Submanifold Theory.

In this chapter we review some basic facts concerning connections and the existence theory for systems of first order partial differential equations. These are basic tools for the study of submanifold geometry. A connection is defined both globally as a differential operator (Koszul's definition) and locally as connection 1-forms (Cartan's formulation). While the global definition is better for interpreting the geometry, the local definition is easier to compute with. A first order system of partial differential equations can be viewed as a system of equations for differential 1-forms, and the associated existence theory is referred to as the Frobenius theorem.

1.1. Connections on a vector bundle

Let M be a smooth manifold, ξ a smooth vector bundle of rank k on M, and $C^\infty(\xi)$ the space of smooth sections of ξ.

1.1.1. Definition. A *connection* for ξ is a linear operator

$$\nabla : C^\infty(\xi) \to C^\infty(T^*M \otimes \xi)$$

such that

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for every $s \in C^\infty(\xi)$ and $f \in C^\infty(M)$. We call $\nabla(s)$ the *covariant derivative* of s.

If ξ is trivial, i.e., $\xi = M \times \boldsymbol{R}^k$, then $C^\infty(\xi)$ can be identified with $C^\infty(M, \boldsymbol{R}^k)$ by $s(x) = (x, f(x))$. The differential of maps gives a trivial connection on ξ, i.e., $\nabla s(x) = (x, df_x)$. The collection of all connections on ξ can be described as follows. We call k smooth sections $s_1, \ldots, s_k$ of ξ a *frame field* of ξ if $s_1(x), \ldots, s_k(x)$ is a basis for the fiber ξ_x at every $x \in M$. Then every section of ξ can be uniquely written as a sum $f_1 s_1 + \ldots + f_k s_k$, where f_i are uniquely determined smooth functions on M. A connection ∇ on ξ is uniquely determined by $\nabla(s_1), \ldots, \nabla(s_k)$, and these can be completely arbitrary smooth sections of the bundle $T^*M \otimes \xi$. Each of the sections $\nabla(s_i)$ can be written uniquely as a sum $\sum \omega_{ij} \otimes s_j$, where (ω_{ij}) is an arbitrary $n \times n$

matrix of smooth real-valued one forms on M. In fact, given $\nabla(s_1), \ldots, \nabla(s_k)$ we can define ∇ for an arbitrary section by the formula

$$\nabla(f_1 s_1 + \cdots + f_k s_k) = \sum (df_i \otimes s_i + f_i \nabla(s_i)).$$

(Here and in the sequel we use the convention that $\sum$ always stands for the summation over all indices that appear twice).

Suppose U is a small open subset of M such that $\xi|U$ is trivial. A frame field $s_1, \ldots, s_k$ of $\xi|U$ is called a *local frame field* of ξ on U.

It follows from the definition that a connection ∇ is a local operator, that is, if s vanishes on an open set U then ∇s also vanishes on U. In fact, since $s(p) = 0$ and $ds_p = 0$ imply $\nabla s(p) = 0$, ∇ is a first order differential operator ([Pa3]).

Since a connection is a local operator, it makes sense to talk about its restriction to an open subset of M. If a collection of open sets U_α covers M such that $\xi|U_\alpha$ is trivial, then a connection ∇ on ξ is uniquely determined by its restrictions to the various U_α. Let $s_1, \ldots, s_k$ be a local frame field on U_α, then there exists unique $n \times n$ matrix of smooth real-valued one forms (ω_{ij}) on U_α such that $\nabla(s_i) = \sum \omega_{ij} \otimes s_j$.

Let $\boldsymbol{GL}(k)$ denote the Lie group of the non-singular $k \times k$ real matrices, and $gl(k)$ its Lie algebra. If s_i and s_i^* are two local frame fields of ξ on U, then there is a uniquely determined smooth map $g = (g_{ij}) : U \to \boldsymbol{GL}(k)$ such that $s_i^* = \sum g_{ij} s_j$. Let $g^{-1} = (g^{ij})$ denote the inverse of g, so that $s_i = \sum g^{ij} s_j^*$. Suppose

$$\nabla s_i = \sum \omega_{ij} \otimes s_j, \qquad \nabla s_i^* = \sum \omega_{ij}^* \otimes s_j^*.$$

Let $\omega = (\omega_{ij})$ and $\omega^* = (\omega_{ij}^*)$. Since

$$\begin{aligned}
\nabla s_i^* &= \nabla(\sum g_{im} s_m) = \sum dg_{im} s_m + g_{im} \nabla s_m \\
&= \sum_m (dg_{im} + \sum_k g_{ik} \omega_{km}) s_m \\
&= \sum_j (\sum_m dg_{im} g^{mj} + \sum_{m,k} g_{ik} \omega_{km} g^{mj}) s_j^* \\
&= \sum_j \omega_{ij}^* s_j^*,
\end{aligned}$$

we have

$$\omega^* = (dg) g^{-1} + g \omega g^{-1}.$$

Given an open cover U_α of M and local frame fields $\{s_i^\alpha\}$ on U_α, suppose $s_i^\alpha = \sum (g_{ij}^{\alpha\beta}) s_j^\beta$ on $U_\alpha \cap U_\beta$. Let $g^{\alpha\beta} = (g_{ij}^{\alpha\beta})$. Then a connection on ξ is defined by a collection of $gl(k)-$valued 1-forms ω^α on U_α, such that on $U_\alpha \cap U_\beta$ we have $\omega^\beta = (dg^{\alpha\beta})(g^{\alpha\beta})^{-1} + g^{\alpha\beta} \omega^\alpha (g^{\alpha\beta})^{-1}$.

Identify $T^*M \otimes \xi$ with $L(TM, \xi)$, and let $\nabla_X s$ denote $(\nabla s)(X)$. For $X, Y \in C^\infty(TM)$ and $s \in C^\infty(\xi)$ we define

$$K(X,Y)(s) = -(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(s). \tag{1.1.1}$$

It follows from a direct computation that

$$K(Y,X) = -K(X,Y),$$

$$K(fX,Y) = K(X,fY) = fK(X,Y),$$
$$K(X,Y)(fs) = fK(X,Y)(s).$$

Hence K is a smooth section of $L(\xi \otimes \bigwedge^2 TM, \xi) \simeq L(\xi, \bigwedge^2 T^*M \otimes \xi)$.

1.1.2. Definition. This section K of the vector bundle $L(\xi, \bigwedge^2 T^*M \otimes \xi)$ is called the *curvature* of the connection ∇.

Recall that the bracket operation on vector fields and the exterior differentiation on p forms are related by

$$\begin{aligned} d\omega(X_0, \ldots, X_p) &= \sum_i (-1)^i X_i \omega(X_0, \ldots, \hat{X}_i, \ldots, X_p) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned} \tag{1.1.2}$$

Suppose $s_1, \ldots, s_k$ is a local frame field on U, and $\nabla s_i = \sum \omega_{ij} \otimes s_j$. Then there exist 2-forms Ω_{ij} such that

$$K(s_i) = \sum \Omega_{ij} \otimes s_j.$$

Since

$$\begin{aligned} -K(X,Y)(s_i) &= \nabla_X \nabla_Y s_i - \nabla_Y \nabla_X s_i - \nabla_{[X,Y]} s_i \\ &= \nabla_X (\sum \omega_{ij}(Y) s_j) - \nabla_Y (\sum \omega_{ij}(X) s_j) \\ &\quad - \sum \omega_{ij}([X,Y]) s_j \\ &= \sum (X(\omega_{ij}(Y)) - Y(\omega_{ij}(X)) - \omega_{ij}([X,Y])) s_j \\ &\quad + \sum (\omega_{ij}(Y)\omega_{jk}(X) - \omega_{ij}(X)\omega_{jk}(Y)) s_k \\ &= \sum (d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj})(X,Y)\, s_j, \end{aligned}$$

we have

$$-\Omega_{ij} = d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj}.$$

Thus K can is locally described by the $k \times k$ matrix $\Omega = (\Omega_{ij})$ of 2-forms just as ∇ is defined locally by the matrix $\omega = (\omega_{ij})$ of 1-forms. In matrix notation, we have

$$-\Omega = d\omega - \omega \wedge \omega. \tag{1.1.3}$$

Let $g = (g_{ij}) : U \to \boldsymbol{GL}(k)$ be a smooth map and let $\omega = (dg)g^{-1}$. Then ω is a $gl(k)-$ valued 1-form on U, satisfying the so-called *Maurer-Cartan equation*

$$d\omega = \omega \wedge \omega.$$

Conversely, given a $gl(k)-$ valued 1-form on U with $d\omega = \omega \wedge \omega.$, it follows from Frobenius theorem (cf. 1.4) that given any $x_0 \in U$ and $g_0 \in \boldsymbol{GL}(k)$ there is a neighborhood U_0 of x_0 in U and a smooth map $g = (g_{ij}) : U_0 \to \boldsymbol{GL}(k)$ such that $g(x_0) = g_0$ and $(dg)g^{-1} = \omega$. Thus $d\omega = \omega \wedge \omega$ is a necessary and sufficient condition for being able to solve locally the system of first order partial differential equations:

$$dg = \omega g. \tag{1.1.4}$$

Let e_i denote the i^{th} row of the matrix g and $\omega = (\omega_{ij})$. Then (1.1.4) can be rewritten as

$$de_i = \sum_j \omega_{ij} \otimes e_j.$$

1.1.3. Definition. A smooth section s of $\xi|U$ is *parallel* with respect to ∇ if $\nabla s = 0$ on U.

1.1.4. Definition. A connection is *flat* if its curvature is zero.

1.1.5. Proposition. *The connection ∇ on ξ is flat if and only if there exist local parallel frame fields.*

PROOF. Let s_i and $\omega = (\omega_{ij})$ be as before. Suppose $\Omega = 0$, then ω satisfies the Maurer- Cartan equation $d\omega = \omega \wedge \omega$. So locally there exists a $\boldsymbol{GL}(k)-$ valued map $g = (g_{ij})$ such that $(dg)g^{-1} = \omega$. Let $g^{-1} = (g^{ij})$, and $s_i^* = \sum g^{ij} s_j$. Then $\nabla s_i^* = \sum \omega_{ij}^* \otimes s_j^*$, and

$$\begin{aligned} \omega^* &= d(g^{-1})g + g^{-1}\omega g \\ &= -g^{-1}(dg)g^{-1}g + g^{-1}(dg)g^{-1}g = 0 \end{aligned}.$$

So s_i^* is a parallel frame. ∎

1.1.6. Definition. A connection ∇ on ξ is called globally flat if there exists a parallel frame field defined on the whole manifold M.

1.1.7. Example. Let ξ be the trivial vector bundle $M \times \boldsymbol{R}^k$, and ∇ the trivial connection on ξ given by the differential of maps. Then a section $s(x) = (x, f(x))$ is parallel if and only if f is a constant map, so ∇ is globally flat.

1.1.8. Remarks.

(1) If ξ is not a trivial bundle then no connection on ξ can be globally flat.

(ii) A flat connection need not be globally flat. For example, let M be the Möbius band $[0,1] \times \boldsymbol{R}/\sim$ (where $(0,t) \sim (1,-t)$)). Then the trivial connection on $[0,1] \times R$ induces a flat connection on TM. But since TM is not a product bundle this connection is not globally flat.

Given $x_0 \in M$, a smooth curve $\alpha : [0,1] \to M$ such that $\alpha(0) = x_0$ and $v_0 \in \xi_{x_0}$ (the fiber of ξ over x_0), then the following first order ODE

$$\nabla_{\alpha'(t)} v = 0, \;\; v(0) = v_0, \tag{1.1.5}$$

has a unique solution. A solution of (1.1.5) is called a parallel field along α, and $v(1)$ is called the parallel translation of v_0 along α to $\alpha(1)$. Let $P(\alpha) : \xi_{x_0} \to \xi_{x_0}$ be the map defined by $P(\alpha)(v_0) = v(1)$ for closed curve α such that $\alpha(0) = \alpha(1) = x_0$. The set of all these $P(\alpha)$ is a subgroup of $\boldsymbol{GL}(\xi_{x_0})$, that is called the *holonomy group* of ∇ with respect to x_0. It is easily seen that ∇ is globally flat if and only if the holonomy group of ∇ is trivial.

1.1.9. Definition. A local frame s_i of vector bundle ξ is called parallel at a point x_0 with respect to the connection ∇, if $\nabla s_i(x_0) = 0$ for all i.

1.1.10. Proposition. *Let ∇ be a connection on the vector bundle ξ on M. Given $x_0 \in M$, then there exist an open neighborhood U of x_0 and a frame field defined on U, that is parallel at x_0.*

PROOF. Let s_i be a local frame field, $\nabla s_i = \sum_j \omega_{ij} \otimes s_j$, and $\omega = (\omega_{ij})$. Let $x_1, \ldots, x_n$ be a local coordinate system near x_0, and $\omega = \sum_i f_i(x)\, dx_i$, for some smooth $gl(k)$ valued maps f_i. Let $a_i = f_i(x_0)$. Then $a_i \in gl(k)$, and $g^{-1}dg + \omega = 0$ at x_0, where $g(x) = \exp(\sum_i x_i a_i)$. So we have $dg\, g^{-1} + g\omega g^{-1} = 0$ at x_0, i.e., $s_i^* = \sum g_{ij} s_j$ is parallel at x_0, where $g = (g_{ij})$. ∎

Let $\boldsymbol{O}(m,k)$ denote the Lie group of linear isomorphism that leave the following bilinear form on $\boldsymbol{R}^{m+k}$ invariant:

$$(x,y) = \sum_{i=1}^{m} x_i y_i - \sum_{j=1}^{k} x_{m+j} y_{m+j}.$$

So an $(m+k) \times (m+k)$ matrix A is in $\boldsymbol{O}(m,k)$ if and only if

$$A^t E A = E, \quad \text{where} \quad E = \quad \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1),$$

and its Lie algebra is:

$$o(m,k) = \{A \in gl(m+k) \mid A^t E + EA = 0\}.$$

1.1.11. Definition. A rank $(m+k)$ vector bundle ξ is called an $\boldsymbol{O}(m,k)-$bundle (an orthogonal bundle if $k=0$) if there is a smooth section g of $S^2(\xi^*)$ such that $g(x)$ is a non-degenerate bilinear form on ξ_x of index k for all $x \in M$. A connection ∇ on ξ is said to be *compatible* with g if

$$X(g(s,t)) = g(\nabla_X s, t) + g(s, \nabla_X t),$$

for all $X \in C^\infty(TM), s, t \in C^\infty(\xi)$.

Suppose $s_1, \ldots, s_{m+k}$ is a local frame field, $g(s_i, s_j) = g_{ij}$, and

$$\nabla s_i = \sum_j \omega_{ij} \otimes s_j.$$

Then ∇ is compatible with g if and only if

$$\omega G + G\omega^t = dG,$$

where $\omega = (\omega_{ij})$ and $G = (g_{ij})$. In particular, if $G = E$ as above, then

$$\omega E + E\omega^t = 0, \tag{1.1.6}$$

i.e., ω is an $o(m,k)-$ valued 1-form on M.

The collection of all connections on ξ does not have natural vector space structure. However it *does* have a natural affine structure. In fact if ∇_1 and ∇_2 are two connections on ξ and f is a smooth function on M then the linear combination $f\nabla_1 + (1-f)\nabla_2$ is again a well-defined connection on ξ, and $\nabla_1 - \nabla_2$ is a smooth section of $L(\xi, T^*M \otimes \xi)$.

Next we consider connections on induced vector bundles. Given a smooth map $\varphi : N \to M$ we can form the induced vector bundle $\varphi^*\xi$. Note that there are canonical maps

$$\varphi^* : C^\infty(\xi) \to C^\infty(\varphi^*\xi),$$

$$\varphi^* : C^\infty(T^*M) \to C^\infty(T^*N).$$

So there is also a canonical map

$$\varphi^* : C^\infty(T^*M \otimes \xi) \to C^\infty(T^*N \otimes \varphi^*\xi).$$

1.1.12. Lemma. *To each connection ∇ on ξ there corresponds a unique connection $\varphi^*\nabla$ on the induced bundle $\varphi^*\xi$ so that*

$$(\varphi^*\nabla)(\varphi^*s) = \varphi^*(\nabla s).$$

For example, given a local frame field $s_1, \ldots, s_k$ over an open subset U of M with $\nabla(s_i) = \sum \omega_{ij} \otimes s_j$, then

$$(\varphi^*\nabla)(\varphi^*s_i) = \sum \varphi^*\omega_{ij} \otimes \varphi^*s_j,$$

i.e., the connection 1-form for $\varphi^*\nabla$ is $\varphi^*\omega_{ij}$.

Suppose ∇_1 and ∇_2 are connections on the vector bundles ξ_1 and ξ_2 over M. Then there is a natural connection ∇ on $\xi_1 \otimes \xi_2$ that satisfies the usual "product rule", i.e.,

$$\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2)$$

1.2. Levi-Civita Connections

Let M be an n-dimensional smooth manifold, and g a smooth metric on M, i.e., $g \in C^\infty(S^2T^*M)$, such that $g(x)$ is positive definite for all $x \in M$ (or equivalently, TM is an orthogonal bundle). Suppose ∇ is a connection on TM, and given vector fields X and Y on M let

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

It follows from a direct computation that we have

$$T(fX, Y) = T(X, fY) = fT(X, Y), \qquad T(X, Y) = -T(Y, X).$$

So T is a section of $\bigwedge^2 T^*M \otimes TM$, called the *torsion tensor* of ∇.

1.2.1. Definition. A connection ∇ on TM is said to be *torsion free* if its torsion tensor T is zero.

Let $e_1, \ldots, e_n$ be a local orthonormal tangent frame field on an open subset U of M, i.e., $e_1(x), \ldots, e_n(x)$ forms an orthonormal basis for TM_x for all $x \in M$. We denote by $\omega_1, \ldots, \omega_n$ the 1-forms in U dual to $e_1, \ldots, e_n$, i.e., satisfying $\omega_i(e_j) = \delta_{ij}$. Suppose

$$\nabla e_i = \sum \omega_{ij} \otimes e_j.$$

It follows from (1.1.6) that ∇ is compatible with g if and only if $\omega_{ij} + \omega_{ji} = 0$. The torsion is zero if and only if

$$[e_i, e_j] = \sum(\omega_{jk}(e_i) - \omega_{ik}(e_j))e_k. \tag{1.2.1}$$

Then (1.1.2) and (1.2.1) imply that

$$d\omega_k = \sum \omega_l \wedge \omega_{lk}.$$

Let c_{ijk}, and γ_{ijk} be the coefficients of $[e_i, e_j]$ and ω_{ij} respectively, i.e.,

$$[e_i, e_j] = \sum c_{ijk} e_k,$$

and

$$\omega_{ij} = \sum \gamma_{ijk} \omega_k.$$

Then we have:

$$\gamma_{ijk} = -\gamma_{jik}, \qquad \gamma_{jki} - \gamma_{ikj} = c_{ijk}.$$

This system of linear equations for the γ_{ijk} has a unique solution that is easily found explicitly; namely

$$\gamma_{ijk} = \frac{1}{2}(-c_{ijk} + c_{jki} + c_{kij}).$$

Equivalently, $\nabla_Z X$ is determined by the following equation:

$$\begin{aligned} g(\nabla_Z X, Y) = \frac{1}{2}\{ & g([Y,Z],X) + g([Z,X],Y) - g([X,Y],Z) \\ & + X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\} \end{aligned}, \tag{1.2.2}$$

for all smooth vector field Y on M. So we have:

1.2.2. Theorem. *There is a unique connection ∇ on a Riemannian manifold (M, g) that is torsion free and compatible with g. This connection is called the Levi-Civita connection of g. If $e_1, \ldots, e_n$ is a local orthonormal frame field of TM and $\omega_1, \ldots, \omega_n$ is its dual coframe, then the Levi-Civita connection 1-form ω_{ij} of g are characterized by the following "structure equations":*

$$d\omega_i = \sum \omega_j \wedge \omega_{ji}, \qquad \omega_{ij} + \omega_{ji} = 0,$$

or equivalently

$$d\omega_i = \sum \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0. \tag{1.2.3}$$

1.2.3. Definition. The curvature of the Levi-Civita connection of (M, g) is called the *Riemann tensor* of g.

Let $\omega = (\omega_{ij})$ be the Levi-Civita connection 1-form of g, and $\Omega = (\Omega_{ij})$ the Riemann tensor. It follows from (1.1.3) that we have

$$d\omega - \omega \wedge \omega = -\Omega. \tag{1.2.4}$$

This is called the *curvature equation*. Write

$$\Omega_{ij} = \frac{1}{2} \sum_{k \neq l} R_{ijkl} \omega_k \wedge \omega_l \tag{1.2.5}$$

with $R_{ijkl} = -R_{ijlk}$. It is easily seen that

$$R_{klij} = g(K(e_i, e_j)(e_k), e_l).$$

Next we will derive the first Bianchi identity. Taking the exterior derivative of (1.2.3) and using (1.2.4), we get

$$\sum_j \Omega_{ij} \wedge \omega_j = \frac{1}{2} \sum_{j,k,l} R_{ijkl} \omega_k \omega_l \omega_j = 0,$$

which implies the first Bianchi identity

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{1.2.6}$$

If the dimension of M is 2 and $\Omega_{12} = K\omega_1 \wedge \omega_2$, then K is a well-defined smooth function on M, called the *Gaussian curvature* of g. The curvature equation (1.2.4) gives

$$d\omega_{12} = -K\omega_1 \wedge \omega_2.$$

Let M be a Riemannian n-manifold, E a linear 2-plane of TM_p and v_1, v_2 is an orthonormal basis of E. Then $g(K(v_1, v_2)(v_1), v_2)$ is independent of the choice of v_1, v_2 and depends only on E; it is called the *sectional curvature* $K(E)$ of the 2-plane E with respect to g. In fact $K(E)$ is equal to the Gaussian curvature of the surface $\exp_p(B)$ at p with induced metric from M, where B is a small disk centered at the origin in E. The metric g is said to have *constant sectional curvature* c if $K(E) = c$ for all two planes. It is easily seen that g has constant sectional curvature c if and only if

$$\Omega_{ij} = c\,\omega_i \wedge \omega_j.$$

The metric g has positive sectional curvature if $K(E) > 0$ for all two planes E.

The *Ricci curvature*,

$$\mathrm{Ric} = \sum r_{ij}\,\omega_i \otimes \omega_j,$$

of g is defined by the following contraction of the Riemann tensor Ω:

$$r_{ij} = \sum_k R_{ikjk}.$$

The *scalar curvature*, μ, of g is the trace of the Ricci curvature, i.e.,

$$\mu = \sum_i r_{ii}.$$

It is easily seen that Ric is a symmetric 2-tensor. We say that Ric is positive, negative, non-positive, or non-negative if it has the corresponding property as a quadratic form, e.g, $\mathrm{Ric} > 0$ if $\mathrm{Ric}(X,X) > 0$ for all non-zero tangent vector X. The metric g is called an *Einstein metric*, if the Ricci curvature $Ric = cg$ for some constant c.

The study of constant scalar curvature metrics and Einstein metrics plays very important role in geometry, partial differential equations and physics, for example see [Sc1],[KW] and [Be].

1.2.4. Example. Suppose $g = A^2(x,y)dx^2 + B^2(x,y)dy^2$ is a metric on an open subset U of $\boldsymbol{R}^2$. Set

$$\omega_1 = Adx, \quad \omega_2 = Bdy, \quad \omega_{12} = p\omega_1 + q\omega_2.$$

Then using the structure equations:

$$d\omega_1 = \omega_{12} \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_{12},$$

we can solve p and q explicitly. Let f_x denote $\frac{\partial f}{\partial x}$. We have

$$\omega_{12} = -\frac{A_y}{B}\,dx + \frac{B_x}{A}\,dy,$$

$$K = \frac{-1}{AB}\left[\left(\frac{A_y}{B}\right)_y + \left(\frac{B_x}{A}\right)_x\right].$$

1.2.5. Example. Let $M = \boldsymbol{R}^n$, and $g = dx_1^2 + \ldots + dx_n^2$ the standard metric. A smooth vector field u of $\boldsymbol{R}^n$ can be identified as a smooth map

$u = (u_1, \ldots, u_n) : \boldsymbol{R}^n \to \boldsymbol{R}^n$. Then the constant vector fields $e_i(x) = (0, \ldots, 1, \ldots, 0)$ with 1 at the i^{th} place form an orthonormal frame of $T\boldsymbol{R}^n$, and $\omega_i = dx_i$ are the dual coframe. It is easily seen that $\omega_{ij} = 0$ is the solution of the structure equations (1.2.3). So $\nabla e_i = 0$, and the curvature forms are

$$\Omega = -d\omega + \omega \wedge \omega = 0.$$

If $u = (u_1, \ldots, u_n)$ is a vector field, then $u = \sum u_i e_i$ and

$$\nabla u = \sum du_i \otimes e_i = (du_1, \ldots, du_n),$$

i.e., the covariant derivative of the tangent vector field u is the same as the differential of u as a map.

Exercises.

1. Using the first Bianchi identity and the fact that R_{ijkl} is antisymmetric with respect to ij and kl, show that $R_{ijkl} = R_{klij}$, i.e., if we identify T^*M with TM via the metric, then the Riemann tensor Ω is a self-adjoint operator on $\bigwedge^2 TM$. (Note that if g has positive sectional curvature then Ω is a positive operator, but the converse is not true.)
2. Show that Ricci curvature tensor is a section of $S^2(T^*M)$, i.e., $r_{ij} = r_{ji}$.
3. Suppose (M, g) is a Riemannian 3-manifold. Show that the Ricci curvature Ric determines the Riemann curvature Ω. In fact since Ric is symmetric, there exists a local orthonormal frame e_1, e_2, e_3 such that $\mathrm{Ric} = \sum \lambda_i \omega_i \otimes \omega_i$. Then R_{ijkl} can be solved explicitly in terms of the λ_i from the linear system $\sum_k R_{ikjk} = \lambda_i \delta_{ij}$.
4. Let (M^n, g) be a Riemannian manifold with $n \geq 3$. Suppose that for all 2−plane E_x of TM_x we have $K(E_x) = c(x)$, depending only on x. Show that $c(x)$ is a constant, i.e., independent of x.
5. Let G be a Lie group, V a linear space, and $\rho : G \to \boldsymbol{GL}(V)$ a group homomorphism, i.e., a representation. Then V is called a linear G−space and we let gv denote $\rho(g)(v)$. A linear subspace V_0 of V is G-invariant if $g(V_0) \subseteq V_0$ for all $g \in G$.
   (i) Let V_1, V_2 be linear G-spaces and $T : V_1 \to V_2$ a linear equivariant map, i.e., $T(gv) = gT(v)$ for all $g \in G$ and $v \in V_1$. Show that both $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are G-invariant linear subspaces.
   (ii) If V is a linear G-space given by ρ then the dual V^* is a linear G-space given by ρ^*, where $\rho^*(g)(\ell)(v) = \ell(\rho(g^{-1})(v))$.
   (iii) Suppose V is an inner product and $\rho(G) \subseteq \boldsymbol{O}(V)$. If V_0 is an invariant linear subspace of V then $V_0^{\perp}$ is also invariant.
   (iv) With the same assumption as in (iii), if we identify V^* with V via the inner product then $\rho^* = \rho$.

6. Let M be a smooth (Riemannian) n-manifold, and $F(M)$ ($F_0(M)$) the bundle of (orthonormal) frames on M, i.e., the fiber $F(M)_x$ ($F_0(M)_x$) over $x \in M$ is the set of all (orthonormal) bases of TM_x.
   (i) Show that $F(M)$ is a principal $\boldsymbol{GL}(n)$-bundle.
   (ii) Show that $F_0(M)$ is a principal $\boldsymbol{O}(n)$-bundle,
   (iii) Show that the vector bundle associated to the representation $\rho = id : \boldsymbol{GL}(n) \to \boldsymbol{GL}(n)$ is TM.
   (iv) Find the $\boldsymbol{GL}(n)$-representations associated to the tensor bundles of M, S^2TM and $\bigwedge^p TM$.
7. Let $v_1, \ldots, v_n$ be the standard basis of $\boldsymbol{R}^n$, and

$$V = \{\sum x_{ijkl} v_i \otimes v_j \otimes v_k \otimes v_l \mid$$
$$x_{ijkl} + x_{jikl} = x_{ijkl} + x_{ijlk} = x_{ijkl} + x_{iklj} + x_{iljk} = 0\}.$$

   Let $r : V \to S^2(\boldsymbol{R}^n)$ be defined by $r(x) = \sum x_{ikjk} v_i \otimes v_j$.
   (i) Show that V is an $\boldsymbol{O}(n)$-invariant linear subspace of $\otimes^4 \boldsymbol{R}^n$, and the Riemann tensor Ω is a section of the vector bundle associated to V.
   (ii) Show that r is an $\boldsymbol{O}(n)-$equivariant map, $V = \mathrm{Ker}(r) \oplus S^2(\boldsymbol{R}^n)$ as $\boldsymbol{O}(n)$-spaces, and the Ricci tensor is a section of the vector bundle associated to $S^2(\boldsymbol{R}^n)$, i.e., S^2TM. The projection of Riemann tensor Ω onto the vector bundle associated to $\mathrm{Ker}(r)$ is called the *Weyl tensor* (For detail see [Be]).
   (iii) Write down the equivariant projection of V onto $\mathrm{Ker}(r)$ explicitly. (This gives a formula for the Weyl tensor).
8. Let $M = \boldsymbol{R}^n$ and $g = a_1^2(x)\, dx_1^2 + \ldots + a_n^2(x)\, dx_n^2$. Find the Levi-Civita connection 1-form of (M, g).
9. Let $\boldsymbol{H}^n = \{(x_1, \ldots, x_n) \mid x_n > 0\}$, and $g = \left(dx_1^2 + \ldots + dx_n^2\right)/x_n^2$. Show that the sectional curvature of $(\boldsymbol{H}^n, g)$ is -1.

1.3. Covariant derivative of tensor fields

Let (M, g) be a Riemannian manifold, and ∇ the Levi-Civita connection on TM. There is a unique induced connection ∇ on T^*M by requiring

$$X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y). \tag{1.3.1}$$

Let $e_1, \ldots, e_n$ be a local orthonormal frame field on M, and $\omega_1, \ldots, \omega_n$ its dual coframe. Suppose

$$\nabla e_i = \sum \omega_{ij} \otimes e_j,$$
$$\nabla \omega_i = \sum \tau_{ij} \otimes \omega_j.$$

Then (1.3.1) implies that $\tau_{ij} = -\omega_{ji} = \omega_{ij}$, i.e.,

$$\nabla \omega_i = \sum \omega_{ij} \otimes \omega_j. \tag{1.3.2}$$

So ∇ can be naturally extended to any tensor bundle $\mathcal{T}_s^r = \otimes^r T^*M \otimes^s TM$ of type (r,s) as in section 1.1.

For $r > 0$, $s > 0$, let $C_q^p : \mathcal{T}_s^r \to \mathcal{T}_{s-1}^{r-1}$ denote the linear map such that

$$\begin{aligned} &C_q^p(\omega_{i_1} \otimes \ldots \otimes \omega_{i_r} \otimes e_{j_1} \otimes \ldots \otimes e_{j_s}) \\ &= \omega_{i_1} \otimes \ldots \omega_{i_{p-1}} \otimes \omega_{i_{p+1}} \otimes \ldots \omega_{i_r} \otimes e_{j_1} \otimes \ldots e_{j_{q-1}} \otimes e_{j_{q+1}} \otimes \ldots e_{j_s}. \end{aligned}$$

These linear maps C_q^p are called *contractions*. If we make the standard identification of $\mathcal{T}_1^1$ with $L(TM, TM)$, then for $t = \sum t_{ij}\omega_i \otimes e_j$, we have

$$C_1^1(t) = \sum t_{ii} = \operatorname{tr}(t).$$

Since T^*M can be naturally identified with TM via the metric, the contraction operators are defined for any tensor bundles. For example if $t = \sum t_{ij}\omega_i \otimes \omega_j$, then $C(t) = \sum t_{ii}$ defines a contraction. The induced connections on the tensor bundles commute with tensor product and contractions.

In the following we will demonstrate how to compute the covariant derivatives of tensor fields. Let f be a smooth function on M, and

$$\nabla f = \sum f_i \omega_i = df. \tag{1.3.3}$$

Since $\nabla(df)$ is a section of $\mathcal{T}_0^2$, it can be written as a linear combination of $\{\omega_i \otimes \omega_j\}$:

$$\nabla(df) = \sum f_{ij}\omega_j \otimes \omega_i, \tag{1.3.4}$$

where $\nabla_{e_j}(df) = \sum f_{ij}\omega_i$. Using the product rule, we have

$$\begin{aligned} \nabla(df) &= \sum df_i \otimes \omega_i + f_i \nabla \omega_i \\ &= \sum_i df_i \otimes \omega_i + \sum_{i,j} f_i \omega_{ij} \otimes \omega_j \\ &= \sum_i df_i \otimes \omega_i + \sum_{i,m} f_m \omega_{mi} \otimes \omega_i. \end{aligned} \tag{1.3.5}$$

Compare (1.3.4) and (1.3.5), we obtain

$$\sum_j f_{ij}\omega_j = df_i + \sum_m f_m \omega_{mi}. \tag{1.3.6}$$

Taking the exterior derivative of (1.3.3) and using (1.2.3), (1.3.6), we obtain

$$\begin{aligned} 0 &= \sum df_i \wedge \omega_i + \sum f_i \omega_{ij} \wedge \omega_j \\ &= \sum (\sum f_{ij}\omega_j - f_j \omega_{ji}) \wedge \omega_i + \sum f_i \omega_{ij} \wedge \omega_j \\ &= \sum_{ij} f_{ij} \omega_j \wedge \omega_i, \end{aligned}$$

which implies that $f_{ij} = f_{ji}$. So we have

1.3.1. Proposition. *If $f : M \to \boldsymbol{R}$ is a smooth function then $\nabla^2 f$ is a smooth section of S^2T^*M.*

The *Laplacian* of f is defined to be the trace of $\nabla^2 f$, i.e.,

$$\Delta f = \sum_i f_{ii}.$$

Now suppose that $u = \sum u_{ij}\omega_i \otimes \omega_j$ is a smooth section of $\otimes^2 T^*M$, and

$$\nabla u = \sum u_{ijk}\omega_k \otimes \omega_i \otimes \omega_j,$$

where

$$\nabla_{e_k}(u) = \sum u_{ijk}\omega_i \otimes \omega_j.$$

Since

$$\nabla u = \sum du_{ij} \otimes \omega_i \otimes \omega_j + u_{ij}\nabla\omega_i \otimes \omega_j + u_{ij}\omega_i \otimes \nabla\omega_j,$$

and (1.3.2), we have

$$\sum_k u_{ijk}\omega_k = du_{ij} + \sum_m u_{im}\omega_{mj} + \sum_m u_{mj}\omega_{mi}. \tag{1.3.7}$$

For example, if u is the metric tensor g, then we have $u_{ij} = \delta_{ij}$ and by (1.3.7) we see that $u_{ijk} = 0$, i.e., $\nabla g = 0$ or g is parallel.

In the following we derive the formula for the covariant derivative of the Riemann tensor and the second Bianchi identity. Let $\Omega = \sum R_{ijkl}\omega_i \otimes e_j \otimes \omega_k \otimes \omega_l$ be the Riemann tensor of g. Set

$$\nabla\Omega = \sum R_{ijklm}\omega_m \otimes \omega_i \otimes e_j \otimes \omega_k \otimes \omega_l,$$

where

$$\nabla_{e_m}\Omega = \sum R_{ijklm}\omega_i \otimes e_j \otimes \omega_k \otimes \omega_l.$$

Using an argument similar to the above we find

$$\begin{aligned}\sum_m R_{ijklm}\omega_m = dR_{ijkl} + \sum_m R_{mjkl}\omega_{mi} + \sum_m R_{imkl}\omega_{mj} \\ + \sum_m R_{ijml}\omega_{mk} + \sum_m R_{ijkm}\omega_{ml}.\end{aligned} \tag{1.3.8}$$

Taking the exterior derivative of (1.2.4) and using (1.3.8) we have

$$\sum_{k,l,m} R_{ijklm}\omega_k \wedge \omega_l \wedge \omega_m = 0.$$

So we obtain the second Bianchi identity :

$$R_{ijklm} + R_{ijlmk} + R_{ijmkl} = 0. \tag{1.3.9}$$

Let u be a smooth section of tensor bundle $\mathcal{T}_r^s$. Then $\nabla^2 u$ is a section of $\mathcal{T}_r^s \otimes T^*M \otimes T^*M$. The *Laplacian* of u, $\triangle u$, is the section of $\mathcal{T}_r^s$ defined by contracting on the last two indices of $\nabla^2 u$. For example, if

$$u = \sum u_{ij}\omega_i \otimes \omega_j, \quad \nabla^2 u = \sum_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$

then $(\triangle u)_{ij} = \sum_k u_{ijkk}$.

Exercises.

1. Let μ, Ric be the scalar and Ricci curvature of g respectively, $d\mu = \sum \mu_k \omega_k$ and $\nabla\,\mathrm{Ric} = \sum r_{ijk}\omega_i \otimes \omega_j \otimes \omega_k$. Show that $\mu_k = 2\sum_i r_{iki}$.
2. Suppose (M, g) is an n-dimensional Riemannian manifold, and its Ricci curvature Ric satisfies the condition that $\mathrm{Ric} = fg$ for some smooth function f on M. If $n > 2$, then f must be a constant, i.e., g is Einstein.
3. Let f be a smooth function on M, and $\nabla^3 f = \sum f_{ijk}\omega_i \otimes \omega_j \otimes \omega_k$. Show that
$$f_{ijk} = f_{ikj} + \sum_m f_m R_{mijk}.$$
4. Let $\varphi : \boldsymbol{R} \to \boldsymbol{R}$ and $u : M \to \boldsymbol{R}$ be smooth functions. Show that
$$\triangle(\varphi(u)) = \varphi'(u)\triangle u + \varphi''(u)\|\nabla u\|^2.$$

6. Let (M^n, g) be an orientable Riemannian manifold, $f : M \to \boldsymbol{R}$ a smooth function, and $df = \sum_i f_i\omega_i$. Show that
   (i) there is a unique linear operator $* : \bigwedge^p T^*M \to \bigwedge^{n-p} T^*M$ such that
   $$\omega \wedge *\tau = \langle \omega, \tau \rangle dv$$
   for all p-forms ω and τ, where dv is the volume form of g.
   (ii)
   $$*df = \sum_i (-1)^{i-1} f_i\omega_1 \wedge \ldots \omega_{i-1} \wedge \omega_{i+1} \ldots \wedge \omega_n.$$
   (iii)
   $$\int_M \triangle f dv = \int_{\partial M} *df,$$
   In the following we assume that $\partial M = \emptyset$, show that
   (iv)
   $$\int_M f\triangle f dv = -\int_M \|\nabla f\|^2 dv,$$
   (v) if $\triangle f = \lambda f$ for some $\lambda \geq 0$ then f is a constant.

1.4. Vector fields and differential equations

A time independent system of ordinary differential equations (ODE) for n functions $\alpha = (\alpha_1, \ldots, \alpha_n)$ of one real variable t is given by a smooth map $f : U \to \boldsymbol{R}^n$ on an open subset U of $\boldsymbol{R}^n$. Corresponding to this ODE we have the following "initial value problem" : Given $x_0 \in U$, find $\alpha : (-t_0, t_0) \to U$ for some $t_0 > 0$ such that

$$\begin{cases} \alpha'(t) = f(\alpha(t)), \\ \alpha(0) = x_0. \end{cases} \tag{1.4.1}$$

The map f is a local vector field on $\boldsymbol{R}^n$ and the solutions of (1.4.1) are called the integral curves of the vector field f. As a consequence of the existence and uniqueness theorem of ODE, we have

1.4.1. Theorem. *Suppose M is a compact, smooth manifold, and X is a smooth vector field on M. Then there exists a unique family of diffeomorphisms $\varphi_t : M \to M$ for all $t \in \boldsymbol{R}$ such that*
(i) $\varphi_0 = id, \quad \varphi_{s+t} = \varphi_s \circ \varphi_t$,
(ii) let $\alpha(t) = \varphi_t(x_0)$, then α is the unique solution for the ODE system

$$\begin{cases} \alpha'(t) = X(\alpha(t)), \\ \alpha(0) = x_0. \end{cases}$$

The map $t \mapsto \varphi_t$ from the additive group $\boldsymbol{R}$ to the group $\mathrm{Diff}(M)$ of the diffeomorphisms of M is a group homomorphism, and is called the one-parameter subgroup of diffeomorphisms generated by the vector field X. Conversely, any group homomorphism $\rho : \boldsymbol{R} \to \mathrm{Diff}(M)$ arises this way, namely it is generated by the vector field X, where

$$X(x_0) = \frac{d}{dt}\bigg|_{t=0} (\rho(t)(x_0)).$$

In fact, $\mathrm{Diff}(M)$ is an infinite dimensional Fréchet Lie group and $C^\infty(TM)$ is its Lie algebra.

It follows from Theorem 1.4.1 that if X is a vector field on M such that $X(p) \neq 0$, then there exists a local coordinate system (U, x), $x = (x_1, \ldots, x_n)$ around p such that $X = \frac{\partial}{\partial x_1}$. It is obvious that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, for all i, j. This is also a sufficient condition for any k vector fields being part of coordinate vector fields, i.e.,

1.4.2. Theorem. *Let $X_1, \ldots, X_k$ be k smooth tangent vector fields on an n-dimensional manifold M such that $X_1(x), \ldots, X_k(x)$ are linearly independent for all x in a neighborhood U of p. Suppose $[X_i, X_j] = 0, \ \forall\ 1 \leq i < j \leq k$ on U. Then there exist $U_0 \subset U$ and a coordinate system (x, U_0) around p such that $X_i = \frac{\partial}{\partial x_i}$ for all $1 \leq i \leq k$.*

The following first order system of partial differential equations (PDE) for u,

$$\frac{\partial u}{\partial x_i} = P_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n, \tag{1.4.2}$$

is equivalent to $\tau = du$ for some u, i.e., τ is an exact 1-form, where

$$\tau = \sum_i P_i(x_1, \ldots, x_n) dx_i.$$

So by the Poincaré Lemma, (1.4.2) is solvable if and only if τ is closed, i.e., $d\tau = 0$, or equivalently

$$\frac{\partial P_i}{\partial x_j} = \frac{\partial P_j}{\partial x_i}, \quad \text{for} \quad \text{all} \quad i \neq j.$$

For the more general first order PDE:

$$\frac{\partial u}{\partial x_i} = P_i(x_1, \ldots, x_n, u(x)), \tag{1.4.3}$$

the solvability condition is that "the mixed second order partial derivatives are independent of the order of derivatives". But to check this condition for a complicated system can be tedious, and the Frobenius theorem gives a systematic way to determine whether a system is solvable, that can be stated either in terms of vector fields or differential forms.

1.4.3. Frobenius Theorem. *Let $X_1, \ldots, X_k$ be k smooth tangent fields on an n-dimensional manifold M such that $X_1(x), \ldots, X_k(x)$ are linearly independent for all x in a neighborhood U of p. Suppose*

$$[X_i, X_j] = \sum_{l=1}^{k} f_{ijl} X_l, \quad \forall \quad i \neq j, \tag{1.4.4}$$

on U, for some smooth functions f_{ijl}. Then there exist an open neighborhood U_0 of p and a local coordinate system (x, U_0) such that the span of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$ is equal to the span of $X_1, \ldots, X_k$.

A rank k *distribution* E on M is a smooth rank k subbundle of TM. It is *integrable* if whenever $X, Y \in C^\infty(E)$, we have $[X, Y] \in C^\infty(E)$. Given a rank k distribution, locally there exist k smooth vector fields $X_1, \ldots, X_k$ such that E_x is the span of $X_1(x), \ldots, X_k(x)$. The vector fields $X_1, \ldots, X_k$ satisfies condition (1.4.4) if and only if E is integrable. A submanifold N of M is called an integral submanifold of E, if $TN_x = E_x$ for all $x \in N$. Then Theorem 1.4.3. can be restated as:

1.4.4. Theorem. *If E is a smooth, integrable, rank k distribution of M, then there exists a local coordinate system (x, U) such that*

$$\{q \in U \mid x_{k+1}(q) = c_{k+1}, \ldots, x_n(q) = c_n\}$$

are integral submanifolds for E.

The space $\mathcal{A}$ of all differential forms is an anti-commutative ring under the standard addition and the wedge product. An ideal $\wp$ of $\mathcal{A}$ is called *d-closed* if $d\wp \subseteq \wp$. Given a rank k distribution E, locally there also exist $(n-k)$ linearly independent 1-forms $\omega_{k+1}, \ldots, \omega_n$ such that $E_x = \{u \in TM_x \mid \omega_{k+1}(x) = \cdots = \omega_n(x) = 0\}$. Using (1.1.2), Theorem 1.4.3. can be formulated in terms of differential forms.

1.4.5. Theorem. *Let $\omega_1, \ldots, \omega_m$ be linearly independent 1-forms on M^n, and $\wp$ the ideal in the ring $\mathcal{A}$ of differential forms generated by $\omega_1, \ldots, \omega_m$. Suppose $\wp$ is d-closed. Then given $x_0 \in M$ there exists a local coordinate system (x, U) around x_0 such that $dx_1, \ldots, dx_m$ generates $\wp$.*

1.4.6. Corollary. *With the same assumption as in Theorem 1.4.5, given $x_0 \in M$, there exists an (n-m)-dimensional submanifold N of M through x_0 such that $i^*\omega_j = 0$ for all $1 \leq j \leq m$, where $i : N \to M$ is the inclusion.*

Let $\omega_1, \ldots, \omega_k$ be linearly independent 1-forms on M^n, and $\wp$ the ideal generated by $\omega_1, \ldots, \omega_k$. Then locally we can find smooth 1-forms $\omega_{k+1}, \ldots, \omega_n$ such that $\omega_1, \ldots, \omega_n$ are linear independent. We may assume that

$$d\omega_i = \sum_{j<l} f_{ijl}\omega_j \wedge \omega_l,$$

for some smooth functions f_{ijl}. Then it is easily seen that $\wp$ is d-closed if and only if one of the following conditions holds:
(i) $f_{ijl} = 0$ if $i \leq k$ and $j, l > k$,
(ii) $d\omega_i = 0 \mod (\omega_1, \ldots, \omega_k)$ for all $i \leq k$.

1.4.7. Example. In order to solve (1.4.3), we consider the following 1-form on $\boldsymbol{R}^n \times \boldsymbol{R}$:

$$\omega = dz - \sum_i P(x, z)dx_i.$$

Let $\wp$ be the ideal generated by ω. Then the condition that $\wp$ is d-closed is equivalent to one of the following:
(i) there exists a 1-form τ such that $d\omega = \omega \wedge \tau$,
(ii) $\omega \wedge d\omega = 0$.

If $\wp$ is integrable then there is a smooth function $f(x, z)$ such that $f(x, z) = c$ defines integrable submanifolds of $\wp$. Since df never vanishes and is proportional to ω, $\frac{\partial f}{\partial z} \neq 0$. So it follows from the Implicit Function Theorem that locally there exists a smooth function $u(x)$ such that $f(x, u(x)) = c$. So u is a solution of (1.4.3). In particular, the first order system for $g : U \to \boldsymbol{GL}(n)$:

$$dg = \omega g,$$

is solvable if and only if $d\omega = \omega \wedge \omega$.

Exercises.

1. Let $\{X_1, X_2\}$ be a local frame field around p on the surface M. Show that there exists a local coordinate system (x_1, x_2) around p such that X_i is parallel to $\frac{\partial}{\partial x_i}$.

1.5. Lie derivative of tensor fields

Let $\varphi : M \to N$ be a diffeomorphism. Then the pull back φ^* on vector fields and 1-forms are defined as follows:

$$\varphi^* : C^\infty(TN) \to C^\infty(TM), \quad \varphi^*(X)_p = (d\varphi_p)^{-1}(X(\varphi(p))),$$

$$\varphi^* : C^\infty(T^*N) \to C^\infty(T^*M), \quad \varphi^*(\omega)_p = \omega_{\varphi(p)} \circ d\varphi_p.$$

Hence φ^* is defined for any tensor fields by requiring that

$$\varphi^*(t_1 \otimes t_2) = \varphi^*(t_1) \otimes \varphi^*(t_2),$$

for any two tensor fields t_1 and t_2.

Let X be a vector field on M, and φ_t the one-parameter subgroup of M generated by X. Then the *Lie derivative* of a tensor field u with respect to X is defined to be

$$L_X u = \left.\frac{\partial}{\partial t}\right|_{t=0} (\varphi_t^* u). \tag{1.5.1}$$

Let $\mathcal{T}(M)$ denote the direct sum of all the tensor bundles of M. Then L_X is a linear operator on $\mathcal{T}(M)$, that has the following properties (for proof see [KN] and [Sp]):
(i) If $u \in C^\infty(\mathcal{T}^r_s(M))$, then $L_X u \in C^\infty(\mathcal{T}^r_s(M))$.
(ii) L_X commute with the tensor product and contractions, i.e.,

$$L_X(u_1 \otimes u_2) = (L_X u_1) \otimes u_2 + u_1 \otimes (L_X u_2),$$

$$L_X(C(u)) = C(L_X u),$$

for any contraction operator C.
(iii) $L_X f = Xf = df(X)$, for any smooth function f.
(iv) $L_X Y = [X, Y]$, for any vector field Y.

The *interior derivative*, i_X, is the linear operator

$$i_X : C^\infty\left(\bigwedge^p T^*M\right) \to C^\infty\left(\bigwedge^{p-1} T^*M\right),$$

defined by

$$i_X(\omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}).$$

Then on differential forms we have

$$L_X = i_X\, d + d\, i_X.$$

Let (M, g) be a Riemannian manifold, $e_1, \ldots, e_n$ a local orthonormal frame field, and $\omega_1, \ldots, \omega_n$ its dual coframe. Suppose $X = \sum X_i e_i$, and $\nabla X = \sum X_{ij} e_i \otimes \omega_j$. Using the fact that L_X commutes with contractions, we can easily show that

$$(L_X \omega_i) = \sum_j \omega_{ij}(X)\omega_j + X_{ij}\omega_j.$$

So we have

$$\begin{aligned} L_X g &= L_X \left(\sum \omega_i \otimes \omega_i \right) \\ &= \sum (L_X \omega_i) \otimes \omega_i + \omega_i \otimes (L_X \omega_i) \,. \\ &= \sum_{ij} (X_{ij} + X_{ji}) \omega_i \otimes \omega_j \end{aligned} \qquad (1.5.2)$$

A diffeomorphism $\varphi : M \to M$ is called an *isometry* if $\varphi^* g = g$ for all t, or equivalently $d\varphi_x : TM_x \to TM_{\varphi(x)}$ is a linear isometry for all $x \in M$. If φ_t is a one-parameter subgroup of isometries of M, and X is its vector field, then $\varphi_t^* g = g$ and by definition of $L_X g$ we have $L_X g = 0$. So by (1.5.2), we have

$$X_{ij} + X_{ji} = 0.$$

Any vector field satisfying this condition is called a *Killing vector field* of M. Conversely, if X is a Killing vector field on a complete manifold (M, g), then the 1-parameter subgroup φ_t generated by X consists of isometries.

Exercises.

1. Find all isometries of $(\boldsymbol{R}^n, g)$, where g is the standard metric.
2. If ξ is a Killing vector field and v a smooth tangent vector field on M, then $\langle \nabla_v \xi, v \rangle = 0$.
3. Let X be a smooth Killing vector field on the closed Riemannian manifold M. Show that
   (i)
   $$\frac{1}{2}\triangle(\|X\|^2) = -\operatorname{Ric}(X, X) + \|\nabla X\|^2.$$
   (ii)
   $$\int_M \operatorname{Ric}(\nabla X, \nabla X) dv = \int_M \|\nabla X\|^2 dv.$$
   (iii) If $\operatorname{Ric} \leq 0$ (i.e., $\operatorname{Ric}(X, X) \leq 0$ for all vector field X) then the dimension of the group of isometries of M is 0.