Local Geometry of Submanifolds

Given an immersed submanifold M^n of the simply connected space form $N^{n+k}(c)$ there are three basic local invariants associated to M: the first and second fundamental forms and the normal connection. These three invariants are related by the Gauss, Codazzi and Ricci equations, and they determine the isometric immersion of M into $N^{n+k}(c)$ uniquely up to isometries of $N^{n+k}(c)$.

Local invariants of submanifolds

Let M be an n-dimensional submanifold of an (n+p)-dimensional Riemannian manifold (N, g), and $\bar{\nabla}$ the Levi-Civita connection of g. Let $TM_x^\perp$ denote the orthogonal complement of TM_x in TN_x, and $\nu(M)$ the normal bundle of M in N, i.e., $\nu(M)_x = (TM_x)^\perp$. In this section we will derive the three basic local invariants of submanifolds: the first and second fundamental forms, the induced normal connection, and we will derive the equations that relate them.

Let $i : M \to N$ denote the inclusion. The *first fundamental form*, I, of M is the induced metric i^*g, i.e., the inner product I_x on TM_x is the restriction of the inner product g_x to TM_x.

Let $v \in C^\infty(\nu(M))$ and let $A_v : TM_{x_0} \to TM_{x_0}$ denote the linear map defined by $A_v(u) = -((\bar{\nabla}_u v)(x_0))^T$, the projection of $(\bar{\nabla}_u v)(x_0)$ onto TM_{x_0}. Since

$$\bar{\nabla}_u(fv) = df(u)v + f\bar{\nabla}_u v, \quad \text{for} \quad f \in C^\infty(M, R),$$

and $df(u)v$ is a normal vector, we have

$$A_{fv}(u) = fA_v(u).$$

In particular, if v_1, v_2 are two normal fields on M such that $v_1(x_0) = v_2(x_0)$, then $A_{v_1}(u) = A_{v_2}(u)$ for $u \in TM_{x_0}$. So we have associated to each normal vector $v_0 \in \nu(M)_{x_0}$ a linear operator A_{v_0} on TM_{x_0}, that is called the *shape operator* of M in the normal direction v_0.

2.1.1. Proposition. *The shape operator $A_{v_0} : TM_{x_0} \to TM_{x_0}$ is self-adjoint, i.e., $g(A_{v_0}(u_1), u_2) = g(u_1, A_{v_0}(u_2))$.*

PROOF. Let v be a smooth normal field on M defined on a neighborhood U of x_0 such that $v(x_0) = v_0$, and X_i smooth tangent vector field on U such that $X_i(x_0) = u_i$. Let $\langle \, , \, \rangle$ denote the inner product g_x on TN_x. Then

$$\begin{aligned}\langle A_v(X_1), X_2\rangle &= -\langle(\bar{\nabla}_{X_1}(v))^T, X_2\rangle = -\langle(\bar{\nabla}_{X_1}(v)), X_2\rangle \\ &= -X_1(\langle v, X_2\rangle) \quad + \quad \langle v, \bar{\nabla}_{X_1}X_2\rangle \\ &= \langle v, \bar{\nabla}_{X_1}X_2\rangle.\end{aligned}$$

Similarly, we have

$$\langle A_v(X_2), X_1\rangle = \langle v, \bar{\nabla}_{X_2}X_1\rangle,$$

so

$$\langle A_v(X_1), X_2\rangle - \langle A_v(X_2), X_1\rangle = \langle v, [X_1, X_2]\rangle.$$

Then the proposition follows from the fact that $[X_1, X_2]$ is a tangent vector field. ∎

By identifying T^*M with TM via the induced metric, the shape operator A_v corresponds to a smooth section of $S^2(T^*M) \otimes \nu(M)$. called the *second fundamental form* of M, and denoted by II. Explicitly,

$$\langle II(u_1, u_2), v\rangle = \langle A_v(u_1), u_2\rangle.$$

The third invariant of M is the induced *normal connection* ∇^ν on $\nu(M)$, defined by $(\nabla^\nu)_u(v) = (\bar{\nabla}_u v)^\nu$, the orthogonal projection of $\bar{\nabla}_u v$ onto $\nu(M)$.

In the following we will write the above local invariants in terms of moving frames. A local orthonormal frame field $e_1, \ldots, e_{n+p}$ in N is said to be *adapted* to M if, when restricted to M, $e_1, \ldots, e_n$ are tangent to M. From now on, we shall agree on the following index ranges:

$$1 \le A, B, C \le (n+p), \quad 1 \le i, j, k \le n, \quad (n+1) \le \alpha, \beta, \gamma \le (n+p).$$

Let $\omega_1, \ldots, \omega_{n+p}$ be the dual coframe on N. Then the first fundamental form on M is

$$I = \sum_i \omega_i \otimes \omega_i.$$

The structure equations of N are

$$d\omega_A = \sum \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1.1}$$

and the curvature equation is

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \Theta_{AB}, \tag{2.1.2}$$

$$\Theta_{AB} = \frac{1}{2}\sum_{C,D} K_{ABCD}\,\omega_C \wedge \omega_D, \quad K_{ABCD} = -K_{ABDC},$$

where ω_{AB} and Θ_{AB} are the Levi-Civita connection and the Riemann curvature tensor of g respectively.

For a differential form τ on N, we still use τ to denote $i^*\tau$, where $i : M \to N$ is the inclusion. Restricting ω_α to M, i.e., applying i^* to ω_α, we have

$$\omega_\alpha = 0. \tag{2.1.3}$$

Using (2.1.3), and applying i^* to (2.1.1), we obtain

$$d\omega_i = \sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.1.4}$$

$$d\omega_\alpha = \sum \omega_{\alpha i} \wedge \omega_i = 0. \tag{2.1.5}$$

Note that (2.1.4) implies that the connection 1-form $\{\omega_{ij}\}$ is the Levi-Civita connection ∇ of the induced metric I on M. Set

$$\omega_{i\alpha} = \sum_j h_{i\alpha j}\omega_j. \tag{2.1.6}$$

Then (2.1.5) becomes

$$\sum_{i,j} h_{i\alpha j}\omega_i \wedge \omega_j = 0,$$

which implies that

$$h_{i\alpha j} = h_{j\alpha i}.$$

Note that

$$A_{e_\alpha}(e_i) = -(\bar{\nabla}_{e_i} e_\alpha)^T = -\sum_j \omega_{\alpha j}(e_i)e_j = \sum_j h_{i\alpha j} e_j.$$

So the second fundamental form of M is

$$\begin{aligned} II &= \sum_{i,j,\alpha} h_{i\alpha j}\omega_i \otimes \omega_j \otimes e_\alpha \\ &= \sum_{i,\alpha} \omega_{i\alpha} \otimes \omega_i \otimes e_\alpha. \end{aligned}$$

It follows from the definition of the normal connection that

$$\nabla^\nu(e_\alpha) = \sum_\beta \omega_{\alpha\beta} \otimes e_\beta.$$

Restricting the curvature equations (2.1.2) of N to M, we have

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j} - \Theta_{ij}, \tag{2.1.7}$$

$$d\omega_{i\alpha} = \sum_k \omega_{ik} \wedge \omega_{k\alpha} + \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha} - \Theta_{i\alpha}, \tag{2.1.8}$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \sum_i \omega_{\alpha i} \wedge \omega_{i\beta} - \Theta_{\alpha\beta}. \tag{2.1.9}$$

Then (2.1.7) and (2.1.9) imply that the Riemann curvature tensor Ω of the induced metric I and the curvature Ω^ν of the normal connection ∇^ν (called the normal curvature of M) are:

$$\Omega_{ij} = \sum_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} + \Theta_{ij}, \tag{2.1.10}$$

$$\Omega^\nu_{\alpha\beta} = \sum_i \omega_{i\alpha} \wedge \omega_{i\beta} + \Theta_{\alpha\beta}, \tag{2.1.11}$$

respectively. Equations (2.1.7)-(2.1.9) are called the *Gauss*, *Codazzi*, and *Ricci* equations of the submanifold M.

Henceforth we assume that (N, g) has constant sectional curvature c, i.e.,

$$\Theta_{AB} = c\, \omega_A \wedge \omega_B.$$

So the Gauss, Codazzi and Ricci equations (2.1.7)-(2.1.9) for the submanifold M are

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j} - c\,\omega_i \wedge \omega_j, \tag{2.1.12}$$

$$d\omega_{i\alpha} = \sum_k \omega_{ik} \wedge \omega_{k\alpha} + \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{2.1.13}$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \sum_i \omega_{\alpha i} \wedge \omega_{i\beta}. \tag{2.1.14}$$

And (2.1.10) and (2.1.11) become

$$\Omega_{ij} = \sum_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} + c\;\; \omega_i \wedge \omega_j, \tag{2.1.15}$$

$$\Omega^\nu_{\alpha\beta} = \sum_i \omega_{i\alpha} \wedge \omega_{i\beta}, \tag{2.1.16}$$

Let

$$\Omega_{ij} = \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \qquad \text{with} \quad R_{ijkl} + R_{ijlk} = 0,$$

$$\Omega^\nu_{\alpha\beta} = \frac{1}{2}\sum_{k,l} R^\nu_{\alpha\beta kl}\omega_k \wedge \omega_l, \qquad \text{with} \quad R^\nu_{\alpha\beta kl} + R^\nu_{\alpha\beta lk} = 0.$$

Using $\omega_{i\alpha} = \sum_j h_{i\alpha j}\omega_j$, we have

$$R_{ijkl} = \sum_\alpha (h_{i\alpha k}h_{j\alpha l} - h_{i\alpha l}h_{j\alpha k}) + c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \qquad (2.1.17)$$

$$R^\nu_{\alpha\beta kl} = \sum_i (h_{i\alpha k}h_{i\beta l} - h_{i\alpha l}h_{i\beta k}.) \qquad (2.1.18)$$

By identifying T^*M with TM via the induced metric, then the Ricci equation (2.1.6) becomes $\Omega^\nu_{\alpha\beta} = [A_\alpha, A_\beta]$. So we have

2.1.2. Proposition. *Suppose (N, g) has constant sectional curvature, and M is a submanifold of N. Then the normal curvature Ω^ν of M measures the commutativity of the shape operators. In fact, $\Omega^\nu(u, v) = [A_u, A_v]$.*

A normal vector field v is *parallel* if $\nabla^\nu v = 0$. The normal bundle $\nu(M)$ is *flat* if ∇^ν is flat. Then it follows from Proposition 1.1.5 that $\nu(M)$ is flat if one of the following equivalent conditions holds:
(i) The normal curvature Ω^ν is zero.
(ii) Given $x_0 \in M$, there exist a neighborhood U of x_0 and a parallel normal frame field on U.

The normal bundle $\nu(M)$ is called *globally flat* if ∇^ν is globally flat, or equivalently, there exists a global parallel normal frame on M.

Since there are connections ∇ on TM and ∇^ν on $\nu(M)$, there exists a unique connection ∇ on the vector bundle $\otimes^2 T^*M \otimes \nu(M)$ that satisfies the "product rule", i.e.,

$$\nabla_X(\theta \otimes \tau \otimes v) = (\nabla_X\theta) \otimes \tau \otimes v + \theta \otimes (\nabla_X\tau) \otimes v + \theta \otimes \tau \otimes (\nabla_X v).$$

Set

$$\nabla II = \sum_{i,j,k\alpha} h_{i\alpha jk}\omega_i \otimes \omega_j \otimes \omega_k \otimes e_\alpha,$$

where

$$\nabla_{e_k} II = \sum_{i,j,k,\alpha} h_{i\alpha jk}\omega_i \otimes \omega_j \otimes e_\alpha.$$

Using an argument similar to that in section 1.3, we have

$$\sum_k h_{i\alpha jk}\omega_k = dh_{i\alpha j} + \sum_m h_{m\alpha j}\omega_{mi} + \sum_m h_{i\alpha m}\omega_{mj} + \sum_\beta h_{i\beta j}\omega_{\beta\alpha}. \tag{2.1.19}$$

Taking the exterior derivative of (2.1.6), we obtain

$$\begin{aligned} d\omega_{i\alpha} &= d\left(\sum_j h_{i\alpha j}\omega_j\right) \\ &= \sum_j dh_{i\alpha j}\omega_j + \sum_{j,k} h_{i\alpha j}\omega_{jk}\wedge\omega_k. \end{aligned} \tag{2.1.20}$$

By the Codazzi equation (2.1.13), we have

$$\begin{aligned} d\omega_{i\alpha} &= \sum_j \omega_{ij}\wedge\omega_{j\alpha} + \sum_\beta \omega_{i\beta}\wedge\omega_{\beta\alpha} \\ &= \sum_{j,k} h_{j\alpha k}\omega_{ij}\wedge\omega_k + \sum_{\beta,j} h_{i\beta j}\omega_j\wedge\omega_{\beta\alpha} \\ &= \sum_j\left(\sum_k h_{k\alpha j}\omega_{ik} - \sum_\beta h_{i\beta j}\omega_{\beta\alpha}\right)\wedge\omega_j. \end{aligned} \tag{2.1.21}$$

Equating (2.1.19) and (2.1.20), we get

$$\sum_j (dh_{i\alpha j} + \sum_k\{h_{k\alpha j}\omega_{ki} + h_{i\alpha k}\omega_{kj}\} + \sum_\beta h_{i\beta j}\omega_{\beta\alpha})\wedge\omega_j = 0.$$

So by (2.1.19), we have

$$\sum_{j,k} h_{i\alpha jk}\omega_j\wedge\omega_k = 0,$$

i.e., $h_{i\alpha jk} = h_{i\alpha kj}$. Since $h_{i\alpha j} = h_{j\alpha i}$, $h_{i\alpha jk} = h_{j\alpha ik}$, so we have

2.1.3. Proposition. *Suppose (N, g) has constant sectional curvature c, and M is an immersed submanifold of N. Then ∇II is a section of $S^3T^*M\otimes\nu(M)$, i.e., $h_{i\alpha jk}$ is symmetric in i, j, k.*

Although all our discussion above have been for embedded submanifolds, they hold equally well for immersions. For, locally an immersion $f : M \to N$ is an embedding, and we can naturally identify $TM_x \simeq T(f(M))_{f(x)}$.

The *principal curvatures* of an immersed submanifold M along a normal vector v are the eigenvalues of the shape operator A_v. The *mean curvature* vector H of M in N is the trace of II, i.e.,

$$H = \sum_\alpha H_\alpha e_\alpha, \quad \text{where} \quad H_\alpha = \sum_i h_{i\alpha i}.$$

The mean curvature vector of an immersion $f : M \to N$ is the gradient of the area functional at f. To be more precise, for any immersion $f : M \to N$, we let $dv(f^*g)$ be the volume element given by the induced metric f^*g, and define

$$A(f) = \int_M dv(f^*g),$$

to be the volume of the immersion f. A compact deformation of an immersion f_0 is a smooth family of immersions $\{f_t : M \to N\}$ such that there exists a relatively compact open set U of M with $f_t|(M \setminus U) = f_0|(M \setminus U)$. Then the deformation vector field

$$\xi = \left.\frac{\partial f_t}{\partial t}\right|_{t=0}$$

is a section of $f_0^*(TN)$ with compact support. It is well-known (cf. Exercise 4 below) that

$$\left.\frac{d}{dt}\right|_{t=0} A(f_t) = -\int_M \langle H, \xi \rangle \, dv_0, \tag{2.1.22}$$

where dv_0 is the volume form of f_0^*g and H is the mean curvature vector of the immersion f_0. The immersion f_0 is called a *minimal*, if its mean curvature vector $H = 0$ or equivalently

$$\left.\frac{d}{dt}\right|_{t=0} A(f_t) = 0,$$

for all compact deformations f_t. The study of minimal immersions plays a very important role in differential geometry, for example see [Lw2], [Os], [Ch5] and [Bb].

Exercises.

1. Let M be the graph of a smooth function $u : \boldsymbol{R}^n \to \boldsymbol{R}$, i.e., $M = \{(x, u(x)) \mid x \in \boldsymbol{R}^n\}$. Find I, II and H for M in $\boldsymbol{R}^{n+1}$.
2. Suppose $\alpha(s) = (f(s), g(s))$ is a smooth curve in the $yz-$plane, parametrized by arc length. Let M be the surface of revolution generated by

the curve α, i.e., M is the surface of $\boldsymbol{R}^3$ obtained by rotating the curve α around the $z-$axis.

(i) Find I, II for M.

(ii) Find a curve α such that M has constant Gaussian curvature.

(iii) Find a curve α such that M has constant mean curvature.

3. Let $\gamma : [0, \ell] \to \boldsymbol{R}^n$ be an immersion parametrized by arc length.

(i) If $n = 2$, then $I = ds^2$ and $II = k(s)\, ds^2$, where $k(s)$ is the curvature of the plane curve.

(ii) For generic immersions, show that we can choose an orthonormal frame field e_A on γ such that

$$\omega_{AB} = \begin{cases} 0, & \text{if } |A - B| \neq 1; \\ k_i(s)\, ds, & \text{if } (A, B) = (i, i+1); \\ -k_i(s)\, ds, & \text{if } (AB) = (i+1, i), \end{cases}$$

i.e., (ω_{AB}) is anti-symmetric and tridiagonal. (When $n = 3$, this frame e_A is the Frenet frame for curves in $\boldsymbol{R}^3$ and k_1, k_2 are the curvature and torsion of γ respectively, for more on the theory of curves see [Ch4], [Do]).

4. Let A denote the area functional for immersions of M into N.

(i) If $\varphi : M \to M$ is a diffeomorphism, then $A(f \circ \varphi) = A(f)$, i.e., A is invariant under the group of diffeomorphisms of M.

(ii) Show that $\nabla A(f)$ has to be a normal field along f.

(iii) It suffices to show (2.1.22) for normal deformations, i.e., we may assume that ξ is a normal field for the immersion f.

(iv) Prove (2.1.22) for normal deformations.

5. Let M be an immersed submanifold of $\boldsymbol{R}^m$, $p \in M$, $u \in TM_p$ and $v \in \nu(M)_p$ unit vectors. Let E be the plane spanned by u, v, and σ the curve given by the intersection of M and $p + E$. Show that $\langle II(u, u), v \rangle$ is equal to the curvature the curve σ at p.

6. Let M^n be an immersed submanifold of $N^{n+k}(c)$.

(i) If we identify T^*M with TM via the metric then

$$\mathrm{Ric} = HA - A^2 + (n-1)c\, I,$$

$$\mu = H^2 - \|II\|^2 + cn(n-1).$$

(ii) If M is minimal in $\boldsymbol{R}^{n+k}$ then $\mathrm{Ric}(M) \leq 0$.

Totally umbilic submanifolds

A submanifold M of N is called *totally geodesic* (*t.g.*) if its second fundamental form is identically zero. A smooth curve α of N is called a *geodesic* if as a submanifold of N it is totally geodesic. It is easily seen that if e_A is an adapted frame for M then M is t.g. if and only if $\omega_{i\alpha} = 0$ for all $1 \leq i \leq n$ and $n+1 \leq \alpha \leq n+p$.

2.2.1. Proposition. *Let γ be a smooth curve on N. Then the following statements are equivalent:*
(i) γ is a geodesic,
(ii) the tangent vector field γ' is parallel along γ,
(iii) the mean curvature of γ as a submanifold of N is zero.

PROOF. We may assume that $\gamma(s)$ is parametrized by its arc length and $e_1(\gamma(s)) = \gamma'(s)$. Then γ is a geodesic if and only if $\omega_{1i}(\gamma') = 0$ for all $1 < i \leq n$, (ii) is equivalent to

$$0 = \nabla_{\gamma'}\gamma' = (\nabla_{e_1}e_1)(\gamma(s)) = \sum_{i=2}^{n} \omega_{1i}(\gamma'(s))e_i,$$

and (iii) gives

$$H = \sum_{i>1} \omega_{1i}(\gamma')e_i = 0.$$

So these three statements are equivalent. ∎

2.2.2. Proposition. *A submanifold M of a Riemannian manifold N is totally geodesic if and only if every geodesic of M (with respect to the induced metric) is a geodesic of N.*

PROOF. The proposition follows from $\nabla_{\alpha'}\alpha' = \bar{\nabla}_{\alpha'}\alpha' - (\bar{\nabla}_{\alpha'}\alpha')^{\nu}$, and $(\bar{\nabla}_{\alpha'}\alpha')^{\nu} = II(\alpha', \alpha')$. ∎

A Riemannian manifold with constant sectional curvature is called a *space form*. We have seen in Example 1.2.5 that $\boldsymbol{R}^n$ with the standard metric has constant sectional curvature 0. In the following we will describe complete simply connected space forms with nonzero curvature.

Let $g = dx_1^2 + \ldots + dx_{n+k}^2$ be the standard metric on $\boldsymbol{R}^{n+k}$, and $\hat{\nabla}$ the Levi-Civita connection of g. Then we have seen in Example 1.2.5 that

$$\hat{\nabla}(u) = du,$$

if we identify $C^\infty(T\boldsymbol{R}^{n+k})$ with the space of smooth maps from $\boldsymbol{R}^{n+k}$ to $\boldsymbol{R}^{n+k}$. Let M^n be a submanifold of $(\boldsymbol{R}^{n+k}, g)$, and $X : M \to \boldsymbol{R}^{n+k}$ the inclusion map. Let e_A and ω_A be as in section 2.1. First note that the differential $dX_p : TM_p \to TN_p$ of the map X at $p \in M$ is the inclusion i of TM_p in TN_p. Under the natural isomorphism $L(TM, TN) \simeq T^*M \otimes TN$, i corresponds to $\sum_i \omega_i(p) \otimes e_i(p)$. Hence we have

$$dX = \sum_i \omega_i \otimes e_i. \tag{2.2.1}$$

2.2.3. Example. Let $\boldsymbol{S}^n$ denote the unit sphere of $\boldsymbol{R}^{n+1}$. Note that the inclusion map $X : \boldsymbol{S}^n \to \boldsymbol{R}^{n+1}$ is also the outward unit normal field on $\boldsymbol{S}^n$, i.e., we may choose $e_{n+1} = X$. The exterior derivative of e_{n+1} gives

$$de_{n+1} = \sum \omega_{n+1,i} \otimes e_i.$$

Using (2.2.1), we have

$$\omega_{i,n+1} = -\omega_i,$$

So it follows from the Gauss equation (2.1.15) that $\boldsymbol{S}^n$ has constant sectional curvature 1. This induced metric of $\boldsymbol{S}^n$ is called the standard metric.

2.2.4. Example. Let $\boldsymbol{R}^{n,1}$ denote the Lorentz space (N, g), i.e., $N = \boldsymbol{R}^{n+1}$ and g is the non-degenerate metric $dx_1^2 + \ldots + dx_n^2 - dx_{n+1}^2$ of index 1. So TN is an $\boldsymbol{O}(n,1)-$ bundle, and results similar to those of section 1.1 and 2.1 can be derived. Let ∇ denote the unique connection TN, that is torsion free and compatible with g. Let

$$M = \{x \in \boldsymbol{R}^{n,1} \mid g(x,x) = -1\},$$

and $X : M \to \boldsymbol{R}^{n,1}$ denote the inclusion map. Then the induced metric on M is positive definite, and X is a unit normal field on M, i.e., $g(x, v) = 0$, for all $v \in TM_x$. Let $e_{n+1} = X$ and $e_1, \ldots, e_{n+1}$ a local frame field on $\boldsymbol{R}^{n,1}$ such that

$$g(e_i, e_j) = \epsilon_i \delta_{ij}, \quad \text{where } \epsilon_1 = \ldots = \epsilon_n = -\epsilon_{n+1} = 1.$$

So $e_1(x), \ldots, e_n(x)$ are tangent to M for $x \in M$. Let $\omega^1, \ldots, \omega^{n+1}$ be the dual coframe, i.e., $\omega^A(e_B) = \delta_{AB}$. Let ω_A^B be the connection 1-form corresponding to $\bar{\nabla}$, i.e.,

$$\bar{\nabla} e_A = de_A = \sum_B \omega_A^B \otimes e_B.$$

By (1.1.6), we have

$$\epsilon_A \omega_A^B + \omega_B^A \epsilon_B = 0, \quad \text{and}$$

$$\bar{\nabla} \omega^A = -\sum_B \omega_B^A \otimes \omega^B.$$

Set

$$\omega_A = \epsilon_A \omega^A.$$

Since $e_{n+1} = X$, we have

$$de_{n+1} = \sum_i \omega_{n+1}^i \otimes e_i = dX = \sum_i \omega^i \otimes e_i.$$

So $\omega_{n+1}^i = \omega_i$. By the Gauss equation we have

$$\begin{aligned}\Omega_i^j &= -\omega_i^{n+1} \wedge \omega_{n+1}^j = -\omega_i^{n+1} \wedge \omega_j^{n+1} \\ &= -\omega^i \wedge \omega^j = -\omega_i \wedge \omega^j.\end{aligned}$$

So M has constant sectional curvature -1. From now on we will let $\boldsymbol{H}^n$ denote M with the induced metric from $\boldsymbol{R}^{n,1}$. $\boldsymbol{H}^n$ is also called the hyperbolic n-space.

It is well-known ([KN]) that every simply connected space form of sectional curvature c is isometric to $\boldsymbol{R}^n, \boldsymbol{S}^n, \boldsymbol{H}^n$ if $c = 0, 1$, or -1 respectively. We will let $N^n(c)$ denote these complete, simply connected Riemannian n-manifold with constant sectional curvature c.

2.2.5. Definition. An immersed hypersurface M^n of the simply connected space form $N^{n+1}(c)$ is called *totally umbilic* if $II(x) = f(x)I(x)$ for some smooth function $f : M \to R$.

In the following we will give examples of totally umbilic hypersurface of space forms.

2.2.6. Example. An affine n-plane E of $\boldsymbol{R}^{n+k}$ is totally geodesic. For we can choose e_α to be a constant orthonormal normal frame on E. Then $de_\alpha \equiv 0$. So we have $II \equiv 0$. Let $\boldsymbol{S}^n(x_0, r)$ be the sphere of radius r centered at x_0 in $\boldsymbol{R}^{n+1}$. Then $e_{n+1}(x) = (x - x_0)/r$ is a unit normal vector field on $\boldsymbol{S}^n(x_0, r)$, and $de_{n+1} = (1/r)\sum_i \omega_i \otimes e_i$. So we have $\omega_{i,n+1} = -(1/r)\omega_i$ and $II = -(1/r)I$, i.e., $\boldsymbol{S}^n(x_0, r)$ is totally umbilic, and has constant sectional curvature $\frac{1}{r^2}$.

2.2.7. Example. Let V be an affine hyperplane of $\boldsymbol{R}^{n+2}$, v_0 a unit normal vector of V, $\cos\theta$ the distance from the origin to V, and $M = \boldsymbol{S}^{n+1} \cap V$. Then $e_{n+1} = -\cot\theta\, X + \csc\theta\, v_0$ is a unit normal field to M in $\boldsymbol{S}^{n+1}$. Taking the exterior derivative of e_{n+1}, we obtain

$$de_{n+1} = -\cot\theta\, dX = -\cot\theta \sum \omega_i \otimes e_i,$$

i.e., $\omega_{i,n+1} = \cot\theta\, \omega_i$ and $II = \cot\theta\, I$. So M is totally umbilic in $\boldsymbol{S}^{n+1}$ with sectional curvature equal to $1 + \cot^2\theta = \csc^2\theta$, and M is t.g. in $\boldsymbol{S}^{n+1}$, if $\cos\theta = 0$ (or equivalently V is a linear hyperplane).

2.2.8. Example. Let v_0 be a non-zero vector of the Lorentz space $\boldsymbol{R}^{n+1,1}$, and

$$M = \{x \in \boldsymbol{R}^{n+1,1} \mid \langle x, x\rangle = -1,\ \langle x, v_0\rangle = a\}.$$

Then

$$0 = \langle dX, X\rangle = \sum_i \langle e_i, X\rangle \omega_i,$$

$$0 = \langle dX, v_0\rangle = \sum_i \langle e_i, v_0\rangle \omega_i.$$

So $\langle X, e_i\rangle = \langle v_0, e_i\rangle = 0$, which implies that

$$v_0 = -aX + be_{n+1}, \tag{2.2.2}$$

for some b. Note that

$$\langle v_0, v_0\rangle = -a^2 + b^2.$$

Taking the differential of (2.2.2), we have $\sum_i (a\omega_i + b\omega_{i,n+1})e_i = 0$. So

$$a\omega_i + b\omega_{i,n+1} = 0. \tag{2.2.3}$$

(i) If $\langle v_0, v_0\rangle = 1$, then $-a^2 + b^2 = 1$ and we may assume that $a = \sinh t_0$ and $b = \cosh t_0$. So (2.2.3) implies that $\omega_{i,n+1} = -\tanh\ t_0\ \omega_i$, i.e., $II = -\tanh\ t_0\ I$, i.e., M is totally umbilic with sectional curvature $-1 + \tanh^2 t_0 = -\operatorname{sech}^2 t_0$.

(ii) If $\langle v_0, v_0\rangle = 0$, then $-a^2 + b^2 = 0$, $\omega_{i,n+1} = \omega_i$. So $II = I$, and M is totally umbilic with sectional curvature 0.

(iii) If $\langle v_0, v_0\rangle = -1$, then $-a^2 + b^2 = -1$ and we may assume that $a = \cosh t_0$, $b = \sinh t_0$. Then we have $\omega_{i,n+1} = -\coth t_0$, which implies that $II = -\coth t_0 I$, i.e., M is totally umbilic with sectional curvature $-\operatorname{csch}^2 t_0$.

2.2.9. Proposition. *Suppose $X : M^n \to \boldsymbol{R}^{n+1}$ is an immersed, totally umbilic connected hypersurface, and $n > 1$. Then*

(i) $II = c\,I$ for some constant c

(ii) $X(M)$ is either contained in a hyperplane, or is contained in a standard hypersphere of $\boldsymbol{R}^{n+1}$.

PROOF. Let e_A, ω_i and ω_{AB} as before. By assumption we have

$$\omega_{i\alpha} = f(x)\omega_i. \tag{2.2.4}$$

Taking the exterior derivative of (2.2.4), and using (2.1.4) and (2.1.13), we obtain

$$\begin{aligned} d\omega_{i\alpha} &= df \wedge \omega_i + f \sum_j \omega_{ij} \wedge \omega_j \\ &= \sum_j f_j \omega_j \wedge \omega_i + f \sum_j \omega_{ij} \wedge \omega_j \\ &= \sum_j \omega_{ij} \wedge \omega_{j\alpha} = f \sum_j \omega_{ij} \wedge \omega_j. \end{aligned}$$

So $\sum_j f_j\omega_j \wedge \omega_i = 0$, which implies that $f_j = 0$ for all $j \neq i$. Since $n > 1$, $df = 0$, i.e., $f = c$ a constant.

If $c = 0$, then $\omega_{i\alpha} = 0$. So $de_\alpha = 0$, e_α is a constant vector v_0, and

$$d\langle X, v_0\rangle = \sum_i \langle e_i, v_0\rangle\omega_i = 0,$$

i.e., $X(M)$ is contained in a hyperplane. If $c \neq 0$, then $\omega_{i\alpha} = c\,\omega_i$ and

$$d\left(X + \frac{e_\alpha}{c}\right) = \sum_i \left(\omega_i e_i - \frac{1}{c}\omega_{i\alpha}\right) e_i = 0.$$

So $X + e_\alpha/c$ is equal to a constant vector $x_0 \in \boldsymbol{R}^{n+1}$, which implies that $\|X - x_0\|^2 = (1/c)^2$. ∎

The concept of totally umbilic was generalized to submanifolds in [NR] as follows:

2.2.10. Definition. An immersed submanifold M^n of the simply connected space form $N^{n+k}(c)$ is called *totally umbilic* if $II = \xi I$, where ξ is a parallel normal field on M.

2.2.11. Proposition. *Let $X : M^n \to \boldsymbol{R}^{n+k}$ be a connected, immersed totally umbilic submanifold, i.e., $II = \xi I$, where ξ is a parallel normal field on M. Then either*

(i) $\xi = 0$ and M is contained in an affine n-plane of $\boldsymbol{R}^{n+k}$, or

(ii) $X + (\xi/a)$ is a constant vector x_0, where $a = \|\xi\|$; and M is contained in a standard $n-$sphere of $\boldsymbol{R}^{n+k}$.

Proof. If $\xi = 0$, then $\omega_{i\alpha} = 0$ for all i, α. The Ricci equation (2.1.14) gives $d\omega_{\alpha\beta} = \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}$, which implies that the normal connection is flat. It follows from Proposition 1.1.5 that there exists a parallel orthonormal normal frame e^*_α. So we may assume that e_α are parallel, i.e., $\omega_{\alpha\beta} = 0$. This implies that $de_\alpha = 0$, i.e., the e_α are constant vectors. Then

$$d\langle X, e_\alpha\rangle = \langle dX, e_\alpha\rangle = 0,$$

so the $\langle X, e_\alpha\rangle$ are constant c_α, and M is contained in the $n-$plane defined by $\langle X, e_\alpha\rangle = c_\alpha$.

If $\xi \neq 0$, then $a = \|\xi\|$ is a constant, and we may assume that $\xi = ae_{n+1}$, $\nabla^\nu e_{n+1} = 0$, so

$$\omega_{i,n+1} = a\,\omega_i, \;\; \omega_{i\alpha} = 0, \;\; \omega_{n+1,\alpha} = 0, \tag{2.2.5}$$

for all $\alpha > (n+1)$. Then

$$d\left(X + \frac{e_{n+1}}{a}\right) = \sum_i \left(\omega_i - \frac{1}{a}\omega_{i,n+1}\right) e_i = 0,$$

so $X + (e_{n+1}/a)$ is a constant vector x_0. Using (2.2.5), we have

$$d(e_1 \wedge \ldots \wedge e_{n+1}) = \sum_{i,\alpha>n+1} e_1 \wedge \ldots \omega_{i\alpha} e_\alpha \wedge e_{i+1} \ldots \wedge e_{n+1}$$
$$+ \sum_{\alpha>n+1} e_1 \wedge \ldots \wedge e_n \wedge \omega_{n+1,\alpha} e_\alpha = 0.$$

Hence the span of $e_1(x), \ldots, e_{n+1}(x)$ is a fixed $(n+1)-$dimensional linear subspace V of $\boldsymbol{R}^{n+k}$ for all $x \in M$. But $X = x_0 - e_{n+1}/a$, so M is contained in the intersection of the affine $(n+1)-$plane $x_0 + V$ and the hypersphere of $\boldsymbol{R}^{n+k}$ of center x_0 and radius $1/a$. ∎

Exercises.

1. Prove the analogue of Proposition 2.2.9 for totally umbilic hypersurfaces of $\boldsymbol{S}^{n+1}$ and $\boldsymbol{H}^{n+1}$.
2. Prove the analogue of Proposition 2.2.11 for totally umbilic submanifolds of $\boldsymbol{S}^{n+1}$ and $\boldsymbol{H}^{n+1}$.

2.3. Fundamental theorem for submanifolds of space forms

Given a submanifold M^n of a complete, simply connected space form, we have associated to M an orthogonal bundle (the normal bundle $\nu(M)$) with a compatible connection, and also the first and second fundamental forms of M. Together these satisfy the Gauss, Codazzi and Ricci equations. In the following, we will show that these data determine the submanifold up to isometries of of the space form.

2.3.1. Theorem. *Suppose (M^n, g) is a Riemannian manifold, ξ is a smooth rank k orthogonal vector bundle over M with a compatible connection ∇^1, and $A : \xi \to S^2T^*M$ is a vector bundle morphism. Let $e_1, \ldots, e_n$ be a local orthonormal frame field on TM, $\omega_1, \ldots, \omega_n$ its dual coframe, and ω_{ij}*

the corresponding Levi-Civita connection 1-form, i.e., ω_{ij} *is determined by the structure equations*

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0. \tag{2.3.1}$$

Let $e_{n+1}, \ldots, e_{n+k}$ *be an orthonormal local frame field of* ξ, *and* $\omega_{\alpha\beta}$ *is the* $o(k)-$*valued 1-form corresponds to* ∇^1. *Let* $\omega_{i\alpha}$ *be the 1-forms determined by the vector bundle morphism* A*:*

$$A(e_\alpha) = \sum_i \omega_{i\alpha} \otimes \omega_i.$$

Set $\omega_{\alpha i} = -\omega_{i\alpha}$, *and suppose* ω_{AB} *satisfy the Gauss, Codazzi and Ricci equations:*

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j}, \tag{2.3.2}$$

$$d\omega_{i\alpha} = \sum_k \omega_{ik} \wedge \omega_{k\alpha} + \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{2.3.3}$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \sum_i \omega_{\alpha i} \wedge \omega_{i\beta}. \tag{2.3.4}$$

Then given $x_0 \in M$, $p_0 \in \boldsymbol{R}^{n+k}$, *and an orthonormal basis* $v_1, \ldots, v_{n+k}$ *of* $\boldsymbol{R}^{n+k}$, *for small enough connected neighborhoods* U *of* x_0 *in* M *there is a unique immersion* $f : U \to \boldsymbol{R}^{n+k}$ *and vector bundle isomorphism* $\eta : \xi \to \nu(M)$ *such that* $f(x_0) = p_0$ *and* $v_1, \ldots, v_n$ *are tangent to* $f(U)$ *at* p_0, g *is the first fundamental form,* $A(\eta(e_\alpha))$ *are the shape operators of the immersion, and* ∇^1 *corresponds to the induced normal connection under the isomorphism* η.

PROOF. It follows from the definition of ω_{AB} that $\varpi = (\omega_{AB})$ is an $o(n+k)-$valued 1-form on M. Then (2.3.2)-(2.3.4) imply that ϖ satisfies Maurer-Cartan equation:

$$d\varpi = \varpi \wedge \varpi,$$

which is the integrability condition for the first order system

$$d\varphi = \varpi\varphi.$$

So there exist a small neighborhood U of x_0 in M and maps $e_A : U \to \boldsymbol{R}^{n+k}$ such that

$$de_A = \sum_B \omega_{AB} \otimes e_B,$$

where $e_A(x_0) = v_A$ and $\{e_A(x)\}$ is orthonormal for all $x \in U$. To solve the system

$$dX = \sum_i \omega_i \otimes e_i,$$

we prove the right hand side is a closed 1-form as follows:

$$\begin{aligned} d\left(\sum_i \omega_i \otimes e_i\right) &= \sum_i d\omega_i \otimes e_i - \omega_i \wedge \sum_A \omega_{iA} \otimes e_A \\ &= \sum_i (d\omega_i - \sum_j \omega_{ij} \wedge \omega_j) \otimes e_i - \sum_{i,j,\alpha} (\omega_i \wedge \omega_{i\alpha}) \otimes e_\alpha, \\ &= \sum_i (d\omega_i - \sum_j \omega_{ij} \wedge \omega_j) \otimes e_i - \sum_{i,j,\alpha} h_{i\alpha j} \omega_i \wedge \omega_j \otimes e_\alpha, \end{aligned} \tag{2.3.5}$$

the structure equations (2.3.1) implies the first term of (2.3.5) is zero and $h_{i\alpha j} = h_{j\alpha i}$ implies the second term is zero. ■

2.3.2. Corollary. *Let $\varphi_0 : (M, g) \to \boldsymbol{R}^{n+k}$ and $\varphi_1 : (M, g) \to \boldsymbol{R}^{n+k}$ be immersions. Suppose that they have the same first, second fundamental forms and the normal connections. Then there is a unique orthogonal transformation B and a vector $v_0 \in \boldsymbol{R}^{n+k}$ such that $\varphi_0(x) = B(\varphi_1(x)) + v_0$.*

The group G_m of isometries of $\boldsymbol{R}^m$ is the semi-direct product of the orthogonal group $\boldsymbol{O}(m)$ and the translation group $\boldsymbol{R}^m$; $gT_v g^{-1} = T_{g(v)}$, where $g \in \boldsymbol{O}(m)$ and T_v is the translation defined by v. So its Lie algebra $\mathcal{G}_m$ can be identified as the Lie subalgebra of $gl(m+1)$ consisting of matrices of the form

$$\begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix},$$

where $A \in o(m)$, and v is an $m \times 1$ matrix.

Let M, ω_i, ω_{AB} be as in Theorm 2.3.1. Let τ denote the following $gl(n+k+1)-$valued 1-form on M:

$$\tau = \begin{pmatrix} \varpi & \theta \\ 0 & 0 \end{pmatrix},$$

where $\varpi = (\omega_{AB})$ is an $o(n+k)-$valued 1-form, and θ is an $(n+k)\times 1$-valued 1-form $(\omega_1, \ldots, \omega_n, 0, \ldots, 0)^t$.

Then τ is a $\mathcal{G}_{n+k}-$valued 1-form on M. The Gauss, Codazzi and Ricci equations are equivalent to the Maurer-Cartan equations

$$d\tau = \tau \wedge \tau.$$

Hence there exists a unique map $F : U \to \boldsymbol{GL}(n+k+1)$ such that $dF = \tau F$, the m^{th} row of $F(x_0)$ is $(v_m, 0)$ for $m \leq (n+k)$, and the $(n+k+1)^{st}$ row of $F(x_0)$ is $(p_0, 1)$. Then the $(n+k+1)^{st}$ row is of the form $(X, 1)$,and X is the immersion of M into $\boldsymbol{R}^{n+k}$.

A similar argument will give the fundamental theorem for submanifolds of the sphere and the hyperbolic space. For $\boldsymbol{S}^{n+k}$, we have $F : U \to \boldsymbol{O}(n+k+1)$, and the $(n+k+1)^{st}$ row of F gives the immersion of M into $\boldsymbol{S}^{n+k}$. For $\boldsymbol{H}^{n+k}$, we have $F : U \to \boldsymbol{O}(n+k, 1)$, and the $(n+k+1)^{st}$ row of F gives the immersion of M into $\boldsymbol{H}^{n+k}$.

2.3.3. Theorem. *Given $(M, g), \xi, \nabla^1, A, \omega_i, \omega_{AB}$ as in Theorem 2.3.1. Let c denote the integer $0, 1$ or -1. Set*

$$\tau_c = \begin{pmatrix} \varpi & \theta \\ -c\theta^t & 0 \end{pmatrix},$$

where $\varpi = (\omega_{AB})$ is an $o(n+k)$ valued 1-form, and θ is the $(n+k) \times 1$ valued 1-form $(\omega_1, \ldots, \omega_n, 0, \ldots, 0)^t$ on M. Then

(i) τ_c is a $\mathcal{G}_{n+k}, o(n+k+1)$, or $o(n+k, 1)-$valued 1-form on M for $c = 0, 1$ or -1 respectively.

(ii) If τ_c satisfies the Maurer-Cartan equations

$$d\tau_c = \tau_c \wedge \tau_c,$$

then

(1) the system

$$dF = \tau_c F \tag{2.3.6}$$

for the $\boldsymbol{GL}(n+k+1)-$valued map F is solvable,

(2) if F is a solution for (2.3.6) and X denotes the $(n+k+1)^{st}$ row, then $X : M \to N^{n+k}(c)$ is an isometric immersion such that g, ξ, ∇^1, A are the first fundamental form, normal bundle, induced normal connection, and the shape operators respectively for the immersion X.

(3) The data g, ξ, ∇^1, A determine the isometric immersions of M into $N^{n+k}(c)$ uniquely up to isometries of $N^{n+k}(c)$.

Exercises.

1. Show that the group of isometries of $(\boldsymbol{S}^n, g)$ is $\boldsymbol{O}(n+1)$, where g is the standard metric of $\boldsymbol{S}^n$.
2. Show that the group of isometries of the hyperbolic space $(\boldsymbol{H}^n, g)$ is $\boldsymbol{O}(n, 1)$.

3. Prove Theorem 2.3.3 for $\boldsymbol{S}^{n+k}$ and $\boldsymbol{H}^{n+k}$.
4. Show that the $n-1$ smooth functions $k_1(s), \ldots, k_{n-1}(s)$ obtained in Ex. 3 of section 2.1 determine the curve uniquely up to rigid motions (this is the classical fundamental theorem for curves in $\boldsymbol{R}^n$).