# Focal Points

One important method for obtaining information on the topology of an immersed submanifold $M^n$ of $\boldsymbol{R}^{n+k}$ is applying Morse theory to the Euclidean distance functions of $M$. This is closely related to the focal structure of the submanifold. In this chapter, we give the definition of focal points and calculate the gradient and the Hessian of the height and Euclidean distance functions in terms of the geometry of the submanifolds.

**Height and Euclidean distance functions**

In the following we will assume that $M^n$ is an immersed submanifold of $\boldsymbol{R}^{n+k}$, and $X$ is the immersion. For $v \in \boldsymbol{R}^{n+k}$, we let $v^{T_x}$ and $v^{\nu_x}$ denote the orthogonal projection of $v$ onto $TM_x$ and $\nu(M)_x$ respectively.

**4.1.1. Proposition.** *Let $a$ denote a non-zero fixed vector of $\boldsymbol{R}^{n+k}$. and $h_a : M \to \boldsymbol{R}$ denote the restriction of the height function of $\boldsymbol{R}^{n+k}$ to $M$, i.e., $h_a(x) = \langle x, a\rangle$. Then we have*

*(i) $\nabla h_a(x) = a^{T_x}$, by identifying $T^*M$ with $TM$,*

*(ii) $\nabla^2 h_a(X) = \langle II(x), a\rangle$, which is equal to $A_{a^{\nu_x}}$ if we identify $\otimes^2 T^*M$ with $L(TM, TM)$,*

*(iii) $\triangle h_a = \langle H, a\rangle$, where $H$ is the mean curvature vector of $M$.*

PROOF. Since $dh_a = \langle dX, a\rangle = \sum \omega_i \langle e_i, a\rangle = \sum (h_a)_i \omega_i$, we have $(h_a)_i = \langle e_i, a\rangle$. So $\nabla h_a = \sum_i \langle e_i, a\rangle \omega_i$. If we identify $T^*M$ with $TM$ via the metric, then $\nabla h_a = \sum_i \langle e_i, a\rangle e_i = a^{T_x}$. Using (1.3.6), we have

$$\sum_j (h_a)_{ij}\omega_j = d(\langle e_i, a\rangle) + \sum_m \langle e_m, a\rangle \omega_{mi}.$$

Since

$$de_i = \sum_j \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha} e_\alpha,$$

we have

$$(h_a)_{ij} = h_{i\alpha j}\langle e_\alpha, a\rangle,$$

which proves (ii), and (iii) follows from the definition of the Laplacian. ∎

**4.1.2. Corollary.** *With the same assumptions as in Proposition 4.1.1,*
*(i) A point $x_0 \in M$ is a critical point of $h_a$ if and only if $a \in \nu(M)_{x_0}$.*
*(ii) The index of $h_a$ at the critical point $x_0$ is the sum of the dimension of the negative eigenspace of $A_a$.*

**4.1.3. Corollary.** *Let $X = (u_1, \ldots, u_{n+k}) : M \to \boldsymbol{R}^{n+k}$ be an immersion. Then*

$$\triangle X = H,$$

*where $\triangle$ is the Laplacian on smooth functions on $M$ given by the induced metric, and $\triangle X = (\triangle u_1, \ldots, \triangle u_{n+k})$.*

**4.1.4. Corollary.** *A closed (i.e., compact without boundary) n-manifold can not be minimally immersed in $\boldsymbol{R}^{n+k}$.*

PROOF. It follows from Stoke's theorem that if $M$ is closed and $f : M \to \boldsymbol{R}$ is a smooth function satisfying $\triangle f = 0$, then $f$ is a constant (cf. Exercise 6(iv) of section 1.3). If $M$ is minimal, then $\triangle h_a = 0$, so $X$ is constant, contradicting that $X$ is an immersion. ■

A similar argument as for 4.1.1. gives

**4.1.5. Proposition.** *Let $a$ denote a fixed vector of $\boldsymbol{R}^{n+k}$, and $f_a : M \to \boldsymbol{R}$ the restriction of the square of the Euclidean distance function of $\boldsymbol{R}^{n+k}$ to $M$, i.e., $f_a(x) = \|x - a\|^2$. Then we have*
*(i) $\nabla f_a(x) = 2(x-a)^{T_x}$, if we identify $T^*M$ with $TM$.*
*(ii) $\frac{1}{2}\nabla^2 f_a(x) = I(x) + \langle II(x), (x-a)\rangle$, and by identifying $\otimes^2 T^*M$ with $L(TM, TM)$, we have $\nabla^2 f_a(x) = Id - A_{(a-x)^{\nu_x}}$,*
*(iii) $\triangle f_a(x) = n - \langle H, (a-x)\rangle$, where $H$ is the mean curvature vector of $M$.*

In Part II, Chapter 9, we define the Hessian of a smooth function $f$ at a critical point $x_0$. Given two smooth vector fields $X$ and $Y$, $X(Yf)(x_0)$ depends only on the value of $X, Y$ at $x_0$, so it defines a bilinear form $\mathrm{Hess}(f, x_0)$ on $TM_{x_0}$. Moreover, because $XY - YX = [X, Y]$ is a tangent vector field and $df_{x_0} = 0$, $\mathrm{Hess}(f, x_o)$ is a symmetric bilinear form.

**4.1.6. Corollary.** *With the same assumption as in Proposition 4.1.5,*
*(i) a point $x_0 \in M$ is a critical point of $f_a$ if and only if $(a - x_0) \in \nu(M)_{x_0}$.*
*(ii) If $x_0$ is a critical point of $f$, then $\mathrm{Hess}(f, x_0) = \nabla^2 f(x_0)$.*
*(iii) The index of $f_a$ at the critical point $x_0$ is the sum of the dimension of the eigenspace $E_\lambda$ of $A_a$ corresponding to the eigenvalue $\lambda > 1$.*

The critical points of $h_a$ and $f_a$ are closely related to the singular points of the normal maps and the endpoint maps of $M$, which are defined as follows:

**4.1.7. Definition.** The *normal map* $N : \nu(M) \to \boldsymbol{R}^{n+k}$ and the *endpoint map* $Y : \nu(M) \to \boldsymbol{R}^{n+k}$ of an immersed submanifold $M$ of $\boldsymbol{R}^{n+k}$ are defined respectively by $N(v) = v$, and $Y(v) = x + v$, for $v \in \nu(M)_x$.

**4.1.8. Proposition.** *Let $M$ be an immersed submanifold of $\boldsymbol{R}^{n+k}$, and $N, Y$ the normal map and the endpoint map of $M$ respectively. Suppose $v \in \nu(M)_{x_0}$, and $e_\alpha$ is an orthonormal frame field of $\nu(M)$ defined on a neighborhood $U$ of $x_0$, which is parallel at $x_0$ (i.e., $\nabla^\nu e_\alpha(x_0) = 0$ for all $\alpha$). Then using the trivialization $\nu(M)|U \simeq U \times \boldsymbol{R}^k$ via the frame field $e_\alpha$, we have*

*(i) $dN_v(u, z) = (-A_v(u), z)$,*

*(ii) $dY_v(u, z) = (I - A_v(u), z)$.*

PROOF. Let $X$ denote the immersion of $M$ into $\boldsymbol{R}^{n+k}$. Then $N = \sum_\alpha z_\alpha e_\alpha$, and $Y = X + \sum_\alpha z_\alpha e_\alpha$. So

$$dN = \sum_{\alpha,i} z_\alpha \omega_{\alpha i} \otimes e_i + \sum_{\alpha,\beta} z_\alpha \omega_{\alpha\beta} \otimes e_\beta + \sum_\alpha dz_\alpha \otimes e_\alpha,$$

$$dY = dX + dN = \sum_i \omega_i \otimes e_i + dN.$$

Then the proposition follows from the fact that $e_\alpha$ is parallel at $x_0$, i.e., $\omega_{\alpha\beta}(x_0) = 0$. ∎

**4.1.9. Corollary.** *With the same assumption as in Proposition 4.1.8. Then for $v \in \nu(M)_x$ we have*

*(i) $v$ is a singular point of the normal map $N$ (i.e., the rank of $dN_v$ is less than $(n+k)$) if and only if $A_v$ is singular; in fact the dimension of $\mathrm{Ker}(dN_v)$ and $\mathrm{Ker}\, A_v$ are equal.*

*(ii) $v$ is a singular point of the end point map $Y$ if and only if $I - A_v$ is singular; in fact the dimension of $\mathrm{Ker}(dY_v)$ and $\mathrm{Ker}(I - A_v)$ are equal.*

Let $X : M \to \boldsymbol{S}^{n+k} \subset \boldsymbol{R}^{n+k+1}$ be an immersion. We may choose a local orthonormal frame $e_0, e_1, \ldots, e_{n+k}$ such that $e_1, \ldots, e_n$ are tangent to $M$, $e_0 = X$, and $e_{n+1}, \ldots, e_{n+k}$ are normal to $M$ in $\boldsymbol{S}^{n+k}$. Then we have

$$de_0 = dX = \sum_i \omega_i \otimes e_i,$$

so $\omega_{0i} = \omega_i$, and $\omega_{0\alpha} = 0$. Since

$$de_i = \sum_j \omega_{ij} \otimes e_j + \sum_\alpha \omega_{i\alpha} \otimes e_\alpha + \omega_{i0} \otimes e_0,$$

we have:

**4.1.10. Proposition.** *Let $X : M \to \boldsymbol{S}^{n+k}$ be an immersion, and $a \in \boldsymbol{S}^{n+k}$. Then*

*(i) $\nabla h_a(x) = a^{T_x}$, by identifying $T^*M$ with $TM$,*

*(ii) $\nabla^2 h_a = -h_a\, I + \langle II, a\rangle$, if we identify $\otimes^2 T^*M$ with $L(TM, TM)$, then $\nabla^2 h_a(x) = -h_a\, I + A_{a^{\nu_x}}$,*

*(iii) $\triangle h_a = -nh_a + \langle H, a\rangle$, where $H$ is the mean curvature vector of $M$ in $\boldsymbol{S}^{n+k}$.*

*(iv) $\triangle X = -nX + H$.*

**4.1.11. Corollary.** *Let $X = (u_1, \ldots, u_{n+k+1}) : M^n \to \boldsymbol{S}^{n+k}$ be an isometric immersion. Then $M$ is minimal in $\boldsymbol{S}^{n+k}$ if and only if $\triangle u_i = -nu_i$ for all $i$, where $\triangle$ is the Laplacian with respect to the metric on $M$.*

Let $X : M^n \to \boldsymbol{H}^{n+k} \subseteq \boldsymbol{R}^{n+k,1}$ be an isometric immersion, and $e_A$ as above. Since

$$\omega_{0i} = \omega_{i0} = \omega_i \quad ,$$

we have

**4.1.12. Proposition.** *Let $X : M \to \boldsymbol{H}^{n+k} \subset \boldsymbol{R}^{n+k,1}$ be an immersion, and $a \in \boldsymbol{H}^{n+k}$. Then*

*(i) $\nabla h_a(x) = a^{T_x}$, by identifying $T^*M$ with $TM$,*

*(ii) $\nabla^2 h_a = h_a\, I + \langle II, a\rangle$, and if we identify $\otimes^2 T^*M$ with $L(TM, TM)$, then $\nabla^2 h_a(x) = h_a\, I + A_{a^{\nu_x}}$,*

*(iii) $\triangle h_a = nh_a + \langle H, a\rangle$, where $H$ is the mean curvature vector of $M$ in $\boldsymbol{H}^{n+k}$.*

*(iv) $\triangle X = nX + H$.*

**4.1.13. Corollary.** *There are no immersed closed minimal submanifolds in the hyperbolic space $\boldsymbol{H}^n$.*

If $M$ is immersed in $\boldsymbol{S}^{n+k}$, then $f_a = 1 + \|a\|^2 - 2h_a$. If $M$ is immersed in $\boldsymbol{H}^{n+k}$, then $f_a = -1 + \|a\|^2 - 2h_a$. It follows that for immersed submanifolds of $\boldsymbol{S}^{n+k}$ or $\boldsymbol{H}^{n+k}$, $f_a$ and $-h_a$ differ only by a constant.

**Exercises.**

1. Let $f : M \to \boldsymbol{R}$ be a smooth function on the Riemannian manifold $M$, and $p$ a critical point of $f$. Show that $\nabla^2 f(p) = \text{Hess}(f)_p$.

## 4.2. The focal points of submanifolds of $\boldsymbol{R}^n$

Let $a \in \boldsymbol{R}^{n+k}$, and define $f_a : M \to \boldsymbol{R}$ by $f_a(x) = \|x - a\|^2$ as in section 4.1. It follows from Proposition 4.1.6 that $q$ is a critical point of $f_a$ if and only if $(a - q) \in \nu(M)_q$, and the Hessian of $f_a$ at a critical point $q$ is $I - A_{(a-q)}$. Note that $I - A_{(a-q)}$ is also the tangential part of $dY_{(q,a-q)}$, where $Y$ is the endpoint map. This leads us to the study of focal points ([Mi1]).

**4.2.1. Definition.** Let $X : M^n \to \boldsymbol{R}^{n+k}$ be an immersion. A point $a = Y(x, e)$ in the image of the endpoint map $Y$ of $M$, is called a *non-focal point* of $M$ with respect to $x$ if $dY_{(x,e)}$ is an isomorphism. If $m = \dim(\text{Ker}\, dY_{(x,e)}) > 0$, then $a$ is called a *focal point* of multiplicity $m$ with respect to $x$. The *focal set* $\Gamma$ of $M$ in $\boldsymbol{R}^{n+k}$ is the set of all focal points of $M$.

Note that $a$ is a focal point of $M$ if and only if $a$ is a critical value of the endpoint map $Y$, and the focal set $\Gamma$ of $M$ is the set of all critical values of $Y$. It follows from Proposition 4.1.8 that

$$\Gamma = \{x + e |\; x \in M,\; e \in \nu(M)_x,\; \text{and}\; \det(I - A_e) = 0\}.$$

**4.2.2. Example.** Let $M^n$ be an immersed hypersurface in $\boldsymbol{R}^{n+1}$, and $\lambda_1, \ldots, \lambda_n$ the principal curvatures of $M$ with respect to the unit normal field $e_\alpha$. Using Proposition 4.1.8, we have $dY_{(x,te_\alpha)} = I - A_{te_\alpha} = I - tA_{e_\alpha}$. So $(x, te_\alpha)$ is a singular point of $Y$ if and only if

$$\det(dY_{(x,te_\alpha)}) = \prod_i (1 - t\lambda_i) = 0.$$

Therefore $\Gamma \cap (x + \nu(M)_x)$ is equal to the finite set $\{x + \frac{1}{\lambda_i(x)} e_\alpha(x) |\;\; \lambda_i \neq 0\}$. For example if $M^n$ is the sphere of radius $r$ and centered at $a_0$ in $\boldsymbol{R}^{n+1}$, then $\Gamma = \{a_0\}$; and if $M = \boldsymbol{S}^1 \times \boldsymbol{R} \subseteq \boldsymbol{R}^3$, a right cylinder based on the unit circle, then $\Gamma = 0 \times \boldsymbol{R}$.

**4.2.3. Example.** Let $M^n$ be an immersed submanifold of $\boldsymbol{R}^{n+k}$, and $\{e_\alpha\}$ a local orthonormal normal frame field. Then it follows from Proposition 4.1.8 that

$$\det(dY_{(x,e)}) = \det(I - \sum z_\alpha A_{e_\alpha}), \qquad (4.2.1)$$

where $e = \sum_\alpha z_\alpha e_\alpha$, and $A_{e_\alpha}$ is the shape operator in the normal direction $e_\alpha$. Note that (4.2.1) is a degree $k$ polynomial with real coefficients, and in general it can not be decomposed as a product of degree one polynomials. Hence the focal set $\Gamma$ of $M$ can be rather complicated.

**4.2.4. Example.** Let $M^n$ be an immersed submanifold of $\boldsymbol{R}^{n+k}$ with flat normal bundle. It follows from Proposition 2.1.2 that $\{A_e | e \in \nu(M)_x\}$ is a family of commuting self-adjoint operators on $TM_x$. So there exist a common eigendecomposition $TM_x = \bigoplus_{i=1}^p E_i$ and $p$ linear functionals $\alpha_i$ on $\nu(M)_x$ such that $A_e|E_i = \alpha_i(e) id_{E_i}$. Since $\nu(M)_x^*$ can be identified as $\nu(M)_x$, there exist $v_i \in \nu(N)_x$ such that $\alpha_i(e) = \langle e, v_i \rangle$. So we have

$$A_e|E_i = \langle e, v_i \rangle id_{E_i},$$

$$\det(dY_e) = \det(I - A_e) = \prod_{i=1}^p (1 - \langle v_i, e \rangle)^{m_i}.$$

So $\Gamma \cap \nu_x$ is the union of $p$ hyperplanes $\ell_i$ in $\nu_x$, where $\nu_x$ is the affine normal plane $x + \nu(M)_x$. We call the normal vectors $v_i$ the *curvature normals* and $\ell_i$ the *focal hyperplanes* at $x$. In general, the focal hyperplanes at $x$ do not have common intersection points. But if $M$ is contained in a sphere centered at $a$, then $a \in \nu_x$ and is a focal point of $M$ with respect to $x$ with multiplicities $n$ for all $x \in M$. Moreover, if $k = 2$, $M$ is contained in $\boldsymbol{S}^{n+1}$ , and $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of $M$ as a hypersurface of $\boldsymbol{S}^{n+1}$, then let $e_{n+1}$ be the normal of $M$ in $\boldsymbol{S}^{n+1}$, and $e_{n+2}(x) = x$, we have $\lambda_{i,n+1} = \lambda_i$, $\lambda_{i,n+2} = -1$, and $\ell_i$ is the line that passes through the origin with slope $1/\lambda_i$.

**4.2.5. Proposition.** *If $M^n$ is an immersed submanifold of codimension $k$ in $\boldsymbol{S}^{n+k}$ with flat normal bundle, then, as an immersed submanifold of codimension $k+1$ in $\boldsymbol{R}^{n+k+1}$, $M^n$ also has flat normal bundle.*

PROOF. Let $X : M \to \boldsymbol{S}^{n+k}$ be the immersion, and $\{e_A\}$ be an adapted local orthonormal frame for $M$ such that $\{e_\alpha\}$ is parallel with respect to the induced normal connection of $M$, i.e., $\omega_{\alpha\beta} = 0$. Set $e_0 = X$. Then $\{e_{n+1}, \ldots, e_{n+k}, e_0\}$ is an orthonormal frame field for the normal bundle $\nu(M)$ in $\boldsymbol{R}^{n+k+1}$. Since $dX = \sum \omega_i e_i$,

$$\omega_{\alpha 0} = 0.$$

This proves that $\{e_{n+1}, \ldots, e_{n+k}, e_0\}$ is a parallel frame field for $\nu(M)$. ∎

Since a hypersurface always has flat normal bundle, any hypersurface of $\boldsymbol{S}^{n+1}$ is a codimension 2 submanifold of $\boldsymbol{R}^{n+2}$ with flat normal bundle. Proposition 4.2.5 also implies that the study of submanifolds of sphere with flat normal bundles is included in the study of submanifolds of Euclidean space with flat normal bundles.

**4.2.6. Theorem.** *Let $M^n$ be an immersed submanifold of $\boldsymbol{R}^{n+k}$, $q \in M$, $e \in \nu(M)_q$, and $a = Y(q, e) = q + e$. Then* *(i) $q$ is a critical point of $f_a$,*

*(ii) $q$ is a non-degenerate critical point of $f_a$ if and only if $a$ is a non-focal point of $M$,*

*(iii) $q$ is a degenerate critical point of $f_a$ with nullity $m$ if and only if $a$ is a focal point of $M$ with multiplicity $m$ with respect to $q$,*

*(iv)* $\mathrm{Index}(f_a, q)$ *is equal to the number of focal points of $M$ with respect to $q$ on the line segment joining $q$ to $a$, each counted with its multiplicities.*

PROOF. Suppose $A_e$ has eigenvalues $\lambda_1, \ldots, \lambda_r$ with multiplicities $m_i$, and eigenspace $E_i$. Since $\mathrm{Hess}(f_a, q) = \nabla^2 f_a(q) = I - A_e$, the negative space of the Hessian is equal to $\bigoplus\{E_i \mid \lambda_i > 1\}$. If $\lambda_i > 1$, then $0 < 1/\lambda_i < 1$ and $\det(I - A_{e/\lambda_i}) = 0$, which implies that $q + (e/\lambda_i)$ is a focal point with respect to $q$ with multiplicity $m_i$. ∎