrary finite group as the generalized Weyl group. Given any compact group G, a closed subgroup H of G, a finite subgroup W of $N(H)/H$, and a smooth manifold Σ such that W acts faithfully on Σ, we let $\pi : N(H) \to N(H)/H$ be the natural projection map, and $K = \pi^{-1}(W)$, so K acts naturally on Σ. Let

$$M = G \times_K \Sigma = \{(g, \sigma)|\ g \in G,\ \sigma \in \Sigma\}/\sim,$$

where the equivalence relation $\sim$ is defined by $(g, \sigma) \sim (gk^{-1}, k\sigma)$, and define the G-action on M by $\gamma(g, \sigma) = (\gamma g, \sigma)$. Now suppose ds^2 is a metric on M

such that $ds^2|\Sigma$ and $ds^2|\nu(\Sigma)$ are K-invariant. Then G acts on M isometrically with $e \times \Sigma$ as a section, (H) as the principal orbit type, and W as the generalized Weyl group.

Note that any finite group W can be embedded as a subgroup of some $\boldsymbol{SO}(n)$. Thus taking $G = \boldsymbol{SO}(n)$, $H = e$, and $\Sigma = \boldsymbol{S}^{n-1}$ in the the above construction gives a G-manifold admitting sections and having W as its generalized Weyl group. This makes it seem unlikely that there can be a good structure theory for polar actions in complete generality. Nevertheless, Dadok's theorem 5.6.17 gives a classification for the polar actions on $\boldsymbol{S}^n$, and it would be interesting to classify the polar actions for other special classes of Riemannian manifolds, say for arbitrary symmetric spaces.

Although a general structure theory for polar actions is unlikely, we will now see that the special properties in Example 5.6.16, for the isotropy representations of symmetric spaces, continue to hold for all polar actions.

5.6.21. Theorem. *If M is a polar G-manifold and $p \in M$, then the slice representation at p is also polar. In fact, if Σ is a section for M through p then $T\Sigma_p$ is a section and $W(\Sigma)_p = \{\varphi \in W(\Sigma) \mid \varphi(p) = p\}$ is the generalized Weyl group for the slice representation at p.*

PROOF. Let $V = \nu(Gp)_p$ be the space of the slice representation, and $V_0 = T\Sigma_p$. Then, by definition of a section, V_0 is a linear subspace of V. Suppose B is a small ball centered at the origin in V, $S = \exp_p(B)$ is a normal slice at p, and $x = \exp_p(v) \in S$. By Corollary 5.1.13, $G_x \subseteq G_p$ for all $x \in S$. So the isotropy subgroup of the linear G_p-action on V at x is G_x. From this follows the well-known fact that the G_p-orbit of x in V has the same codimension as the G-orbit of x in M. By Proposition 5.6.12 it suffices to show that for each $u \in V_0$ and ξ in the Lie algebra of G_p, $\langle \xi(u), v \rangle = 0$ for all $v \in V_0$. Let g^s be the one parameter subgroup on G_p generated by ξ, and $u(t) = \exp_p(tu)$. Choose $v(t) \in T\Sigma_{u(t)}$ such that as $t \to 0$ $v(t) \to v$ in $T\Sigma$. Since Σ is a section,

$$\langle \xi(u(t)), v(t) \rangle_{u(t)} = 0, \tag{5.6.1}$$

where $\langle\ ,\ \rangle_{u(t)}$ is the inner product on $TM_{u(t)}$. Note that the vector field ξ for the G_p-action on V is given by

$$\begin{aligned}\xi(v) &= \lim_{s \to 0} dg^s_p(u) \\ &= \lim_{s \to 0} \lim_{t \to 0} g^s(u(t)) \\ &= \lim_{t \to 0} \lim_{s \to 0} g^s(u(t)) = \lim_{t \to 0} \xi(u(t)).\end{aligned}$$

Letting $t \to 0$ in (5.6.1), we obtain $\langle \xi(u), v \rangle = 0$.

It remains to prove that $W(V_0) = W(\Sigma)_p$. To see this note that $N(V_0) = N(\Sigma) \cap G_p$ and $Z(V_0) = Z(\Sigma) \cap G_p = Z(\Sigma)$, so $W(V_0) \subseteq W(\Sigma)_p$. Conversely if $gZ(\Sigma) \in W(\Sigma)_p$, then $gp = p$, which implies that $W(\Sigma)_p \subseteq W(V_0)$. ∎

5.6.22. Corollary. *Let M be a polar G-manifold. If M has a fixed point then the generalized Weyl group of M is a Weyl group.*

5.6.23. Corollary. *If M is a polar G-manifold then for any $p \in M$, G_p acts transitively on the set of sections of M that contains p.*

PROOF. Let Σ_1 and Σ_2 be sections through p and let x be a regular point of Σ_1 near p. We may regard Σ_2 as a section for the slice representation at p, so it meets $G_p x$, i.e., there exists $g \in G_p$ such that $gx \in \Sigma_2$. Since $g\Sigma_1$ and Σ_2 are both sections of M containing the regular point gx they are equal by Theorem 5.6.7 (5). ∎

5.6.24. Corollary. *Let M be a polar G-manifold, Σ a section of M, and $W = W(\Sigma)$ its generalized Weyl group. Then for $x \in \Sigma$ we have $Gx \cap \Sigma = Wx$.*

PROOF. It is obvious that $Wx \subseteq Gx \cap \Sigma$. Conversely suppose $x' = gx \in \Sigma$. Then $g\Sigma$ is a section at x', so by Corollary 5.6.23 there is $\gamma \in G_{x'}$ such that $\gamma g \Sigma = \Sigma$. Thus $\gamma g \in N(\Sigma)$ so $x' = \gamma x' = \gamma g x$ is in $N(\Sigma)x = Wx$. ∎

For a K-manifold N, we let $C^0(N)^K$ and $C^\infty(N)^K$ denote the space of all continuous and smooth K-invariant functions on N. As a consequence of Corollary 5.6.24 we see that if M is a polar G-manifold with Σ as a section and W as its generalized Weyl group, then the restriction map r from $C^0(M)^G$ to $C^0(\Sigma)^W$ defined by $r(f) = f|\Sigma$ is an isomorphism. Moreover, it follows from Theorem 5.6.21, Corollary 5.6.18, and a theorem of G. Schwarz [Sh] (if G is a subgroup of $\boldsymbol{O}(n)$ then every smooth G-invariant function on $\boldsymbol{R}^n$ can be written as a smooth functions of invariant polynomials) that the Chevalley restriction theorem can be generalized to smooth invariant functions of a polar action, i.e.,

5.6.25. Theorem [PT2]. *Suppose M is a polar G-manifold, Σ is a section, and $W = W(\Sigma)$ is its generalized Weyl group. Then the restriction map $C^\infty(M)^G \to C^\infty(\Sigma)^W$ defined by $f \mapsto f|\Sigma$ is an isomorphism.*

5.7. Submanifold geometry of orbits

One important problem in the study of submanifolds of $N^p(c)$ is to determine submanifolds which have simple local invariants. The submanifolds with the simplest invariants are the totally umbillic submanifolds, and these have been completely classified (see section 2.2). Another interesting class consists of the compact submanifolds with parallel second fundamental forms. It is not surprising that the first examples of the latter arise from group theory. Ferus ([Fe]) noted that if M is an orbit of the isotropy representation of a symmetric space G/K and if M is itself a symmetric space with respect to the metric induced on it as a submanifold of the Euclidean space $T(G/K)_{eK}$, then the second fundamental form of M is parallel with respect to the induced normal connection (defined in section 2.1). Conversely, Ferus ([Fe]) showed that these are the only submanifolds of Euclidean spaces (or spheres) whose second fundamental forms are parallel. These results might lead one to think that orbits of isometric action on $\boldsymbol{S}^n$ may not be too difficult to characterize in terms of their local geometric invariants as submanifolds. But in fact, this turns out to be a rather complicated problem.

Let N be a Riemannian G-manifold, and $M = Gx_0$ a principal orbit in N. If v is a G-equivariant normal field on M, then by Proposition 5.4.12 (3), $M_v = \{\exp(v(x)) \mid x \in M\}$ is the G-orbit through $x = \exp_{x_0}(v(x_0))$. The map $M \to M_v$ defined by $gx_0 \to \exp_{gx_0}(v(gx_0))$ is a fibration. Moreover every orbit is of the form M_v for some equivariant normal field v. So in order to understand the submanifold geometry of orbits of N, it suffices to consider principal orbits.

It follows from Proposition 5.4.12, Corollary 5.4.11 and Theorem 5.6.7 that we have

5.7.1. Theorem. *Suppose M is a principal orbit of an isometric polar G-action on N. Then*

(1) a G-equivariant normal field is parallel with respect to the induced normal connection,

(2) $\nu(M)$ is globally flat,

(3) if v is a parallel normal field on M then the shape operators $A_{v(x)}$ and $A_{v(y)}$ are conjugate for all $x, y \in M$, i.e., the principal curvatures of M along parallel normal field v are constant,

(4) there exists $r > 0$ such that

$$\mathcal{U} = \{\exp_x(v) \mid x \in M,\ v \in \nu(M)_x,\ \|v\| < r\}$$

is a tubular neighborhood of M,

(5) if S_0 is the normal slice at x_0, $\{\exp_{x_0}(v) \mid v \in \nu(M)_{x_0},\ \|v\| < r\}$, with the induced metric from N, then the map $\pi : \mathcal{U} \to S_0$, defined by

$\pi(\exp_x(v(x)) = \exp_{x_0}(v(x_0))$ *for* $x \in M$ *and* v *a parallel normal field, is a Riemannian submersion,*

(6) $\{M_v \mid v \text{ is a parallel normal vector field of } M\}$ *is a singular foliation of* N.

Note that the local invariants (normal and principal curvatures) of principal orbits of polar actions of N are quite simple. So, both from the point of view of submanifold geometry and that of group actions, it is natural to make the following definition:

5.7.2. Definition. A submanifold M of N is called *isoparametric* if $\nu(M)$ is flat and the principal curvatures along any parallel normal field of M are constant.

5.7.3. Example. If N is a polar G-manifold, then the principal G-orbits are isoparametric in N. In particular the principal orbits of the isotropy representation of a symmetric space U/K are isoparametric in the Euclidean space $T(U/K)_{eK}$. But unlike the case of totally umbilic submanifolds and submanifolds with parallel second fundamental forms of $N^p(c)$, there are many isoparametric submanifolds of $N^p(c)$ which are *not* orbits. These submanifolds are far from being classified, but there is a rich theory for such manifolds (for example properties (4-6) of Theorem 5.7.1 hold for these submanifolds), and this will be developed in the later chapters.

Next we will discuss the submanifold geometry of a general Riemannian G-manifold. It again follows from previous discussions that we have

5.7.4. Theorem. *Suppose* M *is a principal orbit of an isometric* G*-manifold* N. *Then*

(1) there exist a tubular neighborhood U *of* M, *a Riemannian manifold* B *and a Riemannian submersion* $\pi : U \to B$ *having* M *as a fiber,*

(2) if v *is a* π*-parallel normal field on* M *then the shape operators* $A_{v(x)}$ *and* $A_{v(y)}$ *are conjugate for all* $x, y \in M$, *i.e., the principal curvatures of* M *along a* π*-parallel normal field* v *are constant,*

(3) if v *is a* π*-parallel normal field on* M *then* M_v *is an embedded submanifold of* N *and the map* $M \to M_v$ *defined by* $x \to \exp_x(v(x))$ *is a fibration,*

(4) $\{M_v \mid v \text{ is a } \pi\text{−parallel normal vector field of } M\}$ *is the orbit foliation on* N *given by* G.

This leads us to make the following definition:

5.7.5. Definition. An embedded submanifold M of N is *orbit-like* if

(i) there exist a tubular neighborhood U of M in N, a Riemannian manifold B and a Riemannian submersion $\pi : U \to B$ having M as a fiber,

(ii) if v is a π-parallel normal field on the fiber $M_b = \pi^{-1}(b)$ then the shape operators $A_{v(x)}$ and $A_{v(y)}$ of M_b are conjugate for all $x, y \in M_b$, i.e., the principal curvatures of M_b along parallel normal field v are constant.

Then the following are some natural questions and problems:

(1) Let M be an orbit-like submanifold of $N^p(c)$, and suppose its Riemannian submersion π is defined on $U = N^p(c)$. Is there a subgroup G of Iso$(N^p(c))$ such that all G-orbits are principal and π is the orbit map?

(2) Do conditions (3) and (4) of Theorem 5.7.4 hold for orbit-like submanifolds? If $\|v\|$ is small then it follows from Definition 5.7.5 that (3) and (4) are true. But it is unknown for large v.

(3) Suppose M^n is a submanifold of $N^{n+k}(c)$ with a global normal frame field $\{e_\alpha\}$ such that the principal curvatures of M along e_α are constant. Are there a "good" necessary and sufficient condition on M that guarantee M is orbit-like.

(4) Develop a theory of isoparametric submanifolds of symmetric spaces.

5.8. Infinite dimensional examples

First we review and set some terminology for manifolds of maps. Let M be a compact Riemannian n-manifold. Then for all k

$$(u, v)_k = \int_M ((I + \triangle)^{\frac{k}{2}} u,\ v) dx$$

defines an inner product on the space $C^\infty(M, \boldsymbol{R}^m)$ of smooth maps from M to $\boldsymbol{R}^m$, where dx is the volume element of M and $(,)$ is the standard inner product on $\boldsymbol{R}^m$. Let $H^k(M, \boldsymbol{R}^m)$ denote the completion of $C^\infty(M, \boldsymbol{R}^m)$ with respect to the inner product $(\ ,\ )_k$. It follows from the Sobolev embedding theorem [GT] that if $k > \frac{n}{2}$ then $H^k(M, \boldsymbol{R}^m)$ is contained in $C^0(M^n, \boldsymbol{R}^m)$ and the inclusion map is compact. Let N be a complete Riemannian manifold isometrically embedded in the Euclidean space $\boldsymbol{R}^m$. If $k > \frac{n}{2}$ then

$$H^k(M, N) = \{u \in H^k(M, \boldsymbol{R}^m) \mid u(M) \subseteq N\}$$

is a Hilbert manifold (for details see [Pa6]). In particular, $H^k(\boldsymbol{S}^1, N)$ is a Hilbert manifold if $k > \frac{1}{2}$.

Let G be a simple compact connected Lie group, T a maximal torus of G, $\mathcal{G}$, $\mathcal{T}$ the corresponding Lie algebras, and b the Killing form on $\mathcal{G}$. Then $(u, v) = -b(u, v)$ defines an inner product on $\mathcal{G}$. Let ξ denote the trivial principal G-bundle on $\boldsymbol{S}^1$. Then the Hilbert group $\hat{G} = H^1(\boldsymbol{S}^1, G)$ is the gauge group, and the Hilbert space $V = H^0(\boldsymbol{S}^1, \mathcal{G})$ is the space of H^0-connections of ξ. The group $\hat{G}$ acts on V by the gauge transformations:

$$g \cdot u = gug^{-1} + g'g^{-1}.$$

5.8.1. Theorem. *Let G be a compact Lie group, T a maximal torus of G, and $\mathcal{G}$, $\mathcal{T}$ the corresponding Lie algebras. Let $\hat{G} = H^1(\boldsymbol{S}^1, G)$ act on $V = H^0(\boldsymbol{S}^1, \mathcal{G})$ by*

$$g \cdot u = gug^{-1} + g'g^{-1}.$$

Then this $\hat{G}$-action is isometric, proper, Fredholm, and admits section. In fact, $\hat{\mathcal{T}}$ = the set of constant maps in V with value in $\mathcal{T}$, is a section, and the associate generalized Weyl group $W(\hat{\mathcal{T}})$ is the affine Weyl group $W \times_s \Lambda$, where $\Lambda = \{t \in \mathcal{T} \mid \exp(t) = e\}$ and

$$(w_1, \lambda_1) \cdot (w_2, \lambda_2) = (w_1 w_2, \lambda_2 + w_2(\lambda_1)).$$

PROOF. Since the Killing form on $\mathcal{G}$ is $Ad(G)$ invariant, the $\hat{G}$-action is isometric (by affine isometries). To see that it is proper, suppose $g_n \cdot u_n \to v$ and $u_n \to u$. Since G is compact, $g_n \cdot u \to v$, i.e., $g_n u g_n^{-1} + g_n' g_n^{-1} \to v$, which implies that $\|u_n + g_n^{-1} g_n'\|_0$ is bounded. So $\|g_n^{-1} g_n'\|$ is bounded. Since G is compact, $\|g_n\|_0$ is bounded. Hence $\|g_n\|_1$ is bounded. It follows from the Sobelov embedding theorem and Rellich's lemma that the inclusion map $H^1(\boldsymbol{S}^1, G) \hookrightarrow C^0(\boldsymbol{S}^1, G)$ is a compact operator, so there exists a subsequence (still denoted by g_n) converging to g_0 in $H^0(\boldsymbol{S}^1, G)$. But

$$\|g_n u g_n^{-1} + g_n' g_n^{-1} - v\|_0 = \|g_n u + g_n' - v g_n\|_0 \to 0,$$

so $g_n \to g_0$ in $H^1(\boldsymbol{S}^1, G)$.

The differential P of the orbit map $g \mapsto gx$ at e is

$$P : H^1(\boldsymbol{S}^1, \mathcal{G}) \to H^0(\boldsymbol{S}^1, \mathcal{G}), \quad u \mapsto u' + [u, x],$$

which is elliptic. So it follows form the standard elliptic theory [GT] that P is Fredholm. This proves that the $\hat{G}$-action is Fredholm.

Next we show that $\hat{\mathcal{T}}$ meets every $\hat{G}$-orbit. Let $\Phi : H^0(\boldsymbol{S}^1, \mathcal{G}) \to G$ be the holonomy map, i.e., given $u \in H^0(\boldsymbol{S}^1, \mathcal{G})$, let $f : R \to G$ be the unique solution for $f' f^{-1} = u$ and $f(0) = e$, then $\Phi(u) = f(2\pi)$. Given $u \in H^0(\boldsymbol{S}^1, \mathcal{G})$, by the maximal torus theorem there exist $s \in G$ and $a \in \mathcal{T}$ such that $s\Phi(u)s^{-1} = \exp(2\pi a)$. Let $\hat{a}$ denote the constant map $\hat{a}(t) = a$. Then

$\hat{a} \in \hat{G} \cdot u$. To see this, let $h(t) = \exp(ta)sf^{-1}(t)$, then $h(0) = h(2\pi) = s$, i.e., $h \in H^1(\boldsymbol{S}^1, G)$ and $h \cdot u = \hat{a}$.

It remains to prove that $\hat{\mathcal{T}}$ is orthogonal to every $\hat{G}$-orbit. Given $t \in \mathcal{T}$, we let $\hat{t} \in H^0(\boldsymbol{S}^1, \mathcal{G})$ denote the constant map with value t. Let $\hat{t}_0 \in \hat{\mathcal{T}}$. Then

$$T(\hat{G} \cdot \hat{t}_0)_{\hat{t}_0} = \{v' + [v, \hat{t}_0] |\ v \in H^1(\boldsymbol{S}^1, \mathcal{G})\}.$$

Given any $\hat{t} \in \hat{\mathcal{T}}$, we have

$$\begin{aligned}(\hat{t}, v' + [v, \hat{t}_0])_0 &= \int_{S^1} (t, v'(\theta) + [v(\theta), t_0])\, d\theta \\ &= \int_{S^1} (t, v'(\theta))\, d\theta + \int_{S^1} (t, [v(\theta), t_0])\, d\theta\,. \\ &= 0 + \int_{S^1} ([t_0, t], v(\theta))\, d\theta = 0\end{aligned}$$

So $\hat{\mathcal{T}}$ is a section. ∎

There is little known about the classification of polar actions on Hilbert spaces.