Weingarten Surfaces in three dimensional space forms

In this chapter we will consider smooth, *oriented* surfaces M in three-dimensional simply-connected space forms $N^3(c)$. Such an M is called a *Weingarten surface* if its two principal curvatures λ_1, λ_2 satisfy a non-trivial functional relation, e.g., surfaces with constant mean curvature or constant Gaussian curvature. We will use the Gauss and Codazzi equations for surfaces to derive some basic properties of Weingarten surfaces.

Let $X : M \to N^3(c)$ be an immersed surface. Using the same notation as in section 2.1, we have

$$dX = \omega_1 \otimes e_1 + \omega_2 \otimes e_2, \tag{3.0.1}$$

$$d\omega_1 = \omega_{12} \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_{12}, \tag{3.0.2}$$

and the Gauss equation (2.1.12), Codazzi equations (2.1.13) become:

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 = -\omega_{13} \wedge \omega_{23} = -(\lambda_1\lambda_2 + c)\omega_1 \wedge \omega_2, \tag{3.0.3}$$

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}, \ \ d\omega_{23} = \omega_{13} \wedge \omega_{12}. \tag{3.0.4}$$

The mean curvature and the Gaussian curvature are given by

$$H = \lambda_1 + \lambda_2, \ \ K = c + \lambda_1\lambda_2.$$

A point $p \in M$ is called an umbilic point if $II_p = \lambda I_p$, i.e., the two principal curvatures at p are equal. The eigendirections of the shape operator of M at a non-umbilic point are called the principal directions. Local coordinates (x, y) on M are called *line of curvature coordinates* if the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are principal directions. If $p \in M$ is not an umbilic point then there is a neighborhood U of p consisting of only non-umbilic points, and the frame field given by the unit eigenvectors of the shape operator is smooth and orthonormal. So it follows form Ex. 1 of section 1.4 that there exist line of curvature coordinates near p. A tangent vector $v \in TM_p$ is called *asymptotic* if $II(v, v) = 0$, and a coordinate system (x, y) is called *asymptotic* if $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are asymptotic.

A local coordinate system on a Riemannian surface is called *isothermal* if the metric tensor is of the form $f^2(dx^2 + dy^2)$. It is well-known that on a Riemannian 2-manifold there always exists isothermal coordinates locally ([Ch2]). If (x, y) and (u, v) are two isothermal coordinate systems on M, then the coordinate change from $z = x + iy$ to $w = u + iv$ is a complex analytic function. Hence every two dimensional Riemannian manifold has a natural complex structure given by the metric.

Constant mean curvature surfaces in $N^3(c)$

In this section we derive a special coordinate system for surfaces of $N^3(c)$ with constant mean curvature, and obtain some immediate consequences.

3.1.1. Theorem. *Let M be an immersed surface in $N^3(c)$ with constant mean curvature H. Suppose $p_0 \in M$ is not an umbilic point. Then there is a local coordinate system (u, v) defined on a neighborhood U of p_0, which is both isothermal and a line of curvature coordinate system for M. In fact, if $\lambda_1 > \lambda_2$ denote the two principal curvatures of M then on U the two fundamental forms are:*

$$\begin{aligned} I &= \frac{2}{(\lambda_1 - \lambda_2)}(du^2 + dv^2), \\ II &= \frac{2}{(\lambda_1 - \lambda_2)}(\lambda_1 du^2 + \lambda_2 dv^2). \end{aligned}$$

PROOF. We will prove this theorem for $H = 0$, and the proof for H being a non-zero constant is similar. We may assume that (x, y) is a line of curvature coordinate system for M near p_0, i.e.,

$$\omega_1 = A(x,y)dx, \quad \omega_2 = B(x,y)dy,$$

$$\omega_{13} = \lambda\omega_1 = \lambda A dx, \quad \omega_{23} = -\lambda\omega_2 = -\lambda B dy, \tag{3.1.1}$$

where λ and $-\lambda$ are the principal curvatures. We may also assume that $\lambda > 0$. By Example 1.2.4 we have

$$\omega_{12} = \frac{-A_y}{B}dx + \frac{B_x}{A}dy. \tag{3.1.2}$$

Substituting (3.1.1) and (3.1.2) to the Codazzi equations (3.0.4) we obtain

$$\lambda_y A + 2\lambda A_y = 0, \quad \lambda_x B + 2\lambda B_x = 0.$$

This implies that

$$(A\sqrt{\lambda})_y = 0, \quad (B\sqrt{\lambda})_x = 0.$$

So $A\sqrt{\lambda}$ is a function $a(x)$ of x alone, and $B\sqrt{\lambda}$ is a function $b(y)$ of y alone. Let (u, v) be the coordinate system defined by

$$du = a(x)dx, \quad dv = b(y)dy.$$

Then we have

$$I = A^2dx^2 + B^2dy^2 = \frac{1}{\lambda}(du^2 + dv^2),$$

$$II = \lambda(A^2dx^2 - B^2dy^2) = du^2 - dv^2. \quad \blacksquare$$

3.1.2. Proposition. *Let U be an open subset of $\boldsymbol{R}^2$ with metric $ds^2 = f^2(dx^2+dy^2)$, and $u : U \to R$ a smooth function. Then*

(i) with respect to the dual frame $\omega_1 = f dx$ and $\omega_2 = f dy$, we have

$$\omega_{12} = -(\log f)_y dx + (\log f)_x dy, \tag{3.1.3}$$

(ii) if $u : U \to \boldsymbol{R}$ is a smooth function then

$$\triangle u = \frac{u_{xx} + u_{yy}}{f^2}, \tag{3.1.4}$$

where $\triangle$ is the Laplacian with respect to ds^2,

(iii) the Gaussian curvature K of ds^2 is

$$K = -\triangle(\log f) = -\frac{(\log f)_{xx} + (\log f)_{yy}}{f^2}. \tag{3.1.5}$$

PROOF. (i) follows from Example 1.2.4. To see (ii), note that

$$du = u_x dx + u_y dy = u_1\omega_1 + u_2\omega_2,$$

so

$$u_1 = u_x/f, \;\; u_2 = u_y/f.$$

Set $\nabla^2 u = \sum u_{ij}\omega_i \otimes \omega_j$, then by (1.3.6)

$$du_1 + u_2\omega_{21} = \sum_i u_{1i}\omega_i, \tag{3.1.6}$$

$$du_2 + u_1\omega_{12} = \sum_i u_{2i}\omega_i. \tag{3.1.7}$$

Comparing coefficients of dx in (3.1.6) and dy in (3.1.7), we obtain

$$u_{11}f = (u_x/f)_x + (u_y f_y/f^2),$$

$$u_{22}f = (u_y/f)_y - (u_x f_x/f^2),$$

which implies that

$$(\triangle u)f = f_{11} + f_{22} = (u_{xx} + u_{yy})/f^2.$$

Since $d\omega_{12} = -K\omega_1 \wedge \omega_2$, (iii) follows. ∎

As a consequence of the Gauss equation (3.0.3), Theorem 3.1.1 and Proposition 3.1.2 we have

3.1.3. Theorem. *Let M be an immersed surface in $N^3(c)$ with constant mean curvature H. Let K be the Gaussian curvature, and $\triangle$ the Laplacian with respect to the induced metric on M. Then K satisfies the following equation:*

$$\triangle \ \log(H^2 - 4K + 4c) = 4K.$$

3.1.4. Theorem. *If M is an immersed surface of $N^3(c)$ with constant mean curvature H, then the traceless part of the second fundamental form of M, i.e., $II - \frac{H}{2}I$, is the real part of a holomorphic quadratic differential. In fact, if $z = x_1 + ix_2$ is an isothermal coordinate on M and $II - \frac{H}{2}I = \sum b_{ij}dx_idx_j$. Then*

(i) $\alpha = b_{11} - ib_{12}$ is analytic,
(ii) $II - \frac{H}{2}I = \ Re(\alpha(z)dz^2)$.

PROOF. We may assume that $\omega_1 = fdx_1$, $\omega_2 = fdx_2$, and $\omega_{i3} = \sum h_{ij}\omega_j$. Then we have

$$\omega_{12} = -(\log f)_y dx + (\log f)_x dy,$$

$$b_{11} = -b_{22} = (h_{11} - \frac{H}{2})f^2,$$

$$b_{12} = h_{12}f^2.$$

Using (1.3.7), and the fact that $h_{11} - h_{22} = 2h_{11} - H$, the covariant derivative of II is given as follows:

$$dh_{11} + 2h_{12}\omega_{21} = \sum h_{11k}\omega_k, \tag{3.1.8}$$

$$dh_{12} + (2h_{11} - H)\omega_{12} = \sum h_{12k}\omega_k. \tag{3.1.9}$$

Equating the coefficient of dx in (3.1.8) and the coefficient of dy in (3.1.9), we obtain

$$(h_{11})_x + 2h_{12}\frac{f_y}{f} = h_{111}f,$$

$$(h_{12})_y + (2h_{11} - H)\frac{f_x}{f} = h_{122}f.$$

Since H is constant and ∇ commutes with contractions, we have $h_{11k}+h_{22k}=0$. Thus $h_{111}=-h_{221}$, which is equal to $-h_{122}$ by Proposition 2.1.3. So

$$(h_{11})_x + 2h_{12}\frac{f_y}{f} = -(h_{12})_y - (2h_{11}-H)\frac{f_x}{f}.$$

It then follows from a direct computation that

$$\begin{aligned}(b_{11})_x &= (h_{11})_x f^2 + 2ff_x(h_{11}-H/2)\\ &= -(h_{12})_y f^2 - 2h_{12}ff_y = -(b_{12})_y.\end{aligned}$$

Similarly, by equating the coefficient of dy in (3.1.8) and the coefficient of dx in (3.1.9), we can prove that

$$(b_{11})_y = (b_{12})_x.$$

These are Cauchy-Riemann equations for α, so α is an analytic function. ∎

Since the only holomorphic differential on $\boldsymbol{S}^2$ is zero ([Ho]), $II - \frac{H}{2}I = 0$ for any immersed sphere in $N^3(c)$ with constant mean curvature H, i.e., they are totally umbilic. Hence we have

3.1.5. Corollary ([Ho]). *If $\boldsymbol{S}^2$ is immersed in $\boldsymbol{R}^3$ with non-zero constant mean curvature H, then $\boldsymbol{S}^2$ is a standard sphere embedded in $\boldsymbol{R}^3$.*

3.1.6. Corollary ([Al],[Cb]). *If $\boldsymbol{S}^2$ is minimally immersed in $\boldsymbol{S}^3$, then $\boldsymbol{S}^2$ is an equator (i.e., totally geodesic)*

3.1.7. Corollary. *If $\boldsymbol{S}^2$ is immersed in $\boldsymbol{S}^3$ with non-zero constant mean curvature H, then $\boldsymbol{S}^2$ is a standard sphere, which is the intersection of $\boldsymbol{S}^3$ and an affine hyperplane of $\boldsymbol{R}^4$.*

Next we discuss the immersions of closed surfaces with genus greater than zero in $N^3(c)$. Given a minimal surface M in $N^3(c)$, we have associated to it a holomorphic quadratic differential Q, and locally we can find isothermal coordinate system (x,y) such that $Q=\alpha(z)dz^2$ for some analytic function $\alpha = b_{11} - ib_{12}$, and

$$I = e^{2u}(dx^2+dy^2), \;\; II = Re(\alpha(z)dz^2). \tag{3.1.10}$$

Then

$$\omega_{12} = -u_y dx + u_x dy,$$

$$b_{11} = h_{11}e^{2u}, \;\; b_{12} = h_{12}e^{2u}.$$

So

$$\det(h_{ij}) = -(b_{11}^2 + b_{12}^2)e^{-4u} = -e^{-4u}\; |\alpha|^2,$$

and the Gaussian curvature is

$$K = \det(h_{ij}) + c. \tag{3.1.11}$$

The Gauss equation (3.0.3) gives

$$u_{xx} + u_{yy} = e^{-2u}\; |\alpha|^2 - ce^{2u}, \tag{3.1.12}$$

and the Codazzi equations are exactly the Cauchy Riemann equations for α. It follows from the Fundamental Theorem 2.3.3 for surfaces in $N^3(c)$, that the following propositions are valid.

3.1.8. Proposition. *Let U be an open subset of the complex plane C, α an analytic function on U, and u a smooth function, which satisfies equation (3.1.12). Then there is a minimal immersion defined on an open subset of U such that its two fundamental forms are given by (3.1.10).*

3.1.9. Proposition ([Lw1]). *Suppose $X : M^2 \to N^3(c)$ is a minimal immersion with fundamental forms I, II, and Q is the associated holomorphic quadratic differential. Then there is a family of minimal immersions X_θ whose fundamental forms are:*

$$I_\theta = I, \;\; II_\theta = Re(e^{i\theta}Q),$$

where θ is a constant.

Let M be a closed complex surface (i.e., a Riemann surface) of genus g. Then it is well-known that there is a metric ds^2 on M, whose induced complex structure is the given one, and that has constant Gaussian curvature $1, 0$, or -1, for $g = 0$, $g = 1$, or $g \geq 1$ respectively.

Now we assume that (M, ds^2) is a closed surface of genus $g \geq 1$ with constant Gaussian curvature k, and Q is a holomorphic quadratic differential on M. Suppose z is a local isothermal coordinate system for M, $ds^2 = f^2|dz|^2$ and $Q = \alpha(z)dz^2$. Then $\|Q\|^2 = |\alpha|^2 f^{-4}$ is a well-defined smooth function on M (i.e., independent of the choice of z), and

$$k = -\triangle \;\; \log f, \tag{3.1.13}$$

where $\triangle$ is the Laplacian of ds^2. If M can be minimally immersed in $\boldsymbol{S}^3$ such that the induced metric is conformal to ds^2, and Q is the quadratic differential associated to the immersion, then there exists a smooth function φ on M such that the induced metric is

$$I = e^{2\varphi} ds^2 = f^2 e^{2\varphi}(dx^2 + dy^2),$$

and

$$K = -e^{-4\varphi}\|Q\|^2 + c.$$

So the conformal equation (1.3.11) implies that φ satisfies the following equation:

$$1 + \triangle\varphi = -e^{2\varphi} + \|Q\|^2 e^{-2\varphi}, \tag{3.1.14}$$

for $g > 1$, or

$$\triangle\varphi = -e^{2\varphi} + \|Q\|^2 e^{-2\varphi}, \tag{3.1.15}$$

for $g = 1$, where $\triangle$ is the Laplacian for the metric ds^2. These equations are the same as the Gauss equation.

If $g = 1$, then M is a torus, so we may assume that $M \simeq \boldsymbol{R}^2/\Lambda$, where Λ is the integer lattice generated by $(1, 0)$, and $(r\cos\theta, r\sin\theta)$, $ds^2 = |dz|^2$, and $\|Q\|^2$ is a constant a. Then equation (3.1.15) become

$$\triangle\varphi = -e^{2\varphi} + ae^{-2\varphi}. \tag{3.1.16}$$

Let $b = \frac{1}{4}\log a$, and $u = \varphi - b$. Then (3.1.16) becomes

$$\triangle u = -2\sqrt{a}\ \sinh(2u).$$

So one natural question that arises from this discussion is: For what values of r and θ is there a doubly periodic smooth solution for

$$u_{xx} + u_{yy} = a\ \sinh u, \tag{3.1.17}$$

with periods $(1, 0)$, and $(r\cos\theta, r\sin\theta)$?

If $g > 1$, then there are two open problems that arise naturally from the above discussion:

(i) Fix one complex structure on a closed surface M with genus $g > 1$, and determine the set of quadratic differentials Q on M such that (3.1.14) admits smooth solutions on M.

(ii) Fix a smooth closed surface M with genus $g > 1$ and determine the possible complex structures on M such that the set in (i) is not empty.

However the understanding of the equation (3.1.14) on closed surfaces is only a small step toward the classification of closed minimal surfaces of $\boldsymbol{S}^3$, because a solution of these equations on a closed surface need not give a closed minimal surface of $\boldsymbol{S}^3$. In the following we will discuss where the difficulties lie. Suppose u is a doubly periodic solution for (3.1.17), i.e., u is a solution on a torus. Then the coefficients τ of the first order system of partial differential equations

$$dF = \tau F, \tag{3.1.18}$$

as in the fundamental theorem 2.2.5 for surfaces in $\boldsymbol{S}^3$, are doubly periodic. But the solution F need *not* to be doubly periodic, i.e., such u need not give an immersed minimal torus of $\boldsymbol{S}^3$. For example, if we assume that u depends only on x, then (3.1.17) reduces to an ordinary differential equation, $u'' = a \sinh u$, which always has periodic solution. But it was proved by Hsiang and Lawson in [HL] that there are only countably many immersed minimal tori in $\boldsymbol{S}^3$, that admit an $\boldsymbol{S}^1$-action. If the closed surface M has genus greater than one, then for a given solution u of (3.1.14), the local solution of the corresponding system (3.1.18) may not close up to a solution on M (the period problem is more complicated than for the torus case).

Let (M, ds^2) be a closed surface with constant curvature k, and $d\tilde{s}^2 = e^{2\varphi}ds^2$. Suppose $(M, d\tilde{s}^2)$ is isometrically immersed in $N^3(c)$ with constant mean curvature H, and Q is the associated holomorphic quadratic differential. Then we have

$$e^{-4\varphi}\|Q\|^2 = -\det(h_{ij}) + H^2/4,$$

and φ satisfies the conformal equation (1.3.11):

$$-k + \triangle\varphi = \|Q\|^2 e^{-2\varphi} - (H^2/4 + c)e^{2\varphi}, \tag{3.1.19}$$

where $\triangle$ is the Laplacian for ds^2. Moreover (3.1.19) is the Gauss equation for the immersion. Note that if $X : M \to \boldsymbol{R}^3$ is an immersion with mean curvature $H \neq 0$ and a is a non-zero constant, then aX is an immersion with mean curvature H/a and the induced metric on M via aX is conformal to that of X. So for the study of constant mean curvature surfaces of $\boldsymbol{R}^3$, we may assume that $H = 2$. Then (3.1.19) is the same as the above equations for minimal surfaces of $\boldsymbol{S}^3$. It is known that the only *embedded* closed surface (no assumption on the genus) with constant mean curvature in $\boldsymbol{R}^3$ is the standard sphere (for a proof see [Ho]), and Hopf conjectured that there is no immersed closed surface of genus bigger than 0 in $\boldsymbol{R}^3$ with non-zero constant mean curvature. Recently Wente found counter examples for this conjecture, he constructed many immersed tori of $\boldsymbol{R}^3$ with constant mean curvature ([We]).

Exercises.

1. Suppose (M, g) is a Riemannian surface, and (x, y), (u, v) are local isothermal coordinates for g defined on U_1 and U_2 respectively. Then

the coordinate change from $z = x + iy$ to $w = u + iv$ on $U_1 \cap U_2$ is a complex analytic function.

3.2. Surfaces of $\boldsymbol{R}^3$ with constant Gaussian curvature

In the classical surface theory, a *congruence of lines* is an immersion $f : U \to Gr$, where U is an open subset of $\boldsymbol{R}^2$ and Gr is the Grassman manifold of all lines in $\boldsymbol{R}^3$ (which need not pass through the origin). We may assume that $f(u,v)$ is the line passes through $p(u,v)$ and parallel to the unit vector $\xi(u,v)$ in $\boldsymbol{R}^3$. Let $t(u,v)$ be a smooth function. Then a necessary and sufficient condition for

$$X(u,v) = p(u,v) + t(u,v)\xi(u,v)$$

to be an immersed surface of $\boldsymbol{R}^3$ such that $\xi(u,v)$ is tangent to the surface at $X(u,v)$ is

$$\det(\xi, X_u, X_v) = 0.$$

This gives the following quadratic equation in t:

$$\det(\xi,\ p_u + t\ \xi_u,\ p_v + t\ \xi_v) = 0,$$

which generically has two distinct roots. So given a congruence of lines there exist two surfaces M and M^* such that the lines of the congruence are the common tangent lines of M and M^*. They are called *focal surfaces* of the congruence. There results a mapping $\ell : M \to M^*$ such that the congruence is given by the line joining $P \in M$ to $\ell(P) \in M^*$. This simple construction plays an important role in the theory of surface transformations.

We rephrase this in more current terminology:

3.2.1. Definition. A *line congruence* between two surfaces M and M^* in $\boldsymbol{R}^3$ is a diffeomorphism $\ell : M \to M^*$ such that for each $P \in M$, the line joining P and $P^* = \ell(P)$ is a common tangent line for M and M^*. The line congruence ℓ is called *pseudo-spherical* (p.s.), or a *Bäcklund transformation*, if

(i) $\|\overrightarrow{PP^*}\| = r$, a constant independent of P.

(ii) The angle between the normals ν_P and ν_{P^*} at P and P^* is a constant θ independent of P.

The following theorems were proved over a hundred years ago:

3.2.2. Bäcklund Theorem. *Suppose $\ell : M \to M^*$ is a p.s. congruence in $\boldsymbol{R}^3$ with distance r and angle $\theta \neq 0$. Then both M and M^* have constant negative Gaussian curvature equal to $-\frac{\sin^2\theta}{r^2}$.*

PROOF. There exists a local orthonormal frame field e_1, e_2, e_3 on M such that $\overrightarrow{PP^*} = re_1$, and e_3 is normal to M. Let

$$\begin{aligned} e_1^* &= -e_1, \\ e_2^* &= \cos\theta\, e_2 + \sin\theta\, e_3, \\ e_3^* &= -\sin\theta\, e_2 + \cos\theta\, e_3. \end{aligned} \tag{3.2.1}$$

Then $\{e_1^*, e_2^*\}$ is an orthonormal frame field for TM^*. If locally M is given by the immersion $X : U \to \boldsymbol{R}^3$, then M^* is given by

$$X^* = X + r\ e_1. \tag{3.2.2}$$

Taking the exterior derivative of (3.2.2), we get

$$\begin{aligned} dX^* &= dX + rde_1 \\ &= \omega_1 e_1 + \omega_2 e_2 + r(\omega_{12} e_2 + \omega_{13} e_3) \\ &= \omega_1 e_1 + (\omega_2 + r\omega_{12}) e_2 + r\omega_{13} e_3. \end{aligned} \tag{3.2.3}$$

On the other hand, letting ω_1^*, ω_2^* be the dual coframe of e_1^*, e_2^*, we have

$$\begin{aligned} dX^* &= \omega_1^* e_1^* + \omega_2^* e_2^*, \ \text{ using}(3.2.1) \\ &= -\omega_1^* e_1 + \omega_2^* (\cos\theta\, e_2 + \sin\theta\, e_3). \end{aligned} \tag{3.2.4}$$

Comparing coefficients of e_1, e_2, e_3 in (3.2.3) and (3.2.4), we get

$$\begin{aligned} \omega_1^* &= -\omega_1, \\ \cos\theta\, \omega_2^* &= \omega_2 + r\omega_{12}, \\ \sin\theta\, \omega_2^* &= r\omega_{13}. \end{aligned} \tag{3.2.5}$$

This gives

$$\omega_2 + r\omega_{12} = r\cot\theta\, \omega_{13}. \tag{3.2.6}$$

In order to compute the curvature, we compute the following 1-forms:

$$\begin{aligned} \omega_{13}^* &= \langle de_1^*, e_3^* \rangle \\ &= -\langle de_1, -\sin\theta\, e_2 + \cos\theta\, e_3 \rangle \\ &= \sin\theta\, \omega_{12} - \cos\theta\, \omega_{13}, \ \text{ using } \ (3.2.6) \\ &= -\frac{\sin\theta}{r}\omega_2, \\ \omega_{23}^* &= \langle de_2^*, e_3^* \rangle \\ &= \langle \cos\theta de_2 + \sin\theta de_3, -\sin\theta e_2 + \cos_\theta\, e_3 \rangle \\ &= \omega_{23}. \end{aligned} \tag{3.2.7}$$

By the Gauss equation (3.0.3), we have

$$\begin{aligned}\Omega^*_{12} &= \omega^*_{13} \wedge \omega^*_{23}, \ \text{ using } \ (3.2.7)\\ &= -\frac{\sin\theta}{r}\omega_2 \wedge \omega_{23}\\ &= \frac{\sin\theta}{r}h_{12}\omega_1 \wedge \omega_2 = \frac{\sin\theta}{r}\omega_1 \wedge \omega_{13}\\ &= -(\frac{\sin\theta}{r})^2\omega^*_1 \wedge \omega^*_2,\end{aligned}$$

i.e., M^* has constant curvature $-(\frac{\sin\theta}{r})^2$. By symmetry, M also has Gaussian curvature $-(\frac{\sin\theta}{r})^2$. ∎

3.2.3. Integrability Theorem. *Let M be an immersed surface of $\boldsymbol{R}^3$ with constant Gaussian curvature -1, $p_0 \in M$, v_0 a unit vector in TM_{p_0}, and r, θ constants such that $r = \sin\theta$. Then there exist a neighborhood U of M at p_0, an immersed surface M^*, and a p.s. congruence $\ell : U \to M^*$ such that the vector joining p_0 and $p_0^* = \ell(p_0)$ is equal to rv_0 and θ is the angle between the normal planes at p_0 and p_0^*.*

PROOF. A unit tangent vector field e_1 on M determines a local orthonormal frame field e_1, e_2, e_3 such that e_3 is normal to M. In order to find the p.s. congruence, it suffices to find a unit vector field e_1 such that the corresponding frame field satisfies the differential system (3.2.6), i.e.,

$$\tau = \omega_2 + \sin\theta\,\omega_{12} - \cos\theta\,\omega_{13} = 0. \qquad (3.2.8)$$

Since the curvature of M is equal to -1, the Gauss equation (3.0.3) implies that

$$d\omega_{12} = \omega_1 \wedge \omega_2, \ \ \omega_{13} \wedge \omega_{23} = -\omega_1 \wedge \omega_2. \qquad (3.2.9)$$

Using (3.2.8) and (3.2.9), we compute directly:

$$\begin{aligned}d\tau &= \omega_{21} \wedge \omega_1 + \sin\theta\,\omega_1 \wedge \omega_2 - \cos\theta\,\omega_{12} \wedge \omega_{23}\\ &= -\omega_{12} \wedge (\omega_1 + \cos\theta\,\omega_{23}) + \sin\theta\,\omega_1 \wedge \omega_2,\\ &\equiv \frac{1}{\sin\theta}(-\cos\theta\,\omega_{13} + \omega_2) \wedge (\omega_1 + \cos\theta\,\omega_{23}) + \sin\theta\,\omega_1 \wedge \omega_2, \ \ \text{mod } \tau,\\ &= \frac{1}{\sin\theta}(-1 + \cos^2\theta + \cos\theta\,h_{12} - \cos\theta\,h_{21})\omega_1 \wedge \omega_2 + \sin\theta\,\omega_1 \wedge \omega_2,\end{aligned}$$

which is 0, because $h_{12} = h_{21}$. Then the result follows from the Frobenius theorem. ∎

The proof of the following theorem is left as an exercise.

3.2.4. Bianchi's Permutability Theorem. *Let $\ell_1 : M_0 \to M_1$ and $\ell_2 : M_0 \to M_2$ be p.s. congruences in $\boldsymbol{R}^3$ with angles θ_1, θ_2 and distance $\sin\theta_1, \sin\theta_2$ respectively. If $\sin\theta_1 \neq \sin\theta_2$, then there exist a unique hyperbolic surface M_3 in $\boldsymbol{R}^3$ and two p.s. congruences $\ell_1^* : M_1 \to M_3$ and $\ell_2^* : M_2 \to M_3$ with angles θ_2, θ_1 respectively, such that $\ell_1^*(\ell_1(p)) = \ell_2^*(\ell_2(p))$ for all $p \in M_0$. Moreover M_3 is obtained by an algebraic method.*

Next we will discuss some special coordinates for surfaces immersed in $\boldsymbol{R}^3$ with constant Gaussian curvature -1, and their relations to the Bäcklund transformations.

3.2.5. Theorem. *Suppose M is an immersed surface of $\boldsymbol{R}^3$ with constant Gaussian curvature $K \equiv -1$. Then there exists a local coordinate system (x, y) such that*

$$I = \cos^2\varphi\, dx^2 + \sin^2\varphi\, dy^2, \tag{3.2.10}$$

$$II = \sin\varphi\cos\varphi(dx^2 - dy^2), \tag{3.2.11}$$

and $u = 2\varphi$ satisfies the Sine-Gordon equation (SGE):

$$u_{xx} - u_{yy} = \sin u. \tag{3.2.12}$$

This coordinate system is called the Tchebyshef curvature coordinate system.

Proof. Since $K = -1$, there is no umbilic point on M. So we may assume (p, q) are line of curvature coordinates and $\lambda_1 = \tan\varphi$, $\lambda_2 = -\cot\varphi$, i.e.,

$$\omega_1 = A(p,q)\, dp, \quad \omega_2 = B(p,q)\, dq,$$

$$\omega_{13} = \tan\varphi\omega_1 = \tan\varphi A\, dp, \quad \omega_{23} = -\cot\varphi\omega_2 = -\cot\varphi B\, dq.$$

By Example 1.2.4, we have

$$\omega_{12} = \frac{-A_q}{B}\, dp + \frac{B_p}{A}\, dq.$$

Substituting the above 1-forms in the Codazzi equations (3.0.4) we obtain

$$A_q\cos\varphi + A\varphi_q\sin\varphi = 0, \quad B_p\sin\varphi - B\varphi_p\cos\varphi = 0,$$

which implies that $\frac{A}{\cos\varphi}$ is a function $a(p)$ of p alone and $\frac{B}{\sin\varphi}$ is a function $b(q)$ of q alone. Then the new coordinate system (x, y), defined by $dx = a(p)\, dp$, $dy = b(q)\, dq$, gives the fundamental forms as in the theorem.

With respect to the coordinates (x, y) we have

$$\omega_{12} = \varphi_y \, dx + \varphi_x \, dy,$$

and the Gauss equation (3.0.3) becomes

$$\varphi_{xx} - \varphi_{yy} = \sin\varphi \cos\varphi,$$

i.e., $u = 2\varphi$ is a solution for the Sine-Gordon equation. ∎

Note that the coordinates (s, t), where

$$x = s + t \quad y = s - t,$$

are asymptotic coordinates, the angle u between the asymptotic curves, i.e., the $s-$curves and $t-$curves, is equal to 2φ, and

$$I = ds^2 + 2\cos u \, ds \, dt + dt^2, \tag{3.2.13}$$

$$II = 2\sin u \, ds \, dt. \tag{3.2.14}$$

(s, t) are called the *Tchebyshef* coordinates. The Sine-Gordon equation becomes

$$u_{st} = \sin u. \tag{3.2.15}$$

3.2.6. Hilbert Theorem. *There is no isometric immersion of the simply connected hyperbolic 2-space $\boldsymbol{H}^2$ into $\boldsymbol{R}^3$.*

PROOF. Suppose $\boldsymbol{H}^2$ can be isometrically immersed in $\boldsymbol{R}^3$. Because $\lambda_1\lambda_2 = -1$, there is no umbilic points on $\boldsymbol{H}^2$, and the principal directions gives a global orthonormal tangent frame field for $\boldsymbol{H}^2$. It follows from the fact that $\boldsymbol{H}^2$ is simply connected that the line of curvature coordinates (x, y) in Theorem 3.2.5 is defined for all $(x, y) \in \boldsymbol{R}^2$, and so is the Tchebyshef coordinates (s, t). They are global coordinate systems for $\boldsymbol{H}^2$. Then using (3.2.10) and (3.2.12), the area of the immersed surface can be computed as follows:

$$\int_{R^2} \omega_1 \wedge \omega_2 = \int_{R^2} \sin\varphi \cos\varphi \, dx \wedge dy$$

$$= -\int_{R^2} \sin(2\varphi) \, ds \wedge dt = -\int_{R^2} 2\varphi_{st} \, ds \wedge dt$$

$$= -\lim_{a\to\infty} \int_{D_a} 2\varphi_{st} \, ds \wedge dt = -\lim_{a\to\infty} \int_{\partial D_a} -\varphi_s \, ds + \varphi_t \, dt,$$

where D_a is the square in the (s,t) plane with $P(-a,-a)$, $Q(a,-a)$, $R(a,a)$ and $S(-a,a)$ as vertices, and ∂D_a is its boundary. The last line integral can be easily seen to be

$$2(\varphi(Q)+\varphi(S)-\varphi(P)-\varphi(R)).$$

Since $I=\cos^2\varphi dx^2+\sin^2\varphi dy^2$ is the metric on $\boldsymbol{H}^2$, $\sin\varphi$ and $\cos\varphi$ never vanish. Hence we may assume that the range of φ is contained in the interval $(0,\pi/2)$, which implies that the area of the immersed surface is less than 4π. On the other hand, the metric on $\boldsymbol{H}^2$ can also be written as $(dx^2+dy^2)/y^2$ for $y>0$ and the area of $\boldsymbol{H}^2$ is

$$\int_{-\infty}^{\infty}\int_0^{\infty}\frac{1}{y}\,dy\,dx,$$

which is infinite, a contradiction. ∎

It follows from the fundamental theorem of surfaces in $\boldsymbol{R}^3$ that there is a bijective correspondence between the local solutions u of the Sine-Gordon equation (3.2.12) whose range is contained in the interval $(0,\pi)$ and the immersed surfaces of $\boldsymbol{R}^3$ with constant Gaussian curvature -1. In fact, using the same proof as for the Fundamental Theorem, we obtain bijection between the global solutions u of the Sine-Gordon equation (3.2.12) and the smooth maps $X:\boldsymbol{R}^2\to\boldsymbol{R}^3$ which satisfy the following conditions:

(i) rank $X\geq 1$ everywhere,

(ii) if X is of rank 2 in an open set U of $\boldsymbol{R}^2$, then $X|U$ is an immersion with Gaussian curvature -1.

Theorem 3.2.3 and 3.2.4 give methods of generating new surfaces of $\boldsymbol{R}^3$ with curvature -1 from a given one. So given a solution u of the SGE (3.2.12), we can use these theorems to obtain a new solution of the SGE by the following three steps:

(1) Use the fundamental theorem of surfaces to construct a hyperbolic surface M of $\boldsymbol{R}^3$ with (3.2.10) and (3.2.11) as its fundamental forms with $\varphi=u/2$.

(2) Solve the first order system (3.2.9) of partial differential equations on M to get a family of new hyperbolic surfaces M_θ in $\boldsymbol{R}^3$.

(3) On each M_θ, find the Tchebyshef coordinate system, which gives a new solution u_θ for the SGE.

However, the first and third steps in this process may not be easier than solving SGE. Fortunately, the following theorem shows that these steps are not necessary.

3.2.7. Theorem. *Let $\ell:M\to M^*$ be a p.s. congruence with angle θ and distance $\sin\theta$. Then the Tchebyshef curvature coordinates of M and M^* correspond under ℓ.*

PROOF. Let (x, y) be the line of curvature coordinates of M as in Theorem 3.2.5, and φ the angle associated to M, i.e.,

$$I = \cos^2\varphi\, dx^2 + \sin^2\varphi\, dy^2, \quad II = \cos\varphi \sin\varphi (dx^2 - dy^2).$$

Let $v_1 = \frac{1}{\cos\varphi}\frac{\partial}{\partial x}$, $v_2 = \frac{1}{\sin\varphi}\frac{\partial}{\partial y}$ (the principal directions), τ_1, τ_2 the dual coframe, and τ_{AB} the corresponding connection 1-forms. Then we have

$$\tau_1 = \cos\varphi\, dx, \quad \tau_2 = \sin\varphi\, dy,$$

$$\tau_{12} = \varphi_y\, dx + \varphi_x\, dy,$$

$$\tau_{13} = \tan\varphi\, \tau_1 = \sin\varphi\, dx, \quad \tau_{23} = -\cot\varphi\, \tau_2 = -\cos\varphi\, dy.$$

Use the same notation as in the proof of Theorem 3.2.2, and suppose

$$e_1 = \cos\alpha v_1 + \sin\alpha v_2, \quad e_2 = -\sin\alpha v_1 + \cos\alpha v_2, \tag{3.2.16}$$

where e_1 is the congruence direction. We will show that the angle associated to M^* is α. It is easily seen that

$$\begin{aligned}
\omega_1 &= \cos\alpha\cos\varphi\, dx + \sin\alpha\sin\varphi\, dy,\\
\omega_2 &= -\sin\alpha\cos\varphi\, dx + \cos\alpha\sin\varphi\, dy,\\
\omega_{13} &= \langle de_1, e_3\rangle = \cos\alpha\sin\varphi\, dx - \sin\alpha\cos\varphi\, dy,\\
\omega_{23} &= \langle de_2, e_3\rangle = -\sin\alpha\sin\varphi\, dx - \cos\alpha\cos\varphi\, dy.
\end{aligned}$$

Using (3.2.5), the first fundamental forms of M^* can be computed directly as follows:

$$\begin{aligned}
I^* &= (\omega_1^*)^2 + (\omega_2^*)^2\\
&= (\omega_1)^2 + (\omega_{13})^2\\
&= (\cos\alpha\cos\varphi\, dx + \sin\alpha\sin\varphi\, dy)^2 + (\cos\alpha\sin\varphi\, dx - \sin\alpha\cos\varphi\, dy)^2\\
&= \cos^2\alpha\, dx^2 + \sin^2\alpha\, dy^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
II^* &= \omega_1^*\omega_{13}^* + \omega_2^*\omega_{23}^*\\
&= \omega_1\omega_2 + \omega_{13}\omega_{23}\\
&= \cos\alpha\sin\alpha(-dx^2 + dy^2). \quad \blacksquare
\end{aligned}$$

Using the same notation as in the proof of Theorem 3.2.7, we have $\tau_{12} = \varphi_y dx + \varphi_x dy$, and $\omega_{12} = \tau_{12} + d\alpha$. Comparing coefficients of dx, dy in (3.2.8), we get the Bäcklund transformation for the SGE (3.2.12):

$$\begin{cases} \alpha_x + \varphi_y &= -\cot\theta\,\cos\varphi\,\sin\alpha + \csc\theta\,\sin\alpha\,\cos\varphi,\\ \alpha_y + \varphi_x &= -\cot\theta\,\sin\varphi\,\cos\alpha - \csc\theta\,\cos\alpha\,\sin\varphi. \end{cases} \tag{3.2.17}$$

The integrability theorem 3.2.3 implies that (3.2.17) is solvable, if φ is a solution for (3.2.12). And Theorem 3.2.7 implies that the solution α for (3.2.17) is also a solution for (3.2.12).

The classical Bäcklund theory for the SGE played an important role in the study of soliton theory (see [Lb]). Both the geometric and analytic aspects of this theory were generalized in [18:39], [Te1] for hyperbolic n-manifolds in $\boldsymbol{R}^{2n-1}$.

É. Cartan proved that a small piece of the simply connected hyperbolic space $\boldsymbol{H}^n$ can be isometrically embedded in $\boldsymbol{R}^{2n-1}$, and it cannot be locally isometrically embedded in $\boldsymbol{R}^{2n-2}$ ([Ca1,2], [Mo]). It is still not known whether the Hilbert theorem 3.2.6 is valid for $n > 2$, i.e., whether or not $\boldsymbol{H}^n$ can be isometrically immersed in $\boldsymbol{R}^{2n-1}$?

Exercises.

1. Let M be an immersed surface in $\boldsymbol{R}^3$. Two tangent vectors u and v of M at x are conjugate if $II(u,v) = 0$. Two curves α and β on M are conjugate if $\alpha'(t)$ and $\beta'(t)$ are conjugate vectors for all t. Let $\ell : M \to M^*$ be a line congruence in $\boldsymbol{R}^3$, e_1 and e_1^* denote the common tangent direction on M and M^* respectively. Then the integral curves of e_1^* and $d\ell(e_1)$ are conjugate curves on M^*.
2. Prove Theorem 3.2.4.
3. Let M_i be as in Theorem 3.2.4, and φ_i the angle associated to M_i. Show that
$$\tan\frac{\varphi_3-\varphi_0}{2} = \frac{\cos\theta_2-\cos\theta_1}{\cos(\theta_1-\theta_2)-1}\tan\frac{\varphi_2-\varphi_1}{2}.$$

3.3. Immersed flat tori in $\boldsymbol{S}^3$

Suppose M is an immersed surface in $\boldsymbol{S}^3$ with $K = 0$. Since $K = 1+\det(h_{ij})$, we have $\det(h_{ij}) = -1$. So using a proof similar to that for Theorem 3.2.5, we obtain the following local results for immersed flat surfaces in $\boldsymbol{S}^3$.

3.3.1. Theorem. *Let M be an immersed surface in $\boldsymbol{S}^3$ with Gaussian curvature 0. Then locally there exist line of curvature coordinates (x,y) such that*

$$I = \cos^2\varphi\,dx^2 + \sin^2\varphi\,dy^2, \tag{3.3.1}$$

$$II = \sin\varphi\cos\varphi(dx^2 - dy^2), \tag{3.3.2}$$

where φ satisfies the linear wave equation:

$$\varphi_{xx} - \varphi_{yy} = 0. \tag{3.3.3}$$

Let $u = 2\varphi$, $x = s + t$, and $y = s - t$. Then we have

3.3.2. Corollary. *Let M be an immersed surface in $\boldsymbol{S}^3$ with Gaussian curvature 0. Then locally there exist asymptotic coordinates (s, t) (Tchebyshef coordinates) such that*

$$I = ds^2 + 2\cos u \, dsdt + dt^2, \tag{3.3.4}$$

$$II = 2\sin u \, dsdt, \tag{3.3.5}$$

where u is the angle between the asymptotic curves and

$$u_{st} = 0, \tag{3.3.6}$$

Suppose M is an immersed surface of $\boldsymbol{S}^3$ with $K = 0$. Let e_A be the frame field such that

$$e_1 = \frac{1}{\cos\varphi}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{\sin\varphi}\frac{\partial}{\partial y}, \quad e_4 = X,$$

and e_3 normal to M in $\boldsymbol{S}^3$. Using the same notation as in section 2.1, we have

$$\begin{aligned} \omega_1 &= \cos\varphi \, dx, \;\; \omega_2 = \sin\varphi \, dy, \;\; \omega_{12} = \varphi_y \, dx + \varphi_x \, dy, \\ \omega_{13} &= \sin\varphi \, dx, \;\; \omega_{23} = -\cos\varphi \, dy, \\ \omega_{14} &= -\cos\varphi \, dx, \;\; \omega_{24} = -\sin\varphi \, dy, \;\; \omega_{34} = 0. \end{aligned}$$

Then

$$dg = \Theta g,$$

where g is the $\boldsymbol{O}(4)$−valued map whose i^{th} row is e_i, and $\Theta = (\omega_{AB})$.

Conversely, given a solution φ of (3.3.3), let I, II be given as in Theorem 3.3.1. Then (3.3.3) implies that the Gaussian curvature of the metric I is 0. Moreover, the Gauss and Codazzi equations are satisfied. So by the fundamental theorem of surfaces in $\boldsymbol{S}^3$ (Theorem 2.2.3), there exists an immersed local surface in $\boldsymbol{S}^3$ with zero curvature. In fact (see section 2.1), the system for $g : \boldsymbol{R}^2 \to \boldsymbol{O}(4)$:

$$dg = \Theta g, \tag{3.3.7}$$

is solvable, and the fourth row of g gives an immersed surface into $\boldsymbol{S}^4$ with I, II as fundamental forms.

Similarly, we can also use the Tchebyshef coordinates and the following frame to write down the immersion equation. Let v_1, v_2, v_3, v_4 be the local orthonormal frame field such that $v_1 = \frac{\partial}{\partial s}$, and $v_3 = e_3$, $v_4 = e_4$. So

$$v_1 = \cos\varphi\, e_1 + \sin\varphi\, e_2, \quad v_2 = -\sin\varphi\, e_1 + \cos\varphi\, e_2.$$

Let τ_i be the dual of v_i, $\tau_{AB} = \langle dv_A, v_B\rangle$, and $u = 2\varphi$. Then we have

$$\begin{aligned} \tau_1 &= ds + \cos u\, dt, \ \tau_2 = -\sin u\, dt, \ \tau_{12} = u_s ds, \\ \tau_{13} &= \sin u\, dt, \ \tau_{23} = -ds + \cos u\, dt, \\ \tau_{14} &= -\tau_1, \ \tau_{24} = -\tau_2, \end{aligned}$$

where $u = 2\varphi$. The corresponding $o(4)-$valued 1-form as in the fundamental theorem of surfaces in $\boldsymbol{S}^3$ is $\tau = P\, ds + Q\, dt$, where

$$P = \begin{pmatrix} 0 & u_s & 0 & -1 \\ -u_s & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & \sin u & -\cos u \\ 0 & 0 & \cos u & \sin u \\ -\sin u & -\cos u & 0 & 0 \\ \cos u & -\sin u & 0 & 0 \end{pmatrix}.$$

If $u_{st} = 0$, then the following system for $g : \boldsymbol{R}^2 \to \boldsymbol{O}(4)$:

$$dg = \tau g, \tag{3.3.8}$$

is solvable, and the fourth row of g gives an immersed surface into $\boldsymbol{S}^4$ with (3.3.4) (3.3.5) as fundamental forms and $K = 0$. Note that (3.3.8) can be rewritten as

$$\begin{cases} g_s &= Pg, \\ g_t &= Qg. \end{cases} \tag{3.3.9}$$

Every solution of (3.3.6) is of the form $\xi(s) + \eta(t)$, and in the following we will show that (3.3.9) reduces to two ordinary differential equations.

Identifying $\boldsymbol{R}^4$ with the 2−dimensional complex plane $\boldsymbol{C}^2$ via the map

$$F(x_1, \ldots, x_4) = (x_1 + ix_2, ix_3 + x_4),$$

we have

$$P = \begin{pmatrix} i\xi' & -1 \\ 1 & 0 \end{pmatrix}.$$

The first equation of (3.3.9) gives a system of ODE:

$$z' = i\xi' z - w, \quad w' = z, \tag{3.3.10}$$

which is equivalent to the second order equation for $z : \boldsymbol{R} \to \boldsymbol{C}$:

$$z'' + i\xi'\, z' + z = 0. \tag{3.3.11}$$

Identifying $\boldsymbol{R}^4$ with the $\boldsymbol{C}^2$ via the map

$$F(x_1, \ldots, x_4) = (x_1 + ix_2, x_3 + ix_4),$$

we have

$$Q = \begin{pmatrix} 0 & ie^{-iu} \\ ie^{iu} & 0 \end{pmatrix}.$$

And the second equation in (3.3.9) gives a system of ODE:

$$z' = ie^{-iu} w, \quad w' = ie^{iu} z, \tag{3.3.12}$$

which is equivalent to the second order equation for $z : \boldsymbol{R} \to \boldsymbol{C}$:

$$z'' + i\eta' z' + z = 0. \tag{3.3.13}$$

So the study of the flat tori in $\boldsymbol{S}^3$ reduces to the study of the above ODE.

In the following we describe some examples given by Lawson: Let $\boldsymbol{S}^3 = \{(z, w) \in \boldsymbol{C}^2 \mid |z|^2 + |w|^2 = 1\}$. then $\boldsymbol{CP}^1 \simeq \boldsymbol{S}^2$ is obtained by identifying $(z, w) \in \boldsymbol{S}^3$ with $e^{i\theta}(z, w)$, and the quotient map $\pi : \boldsymbol{S}^3 \to \boldsymbol{S}^2$ is the Hopf fibration. If $\gamma = (x, y, z) : \boldsymbol{S}^1 \to \boldsymbol{S}^2$ is an immersed closed curve on $\boldsymbol{S}^2$, then $\pi^{-1}(\gamma)$ is an immersed flat torus of $\boldsymbol{S}^3$. In fact, $X(\sigma, \theta) = e^{i\theta}(x(\sigma), y(\sigma) + iz(\sigma))$ gives a parametrization for the torus. It follows from direct computation that this torus has curvature zero, the $\theta-$curves are asymptotic, the Tchebyshef coordinates (s, t) are given by $t = \sigma$ and $s = \theta + \alpha(\sigma)$ for some function α, and the corresponding angle u as in the above Corollary depends only on t. These $s-$curves are great circles, but the other family of asympotics (the $t-$curves) in general need not be closed curves. It is not known whether these examples are the only flat tori in $\boldsymbol{S}^3$.

3.4. Bonnet transformations

Let M be an immersed surface in $N^3(c)$, and e_3 its unit normal vector. The *parallel set* M_t of constant distance t to M is defined to be $\{\exp_x(te_3(x)) \mid x \in M\}$. Note that

$$\exp_x(tv) = \begin{cases} x + tv, & \text{if } c = 0; \\ \cos t\, x + \sin t\, v, & \text{if } c = 1; \\ \cosh t\, x + \sinh t\, v, & \text{if } c = -1. \end{cases}$$

If M_t is an immersed surface, then we call it a *parallel surface*. The classical Bonnet transformation is a transformation from a surface in $\boldsymbol{R}^3$ to one of its parallel sets. Bonnet's Theorem can be stated as follows:

3.4.1. Theorem. *Let $X : M^2 \to \boldsymbol{R}^3$ be an immersed surface, e_3 its unit normal field, and H, K the mean curvature and Gaussian curvature of M.*

(i) If $H = a \neq 0$, and K never vanishes, then the parallel set $M_{1/a}$ (defined by the map $X^* = X + \frac{1}{a}e_3$) is an immersed surface with constant Gaussian curvature a^2.

(ii) If K is a positive constant a^2 and suppose that M has no umbilic points, then its parallel set $M_{1/a}$ is an immersed surface with mean curvature $-a$.

This theorem is a special case of the following simple result :

3.4.2. Theorem. *Let $X : M^2 \to N^3(c)$ be an immersed surface, e_3 its unit normal field, and A the shape operator of M. Then the parallel set M^* of constant distance t to M defined by*

$$X^* = aX + be_3 \tag{3.4.1}$$

is an immersion if and only if $(aI - bA)$ is non-degenerate on M, where $(a, b) = (1, 0)$ for $c = 0$, $(\cos t, \sin t)$ for $c = 1$, and $(\cosh t, \sinh t)$ for $c = -1$. Moreover, $e_3^* = -cbX + ae_3$ is a unit normal field of M^*, and the corresponding shape operator is

$$A^* = (cb + aA)(a - bA)^{-1}. \tag{3.4.2}$$

PROOF. We will consider only the case $c = 0$. The other cases are similar. Let e_A be an adapted local frame for the immersed surface M in $\boldsymbol{R}^3$ as in section 2.3. Taking the differential of (3.4.1), we get

$$\begin{aligned} dX^* &= dX + tde_3, \\ &= \sum \omega_i e_i - t \sum h_{ij}\omega_i e_j, \\ &= \sum (\delta_{ij} - t\, h_{ij})\omega_i e_j = I - tA \end{aligned} \qquad . \tag{3.4.3}$$

Hence X^* is an immersion if and only if $(I - tA)$ is non-degenerate. It also follows from (3.4.3) that e_A is an adapted frame for M^*, and the dual coframe is $\omega_i^* = \sum_j (\delta_{ij} - h_{ij})\omega_j$. Moreover, $\omega_{i3}^* = \langle de_i^*, e_3^* \rangle = \omega_{i3}$, so we have

$$A^* = A(I - tA)^{-1}. \quad \blacksquare \tag{3.4.4}$$

3.4.3. Corollary. *If M^2 is an immersed Weingarten surface in $N^3(c)$ then so is each of its regular parallel surfaces. Conversely, if one of the parallel surface of M is Weingarten then M is Weingarten.*

Let λ_1, λ_2 be the principal curvatures for the immersed surface M in $N^3(c)$, and λ_1^*, λ_2^* the principal curvatures for the parallel surface M^*. Then (3.4.4) becomes

$$\lambda_i^* = (cb + a\lambda_i)/(a - b\lambda_i).$$

As consequences of Theorem 3.4.2, we have

3.4.4. Corollary. *Suppose $X : M^2 \to \boldsymbol{S}^3$ has constant Gaussian curvature $K = (1 + r^2) > 1$, and $t = \tan^{-1}(1/r)$. Then*

$$X^* = \cos t\, X + \sin t\, e_3$$

is a branched immersion with constant mean curvature $(1 - r^2)/r$.

3.4.5. Corollary. *Suppose $X : M^2 \to \boldsymbol{S}^3$ has constant mean curvature $H = r$, and $t = \cot^{-1}(r/2)$. Then*

$$X^* = \cos t\, X + \sin t\, e_3$$

is a branched immersion with constant mean curvature $-r$.

We note that when $r = 0$ the above corollary says that the unit normal of a minimal surface M in $\boldsymbol{S}^3$ gives a branched minimal immersion of M in $\boldsymbol{S}^3$. This was proved by Lawson [Lw1], who called the new minimal surface the *polar variety* of M.

3.4.6. Corollary. *Suppose $X : M^2 \to \boldsymbol{H}^3$ has constant Gaussian curvature $K = (-1 + r^2) > 2$, and $t = \tanh^{-1}(1/r)$. Then*

$$X^* = \cosh t\, X + \sinh t\, e_3$$

is a branched immersion with constant mean curvature $(1 + r^2)/r$.

3.4.7. Corollary. *Suppose $X : M^2 \to \boldsymbol{H}^3$ has constant mean curvature $H = r$, $r > 2$, and $t = \tanh^{-1}(2/r)$. Then*

$$X^* = \cosh t\, X + \sinh t\, e_3$$

is a branched immersion with constant mean curvature $-r$.

Exercises.

1. Prove an analogue of Theorem 3.4.2 for immersed hypersurfaces in $N^{n+1}(c)$.
2. Suppose M^3 is an immersed, orientable, minimal hypersurface of $\boldsymbol{S}^4$ and the Gauss-Kronecker (i.e., the determinant of the shape operator) never vanishes on M. Use the above exercise to show that $\pm e_4 : M^3 \to \boldsymbol{S}^4$ is an immersion, and the induced metric on M has constant scalar curvature 6 ([De]).