3 Glaciers

Glaciers build up over time through the accumulation of snow, which compacts under its weight to form ice. With sufficient weight the ice can flow, by viscous creep or by slipping over the underlying rock. The accumulation generally occurs at high elevations, where the annual balance between snowfall and melting is positive. At lower elevations, melting outpaces snowfall, and a glacier would not be able to form there, were it not for the flow of ice from higher elevations. Of course, latitude is also a big factor in determining where a glacier can form, because air temperatures are considerably colder at higher latitudes so there is less melting there.

Glaciers therefore act in a similar way to rivers, transporting ice from higher elevations to lower ones. In some cases, as for most rivers, the glaciers reach all the way to the ocean and discharge the ice as icebergs into the ocean (sometimes the ice flows into a floating *ice shelf* before breaking off). In other cases the glacier loses all its mass to melting before it reaches the ocean (the analogue for a river would be in a hot environment where evaporation or seepage into the ground (e.g. limestone) causes the river to dry up before it reaches the ocean).

An *ice sheet* is essentially the same as a glacier, but larger. The distinction is blurry, but a glacier usually has a well defined flow direction, whereas an ice sheet flows in all directions (the canonical ice sheet is radially symmetric, whereas a glacier is essentially one-dimensional). An *ice cap* is often used to mean a small ice sheet. There are currently two ice sheets on Earth; one in Antarctica and one in Greenland. During the glacial periods of the last two millennia there were two other large ones; the Laurentide ice sheet (covering much of North America), and the Fennoscandian ice sheet (covering Scandinative, the North Sea, and the British Isles), as well as smaller ice caps in Arctic Russia and the Canadian Archipelago.

The typical speeds of a glacier are around 100 m y^{-1}, but they can in places move at speeds greater than 1 km y^{-1}. This depends upon the temperature of the ice, which controls its effective viscosity (similar to honey), and on the slipperiness of the substrate over which it is moving (the substrate is sometimes rigid bedrock and sometimes water-saturated sediments which can themselves deform). Mountain glaciers (in the Alps, Himalaya, etc.) are typically several hundred meters deep, tens of kilometers long, and have surface slopes on the order of 0.1. The Antarctic and Greenland ice sheets on the other hand are around 3 km deep and 3000 km wide.

In this chapter we describe the equations that govern the flow and melting of glaciers and ice sheets. We will consider steady states, look at how perturbations to the steady states evolve, and how the glaciers can be expected to respond to climate change. We also discuss explanations for some of the more interesting behaviour that is observed: glacier surges, and outburst floods.

3.1 Shallow ice approximation

The starting point for modelling ice flow is the constitutive law, or flow law, that describes viscous deformation. This will be combined with statements of mass and momentum conservation (force balance) to arrive at an evolution equation for the thickness of the glacier. We use an approximate form of the momentum equation appropriate for thin layers of viscous fluid; the necessary approximations can be introduced immediately in the model derivation, or can be justified more systematically using lubrication theory.

3.1.1 Glen's flow law

Glacial ice is a polycrystalline material; it is composed of many crystals that deform under stress by a combination of dislocation creep (the motion of dislocations through the crystal structure) and diffusion creep (the diffusion of molecules along the crystal interfaces). The deformation is usually described by Glen's flow law,

$$\dot{\varepsilon}_{ij} = A\tau^{n-1}\tau_{ij}, \tag{3.1}$$

where $\tau = \sqrt{\frac{1}{2}\tau_{ij}\tau_{ij}}$ is the second invariant of the deviatoric stress tensor τ_{ij} (the summation convention is used here), and where $n \approx 3$, and $A = A(T)$ is a temperature-dependent rate factor. Here $\dot{\varepsilon}_{ij}$ is the strain rate tensor, defined in terms of the velocity $\mathbf{u} = u_i$ by

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3.2}$$

and the deviatoric stress tensor is related to the full (Cauchy) stress tensor σ_{ij} by

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \qquad p = -\tfrac{1}{3}\sigma_{kk}. \tag{3.3}$$

The flow law can be written in the more standard framework of viscous fluid mechanics as

$$\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}, \tag{3.4}$$

where $\eta = 1/2A\tau^{n-1}$ is the *effective* viscosity. For a Newtonian fluid ($n = 1$), the viscosity is constant, but for $n \neq 1$, it depends on the stress. Fluids with $n > 1$, as is the case for ice, are referred to as shear thinning fluids, since the viscosity decreases with increasing stress.

For ice in a glacier, which has a small aspect ratio ($z \ll x$), the dominant component of the deviatoric stress tensor is usually $\tau_{xz} \approx \tau$, and the dominant component of the strain rate tensor is $\dot{\varepsilon}_{xz}$, so the flow law can be written approximately as

$$\frac{\partial u}{\partial z} = 2A\tau^n. \tag{3.5}$$

(Here we have assumed that $\tau_{xz} \approx \tau$ is positive; more generally, the right hand side is $2A|\tau|^{n-1}\tau$.)

The flow law coefficient A depends strongly on temperature, but we will treat it as constant for the rest of this section. This corresponds either to ignoring the temperature dependence, or to assuming that the ice is isothermal.

3.1.2 Mass conservation

We consider a two-dimensional glacier and use coordinates (x, z) for the horizontal and vertical directions. The bed is denoted $z = b(x)$ and the surface is denoted $z = s(x, t)$. We assume for simplicity that the surface elevation is monotonically decreasing, $\partial s/\partial x < 0$.

The mass conservation equation can be derived, as it was for a river, by considering the mass of ice in a section between points x_1 and x_2. This results in

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a, \tag{3.6}$$

where $H(x,t) = s - b$ is the ice thickness, $q(x,t)$ is the flux, and $a(x,t)$ is the net accumulation, being the difference between snowfall and melt.

Ice flux is given in terms of the velocity profile by

$$q(x,t) = \int_b^s u(x,z,t)\ \mathrm{d}z. \tag{3.7}$$

The velocity profile is derived from consideration of force balance.

3.1.3 Force balance

The importance of acceleration in the flow of a glacier is measured by the Reynolds number,

$$Re = \frac{\rho[u][h]}{[\eta]}, \tag{3.8}$$

where $\rho \approx 916$ kg m^{-3} is the density and $[u]$, $[h]$ and $[\eta]$ are typical values of the velocity, depth and viscosity. Taking $[u] = 30$ m y^{-1} $\approx \times 10^{-6}$ m s^{-1}, $[h] = 100$ m, and $[\eta] = 10^{12}$ Pa s, gives $Re \approx 10^{-13}$. We see that the Reynolds number is very small, and the acceleration terms are therefore negligible.

Based on the fact that the aspect ratio is small, we assume that we can also ignore deviatoric longitudinal stresses (compression and extension) and vertical shear stress.

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We can then derive the equations by considering the forces acting on a small rectangle of ice, Δx by Δz. The relevant terms are the pressure p, viscous shear stress $\tau = \tau_{xz}$, and gravity g. Balancing the forces in the x and z directions respectively gives

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z}, \tag{3.9}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g. \tag{3.10}$$

The surface of the glacier $z = s(x,t)$ is assumed stress free, so the boundary conditions that go with these equations are

$$p = \tau = 0 \quad \text{at} \quad z = s(x,t). \tag{3.11}$$

The equations can be integrated immediately to give

$$p = \rho g(s - z), \qquad \tau = -\rho g \frac{\partial s}{\partial x}(s - z). \tag{3.12}$$

We can then substitute the shear stress into the approximate flow law (3.5) and integrate using no-slip condition $u = 0$ at the bed $z = b$, to obtain

$$u = \frac{2A(\rho g)^n}{n+1}\left(-\frac{\partial s}{\partial x}\right)^n \left(H^{n+1} - (s-z)^{n+1}\right). \tag{3.13}$$

Substituting this into (3.7) gives

$$q = \frac{2A(\rho g)^n}{n+2}\left(-\frac{\partial s}{\partial x}\right)^n H^{n+2}. \tag{3.14}$$

Had we not assumed $\partial s/\partial x < 0$, the more general of this relationship that allows flow in both directions is

$$q = -\frac{2A(\rho g)^n}{n+2}\left|\frac{\partial s}{\partial x}\right|^{n-1}\frac{\partial s}{\partial x}H^{n+2}. \tag{3.15}$$

3.1.4 Lubrication theory

A more systematic method of deriving the equations for glacier flow is to start from the Stokes equations (in two dimensions),

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{3.16}$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}, \tag{3.17}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho g, \tag{3.18}$$

together with the full flow law (3.2). The boundary conditions that go with this full problem are the no slip and no penetration conditions at the bed,

$$u = w = 0 \quad \text{at} \quad z = b(x,t), \tag{3.19}$$

the kinematic condition at the surface

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = w + a \quad \text{at} \quad z = s(x,t), \tag{3.20}$$

and the stress-free conditions at the surface

$$-(-p + \tau_{xx})\frac{\partial s}{\partial x} + \tau_{xz} = 0, \tag{3.21}$$

$$-\tau_{xz}\frac{\partial s}{\partial x} + (-p + \tau_{zz}) = 0. \tag{3.22}$$

The neglect of the acceleration terms from the left hand side is justified by the smallness of the Reynolds number, discussed above. If we now non-dimensionalise, with $x \sim [x]$ and $z \sim [H]$, and define the aspect ratio

$$\varepsilon = \frac{[H]}{[x]}, \tag{3.23}$$

it is appropriate to scale the other terms as

$$[p] = \rho g[H], \quad [\tau_{xz}] = [\tau] = \varepsilon\rho g[H], \quad [\tau_{xx}] = [\tau_{zz}] = \varepsilon[\tau], \quad [u] = A[\tau]^n[H], \quad [w] = \varepsilon[u], \tag{3.24}$$

The dimensionless force-balance equations become

$$0 = -\frac{\partial p}{\partial x} + \varepsilon^2 \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}, \tag{3.25}$$

$$0 = -\frac{\partial p}{\partial z} + \varepsilon^2 \frac{\partial \tau_{xz}}{\partial x} + \varepsilon^2 \frac{\partial \tau_{zz}}{\partial z} - 1, \tag{3.26}$$

and the boundary conditions are

$$-(-p + \frac{\varepsilon^2}{\mu}\tau_{xx})\frac{\partial s}{\partial x} + \tau_{xz} = 0, \tag{3.27}$$

$$-\varepsilon^2 \tau_{xz}\frac{\partial s}{\partial x} + (-p + \varepsilon^2 \tau_{zz}) = 0. \tag{3.28}$$

The only relevant component of the flow law that is needed for the leading order problem is the xz component,

$$\frac{1}{2}\left(\frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial w}{\partial x}\right) = \tau^n \tau_{xz}, \tag{3.29}$$

where the second invariant of the stress tensor is

$$\tau^2 = \tau_{xz}^2 + \varepsilon^2 \tau_{xx}^2. \tag{3.30}$$

If we now take the limit $\varepsilon \to 0$, the equations become the same as those used earlier in (3.9) and (3.10) (though already non-dimensionalised),

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z}, \tag{3.31}$$

$$0 = -\frac{\partial p}{\partial z} - 1, \tag{3.32}$$

together with the flow law as in (3.5),

$$\frac{\partial u}{\partial z} = 2\tau^n. \tag{3.33}$$

We can also integrate the pointwise conservation of mass equation (3.16) with the boundary conditions (3.19) and (3.20) to obtain the depth-integrated conservation equation,

$$\int_b^s \frac{\partial u}{\partial x}\, \mathrm{d}z = -\int_b^s \frac{\partial w}{\partial z}\, \mathrm{d}z, \tag{3.34}$$

$$= -[w]_b^s, \tag{3.35}$$

$$\frac{\partial}{\partial x}\left(\int_b^s u\, \mathrm{d}z\right) - u\big|_s \frac{\partial s}{\partial x} + u\big|_b \frac{\partial b}{\partial x} = -\frac{\partial s}{\partial t} - u\big|_s \frac{\partial s}{\partial x} + a, \tag{3.36}$$

$$\frac{\partial}{\partial x}\left(\int_b^s u\, \mathrm{d}z\right) = -\frac{\partial H}{\partial t} + a, \tag{3.37}$$

which is the same as (3.6),

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a. \tag{3.38}$$

3.2 Glaciers

To model a mountain glacier, we take the bed as being an approximately linear slope, with

$$\frac{\partial b}{\partial x} = -\sin\theta. \tag{3.39}$$

Assuming $\partial H/\partial x < \sin\theta$, so that the ice always flows in the positive x direction, the expression for the ice flux (3.14) then becomes

$$q = \frac{2A}{n+2}(\rho g)^n \left(\sin\theta - \frac{\partial H}{\partial x}\right)^n H^{n+2}, \tag{3.40}$$

which we combine with the mass equation

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a. \tag{3.41}$$

3.2.1 Non-dimensionalisation

We non-dimensionalise by writing

$$x = [x]\hat{x}, \qquad t = [t]\hat{t}, \qquad H = [H]\hat{H}, \qquad q = [q]\hat{q}, \qquad a = [a]\hat{a}. \tag{3.42}$$

We assume that the accumulation scale $[a]$ and the length scale $[x]$ are known (from climatic conditions and from the topography of the mountain range). We are then free to choose

$$[q] = [a][x], \qquad [H] = \left(\frac{[q]}{2A(\rho g\sin\theta)^n}\right)^{1/(n+2)}, \qquad [t] = \frac{[H][x]}{[q]}, \tag{3.43}$$

and define

$$\mu = \frac{[H]}{[x]}\operatorname{cosec}\theta. \tag{3.44}$$

The dimensionless mass conservation equation then becomes (dropping hats),

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\left(1 - \mu\frac{\partial H}{\partial x}\right)^n \frac{H^{n+2}}{n+2}\right] = a. \tag{3.45}$$

Typical values are $n = 3$, $A = 10^{-24}$ Pa^{-3} s^{-1}, $\rho = 916$ kg m^{-3}, $g = 9.8$ m s^{-2}, $\sin\theta = 0.1$, $[a] = 1$ m y^{-1}, and $[x] = 10^4$ m, from which $[q] = 3\times10^{-4}$ m^2 s^{-1}, $[H] \approx 200$ m, $[t] = 200$ y, and $\mu \approx 0.2$. Note that $[t] = [x]/[u]$ is the natural advective timescale of the glacier, giving the timescale over which ice moves along its length.

3.2.2 Steady states and surface waves

The governing equation (3.45) is a nonlinear diffusion equation for H, given a prescribed source term a, which we can assume is determined by the local climate. The equation is very similar to the equation that governs the flow of a drop down an inclined plane (indeed, it *is* the same if the drop is that of a shear thinning fluid).

The typical values of μ are relatively small and it is reasonable to neglect the diffusion term, to arrive at a hyperbolic equation (analogous to that which describes the depth of a river),

$$\frac{\partial H}{\partial t} + H^{n+1}\frac{\partial H}{\partial x} = a. \tag{3.46}$$

This is a nonlinear wave equation, with solutions that propagate as waves and, generically, form shocks. Such shocks would in reality be smoothed out by the presence of the diffusive term. If $x = 0$ is the head of the glacier, the appropriate boundary condition is $H = 0$ there, and a similar condition $H = 0$ at $x = x_m$ determines the location of the glacier terminus x_m (this is a form of 'free boundary' - determining the location of the terminus is part of the problem, and this explains why we can effectively have two boundary conditions on a first order equation).

For steady accumulation $a(x)$, the terminus position x_m is given by

$$0 = \int_0^{x_m} a(x)\ \mathrm{d}x. \tag{3.47}$$

The steady state ice depth is given by

$$H_0(x) = \left[(n+2)\int_0^x a(x')\ \mathrm{d}x'\right]^{1/(n+2)}. \tag{3.48}$$

For example, suppose $a = 1 - x$ (for a linearly sloping valley, this reflects the decrease in air temperature, and hence melt rate, with elevation). In this case $q = x - \frac{1}{2}x^2$, $x_m = 2$, and the depth is

$$H_0(x) = \left[(n+2)(x - \tfrac{1}{2}x^2)\right]^{1/(n+2)}. \tag{3.49}$$

We can also consider how perturbations to the steady state propagate. Such perturbations may be caused by a particularly large accumulation event such as an avalanche from the valley sidewalls, or they may be forced by the seasonal cycle of accumulation considered below.

We first consider exact solutions for an initial perturbation, and then consider linearised versions of the equation for small perturbations.

The exact problem (3.46) for $a(x)$ with $H = H_{in}(x)$ at $t = 0$ can be solved using the method of characteristics,

$$\dot{x} = H^{n+1}, \qquad \dot{H} = a, \tag{3.50}$$

with initial data

$$x = x_0 \quad H = H_{in}(x_0), \qquad \text{at} \quad t = 0, \tag{3.51}$$

Dividing the characteristic equations by each other and integrating, recalling the definition of $H_0(x)$ in (3.48), we obtain

$$H(x,t)^{n+2} = H_0(x)^{n+2} - H_0(x_0)^{n+2} + H_{in}(x_0)^{n+2}, \tag{3.52}$$

where $x_0(x,t)$ is defined implicitly by

$$t = \int_{x_0}^{x} \frac{\mathrm{d}x'}{\left[H_0(x')^{n+2} - H_0(x_0)^{n+2} + H_{in}(x_0)^{n+2}\right]^{\frac{n+1}{n+2}}}. \tag{3.53}$$

This provides the exact solution for a given initial condition, but it is not particularly illuminating. More progress can be made if we (reasonably) assume that the initial condition is close to the steady state.

Suppose $H(x,t) = H_0(x) + H_1(x,t)$, where $H_1 \ll H_0$. We can then linearise equation (3.46), giving

$$\frac{\partial H_1}{\partial t} + \frac{\partial}{\partial x}\left[H_0^{n+1}H_1\right] = 0. \tag{3.54}$$

This can be solved by multiplying by $H_0(x)^{n+1}$ to give

$$\frac{\partial}{\partial t}\left[H_0^{n+1}H_1\right] + H_0^{n+1}\frac{\partial}{\partial x}\left[H_0^{n+1}H_1\right] = 0, \tag{3.55}$$

and then defining a distorted space variable,

$$\xi = \int_0^x \frac{\mathrm{d}x'}{H_0(x')^{n+1}}, \tag{3.56}$$

so that

$$\frac{\partial}{\partial t}\left[H_0^{n+1}H_1\right] + \frac{\partial}{\partial \xi}\left[H_0^{n+1}H_1\right] = 0. \tag{3.57}$$

This has the travelling wave solution (using the method of characteristics, or just by inspection) $H_0^{n+1}H_1 = \phi(\xi - t)$ for arbitrary function ϕ determined from the initial condition. This solution takes the form of a travelling wave, moving with constant speed in the distorted space coordinate. The perturbation to the ice depth is therefore

$$H_1(x,t) = \frac{\phi(\xi - t)}{H_0(x)^{n+1}}, \qquad \xi = \int_0^x \frac{\mathrm{d}x'}{H_0(x')^{n+1}}. \tag{3.58}$$

The speed of the perturbations is

$$\frac{\partial x}{\partial \xi} = H_0(x)^{n+1}, \tag{3.59}$$

and is therefore $n+1$ times the surface ice speed (which, from (3.13) after non-dimensionalising, is $H_0^{n+1}/(n+1)$). So the perturbation in surface height moves faster than the ice itself.

There is a problem with this solution, however, because the amplitude of the perturbation blows up at the terminus where H_0 goes to zero. This invalidates the linearisation, since H_1 was assumed small compared to H_0 in deriving (3.54). This issue is related to the fact that the terminus x_m is really a free boundary, the position of which should be perturbed at the same time as the ice depth. Depth perturbations go together with perturbations of the fluid front. A number of methods can be used to study the movement of the front, notably the method of strained coordinates; but we do not go into that here.

An alternative method of linearising the equation is to approximate the characteristic equations

$$\dot{x} = H^{n+1} \approx H_0^{n+1}, \qquad \dot{H} = a \implies \dot{H}_0 = a, \quad \dot{H}_1 = 0. \tag{3.60}$$

This is not quite the same linearisation, as it corresponds to solving the perturbation equation

$$\frac{\partial H_1}{\partial t} + H_0^{n+1}\frac{\partial H_1}{\partial x} = 0, \tag{3.61}$$

which can be solved with the same change of variables as above to give

$$H_1(x,t) = \phi(\xi - t) \qquad \xi = \int_0^x \frac{\mathrm{d}x'}{H_0(x')^{n+1}}, \tag{3.62}$$

where $\phi(\xi - t)$ is determined from the initial condition. This perturbation does not blow up at the terminus. The missing term in this linearisation compared to the earlier one is $(n+1)H_0^n H_1 \partial H_0/\partial x$, and the neglect of this term can be justified if $\partial H_0/\partial x$ is small on the length scale of the perturbation (*i.e.* if $H_0 \partial H_1/\partial x > H_1 \partial H_0/\partial x$).

3.2.3 Seasonal fluctuations

If we resolve annual timescales, the accumulation fluctuates over most of the glacier between positive (during winter) and negative (during summer). We can analyse the seasonal oscillations of a glacier governed by

$$\frac{\partial H}{\partial t} + H^{n+1}\frac{\partial H}{\partial x} = a(x,t), \tag{3.63}$$

by defining the annual average accumulation and ice thickness

$$\overline{a}(x) = \frac{1}{t_y}\int_0^{t_y} a(x,t)\ \mathrm{d}t, \qquad \overline{H}(x) = \frac{1}{t_y}\int_0^{t_y} H(x,t)\ \mathrm{d}t. \tag{3.64}$$

Here t_y is the (dimensionless) length of a year. We also write $a(x,t) = \overline{a}(x) + a_1(x,t)$, $H(x,t) = \overline{H}(x) + H_1(x,t)$ and suppose $H_1 \ll \overline{H}$, so that we can linearise the equation around $\overline{H}$ (using the second of the two linearisation methods above),

$$\frac{\partial H_1}{\partial t} + \overline{H}^{n+1}\frac{\partial \overline{H}}{\partial x} + \overline{H}^{n+1}\frac{\partial H_1}{\partial x} = \overline{a} + a_1. \tag{3.65}$$

Then averaging the equation in time shows that the average state is given by

$$\frac{\partial}{\partial x}\left[\frac{\overline{H}^{n+2}}{n+2}\right] = \overline{a}, \tag{3.66}$$

and the perturbation is governed by

$$\frac{\partial H_1}{\partial t} + \overline{H}^{n+1}\frac{\partial H_1}{\partial x} = a_1. \tag{3.67}$$

For example, if $a_1(x,t) = \cos \omega t$, where $\omega = 2\pi/t_y$, and $H = 0$ at $x = 0$, then

$$H_1 = \frac{2}{\omega}\cos \tfrac{1}{2}\omega(\xi - 2t)\sin \tfrac{1}{2}\omega\xi, \qquad \xi = \int_0^x \frac{\mathrm{d}x'}{\overline{H}(x')^{n+1}}. \tag{3.68}$$

The timescale used to scale the equation was the natural advective timescale of the glacier $[t] = [h]/[u]$, and is much larger than a year, so the dimensionless frequency ω is large. Thus the perturbation H_1 is small, justifying the linearisation. We see that the annual accumulation signal gives rise to small surface waves that propagate down glacier.

3.3 Glacier sliding

In the previous section we assumed that the ice had a no-slip condition at the bed, $u = 0$ at $z = b$. This is appropriate if the ice is frozen to the bed, but is usually not correct if the ice at the bed is at the melting point. In that case there is a thin film of water between the ice and the bedrock, which lubricates the interface and allows slip. Perhaps surprisingly, such conditions of melting ice at the bed are very common, because of geothermal and frictional heating, and because the thick layer of ice insulates the bed from the cold air temperatures.

The slip is parameterised mathematically using a boundary condition of the form

$$\tau_b = f(u_b), \tag{3.69}$$

where $\tau_b = \tau|_{z=b}$ is the basal shear stress, and $u_b = u|_{z=b}$ is the sliding speed. This is referred to as the friction law or sliding law. Note that the basal shear stress is known from (3.12) and is approximately $\rho g H \sin\theta$ for a glacier.

A common version of this sliding law is

$$u_b = C|\tau_b|^{m-1}\tau_b, \tag{3.70}$$

where C and m are constants, which parameterise the detailed small-scale physics that allows for sliding. This is referred to as a Weertman sliding law.

On hard bedrock, sliding is believed to result from lubricated viscous flow around small-scale bedrock obstacles, and by a process called regelation. We discuss both of these below. When a glacier moves over sediments, sliding can result from the same processes, but may additionally be due to the deformation (flow) of the sediments themselves. When water-saturated, the sediments can be considerably less stiff than the ice, and they similarly deform in a viscous manner, driven by the weight of the ice.

Using the sliding law (3.70) as the boundary condition when determining the velocity profile results in an additional term,

$$u_b H = C(\rho g)^m \left(-\frac{\partial s}{\partial x}\right)^m H^{m+1}, \tag{3.71}$$

in the vertically-integrated ice flux (3.14). Following the same non-dimensionalisation as above, the mass conservation equation for the glacier (3.45) then becomes

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\gamma\left(1 - \mu\frac{\partial H}{\partial x}\right)^m \frac{H^{m+1}}{m+1} + \left(1 - \mu\frac{\partial H}{\partial x}\right)^n \frac{H^{n+2}}{n+2}\right] = a, \tag{3.72}$$

(assuming $\partial H/\partial x < 1/\mu$), where the dimensionless sliding coefficient is defined as

$$\gamma = \frac{(m+1)C}{2A}(\rho g \sin\theta)^{m-n}[H]^{m-n-1}. \tag{3.73}$$

Typical values are $m = 3$ and $C = 10^{-22}$ m s^{-1} Pa^{-3}, and using other values defined previously, this gives $\gamma \approx 10$. However, estimates for C are highly variable (it is not expected to take the same values everywhere), and a large range of values of γ are possible. Small γ corresponds to sliding having only a small influence on glacier flow; large γ corresponds to sliding dominating the flow (this is sometimes referred to as *plug* flow), and it may be appropriate to re-scale the equation in this case so that the sliding term balances the accumulation and time-derivative.

In the following sections we discuss the physical basis for adopting a sliding law of the form (3.70), and the additional role that subglacial water may play.

3.3.1 Viscous sliding

The presence of a thin film of water between ice and the underlying substrate means that on a micro scale it is appropriate to apply a zero shear stress condition on the ice flow (compared to the ice, the water is essentially inviscid, so supports negligible shear stress). If the bed were perfectly flat, this would allow very rapid sliding, but this is never likely to be the case in reality. There are small (mm to m scale) bumps over which the ice must deform. Such roughness provides a resistance that supports the large scale shear stress τ_b (which is known from (3.12)).

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Since the microscopic shear stress is zero, the resistance comes from normal stress differences between the upstream and downstream faces of obstacles in the bed. These can be crudely estimated on dimensional grounds from the Glen's flow law: if the sliding velocity is u_b, the strain rates associated with flowing over an obstacle of size a are of order u_b/a, and this gives rise to stress differences of order $(u_b/aA)^{1/n}$. If such obstacles occupy a fraction ν^2 of the bed (by area), this suggests that the flow over such obstacles can be parameterised by a sliding law of the form

$$\tau_b = \nu^2 (aA)^{-1/n} u_b^{1/n}. \qquad (3.74)$$

A more satisfactory theoretical derivation of the sliding law can be performed if the ice is treated as Newtonian. In that case, assuming small amplitude roughness of the bed, a linearised model of the viscous fluid flow can be solved (using Fourier transforms), to find

$$\tau_b = \frac{\eta u_b}{\pi} \int_0^\infty \hat{B}(k) k^3 \, \mathrm{d}k, \qquad (3.75)$$

where the integral is taken over all waveumbers, and $\hat{B}(k)$ is the Fourier transform of the micro-scale bed profile $Z = B(X)$. This makes clear that the basal stress depends on many scales of bed roughness.

3.3.2 Regelation

A second process occurs at the same time as the viscous flow described above, and is more efficient at allowing ice so move past the smallest obstacles (note that the resistance goes to infinity as the obstacle size a goes to zero in the above expressions). This process is referred to as regelation. It relies upon the pressure-dependence of the melting point, which means that the temperature at the ice base (which is constrained to the melting point) is slightly lower on the upstream faces of obstacles than on the downstream faces (due to the normal stress difference alluded to above). This results in melting of the ice on the upstream faces and freezing on the downstream faces. The flow of liquid water through the thin lubricating film, from locations of melting to freezing, effectively allows the ice to move 'through' the obstacles.

The rate at which this process allows the ice (and water) to move is controlled by the rate of melting and freezing. Since the phase changes consume and release energy, this rate is controlled by how quickly the energy released by freezing can conduct back through the obstacle to enable melting; this depends on the conductivity of the obstacle, and will be most efficient for small obstacles.

We can again crudely estimate the sliding rate due to this process by balancing the conductive heat flux through a bump, $k_b \Delta T/a$, with the latent heat required to melt the ice $\rho L u_b$. The temperature difference is due to the stress difference, τ_b/ν^2 if the obstacles occupy a fraction ν^2 of the bed, and the Clapeyron slope $C = -dT_m/dp$ (the gradient of the melting temperature with pressure), giving $\Delta T = C\tau_b/\nu^2$. The energy balance therefore gives a sliding law of the form

$$\tau_b = \nu^2 \frac{\rho L a}{k_b C} u_b. \tag{3.76}$$

3.3.3 Weertman sliding

Writing the result for the two mechanisms (viscous flow and regelation) in the form $\tau_b = \nu^2 R_v a^{-1/n} u_b^{1/n}$ and $\tau_b = \nu^2 R_r a u_b$, we can combine them by taking the minimum,

$$\tau_b = \nu^2 \min\left(R_v a^{-1/n} u_b^{1/n}, R_r a u_b\right). \tag{3.77}$$

Since there are in reality a whole range of obstacle sizes, and the two mechanisms are more efficient at transporting ice past small and large obstacles respectively, there is a controlling obstacle size that provides the dominant resistance. This is given by $a = (R_v/R_r)^{n/(n+1)} u_b^{-(n-1)/(n+1)}$, and leads to the combined sliding law

$$\tau_b = \nu^2 R u_b^{2/(n+1)}, \tag{3.78}$$

where $R = (R_r R_v^n)^{1/(n+1)} = (\rho L/k_b C A)^{1/(n+1)}$. This is the same as (3.70), with $m = (n+1)/2$.

A modification of the linear theory described earlier for viscous sliding can be used to derive a law of this type more systematically. Assuming the ice to be Newtonian, and assuming small amplitude roughness we can solve for the viscous deformation of ice over the linearised bed, and for the conduction of heat through the rock beneath, together with boundary conditions on the interface itself that account for the latent heat of melting. This leads to the expression

$$\tau_b = \frac{\eta u_b}{\pi} \int_0^\infty \frac{\hat{B}(k) k_*^2 k^3}{k_*^2 + k^2}\,\mathrm{d}k, \qquad k_* = \frac{\rho L}{4 C k_b \eta}, \tag{3.79}$$

where $\hat{B}(k)$ is again the power spectrum of the bed roughness, and k_* should be interpreted as a controlling wavenumber, which can be identified with the reciprocal of the controlling obstacle size a (for $n = 1$). This is again the same as (3.70), with $m = 1$.

3.3.4 Sliding on sediments

For a glacier lying on a layer of sediments, the sediments may deform under the action of the shear stress τ_b. If the sediment is treated as viscous, with effective viscosity η_T, and occupies a layer of depth d_T (below which is rigid bedrock), then the lubrication approximation suggests that the velocity profile through the sediment layer is linear, with no-slip at its base,

$$u_T = \frac{\tau_b}{\eta_T}(z - b + d_T), \qquad b - d_T \le z \le b. \tag{3.80}$$

(The till layer is assumed to be relatively thin, so that the shear stress may be taken as constant throughout it; otherwise there may be a quadratic Poiseuille-like component to this profile too, driven by the weight of the sediments). If there is no slip between the ice and the top of the sediments, so $u_b = u_T|_{z=b}$, this gives rise to an effective sliding law

$$\tau_b = \frac{\eta_T}{d_T} u_b. \tag{3.81}$$

The constitutive law for deforming sediments is not well constrained by measurements. There is some suggestion that a power law may be appropriate (as for ice), but other suggestions that a perfectly plastic description is better (this corresponds to taking the power law exponent to infinity), in which case the concept of the effective viscosity η_T is less useful.

In either case, a crucial control appears to be played by the *effective pressure* in the sediments. This is the difference between the overburden pressure (roughly the cryostatic pressure $p = \rho g H$) and the pore water pressure p_w. The effective pressure is commonly denoted $N = p - p_w$, and the till viscosity is thought to be an increasing function of N. This suggests a modified sliding law of the form

$$\tau_b = f(u_b, N), \tag{3.82}$$

with $\partial\tau_b/\partial u_b > 0$ and $\partial\tau_b/\partial N > 0$. The question then is what controls the effective pressure, and this is a subject discussed below in the context of subglacial drainage.

3.3.5 Cavitation

Even for hard-bedded glaciers the water pressure, or more precisely the effective pressure, is thought to exert a strong control on sliding, which is not captured by the Weertman law (3.70). The reason for this is the formation of water-filled 'cavities' at the bed.

DIAGRAM

As ice slides over small obstacles in the bed, we noted earlier that this results in elevated pressure on the upstream faces and reduced pressure on the downstream faces. If the pressure on the downstream face is low enough it causes the ice to separate from the rock, forming a water-filled cavity in the lee side of the obstacle. The size of such cavities depends on the effective pressure N. If the water pressure in the cavities is very low (meaning N is large) then the cavities will be small, or perhaps non-existent. If the water pressure is as large as the average overburden pressure (so $N = 0$) the cavities can enlarge to cover almost the entire bed. Consideration of the viscous flow of the ice over the cavities suggests that the controlling quantity is actually u_b/N^n; cavity size increases with this ratio.

Cavities reduce the contact area between ice and bed, and therefore decrease the resistance to flow for a given sliding speed. Thus, as for sliding over sediments, we expect a sliding law of the form in (3.82). Two such laws that are in common usage are a generalised Weertman law,

$$\tau_b = C u_b^{1/m} N^{1/p}, \tag{3.83}$$

and a Coulomb-limited law,

$$\tau_b = \mu N \left(\frac{u_b}{u_b + \Lambda N^n} \right)^{1/n}, \tag{3.84}$$

which has the property that τ_b can't exceed a maximum value μN regardless of how fast the ice slides. Such behaviour, with $\tau_b = \mu N$ is the same as Coulomb friction, a common model of sliding friction in many other contexts.

These modified sliding laws are only able to provide the necessary link between basal velocity and shear stress if the effective pressure, or equivalently the water pressure, is known. This emphasises the importance of subglacial water, to which we now turn.

3.4 Subglacial drainage

The bottom of a glacier is usually a wet place. Geothermal heating, and frictional heating due to sliding, cause the temperature over most of the bed to be at the melting point, with melting rates from the bottom of the ice on the order of mm to cm per year. In addition, large quantities of water from the ice surface reach the bed through crevasses and moulins (roughly cylindrical shafts). The ice surface often melts at rates of several m per year, and this melting is concentrated in a short period during the summer, so the volumes of water reaching the bed can be large.

Some of the subglacial water may flow into cracks in the underlying rock or into the pore space of subglacial sediments, but most of it must be evacuated by flowing along the ice-bed interface and emanating from the glacier terminus. It usually appears at the terminus as a stream flowing out of a tunnel incised into the ice, and it is thought that most of the drainage beneath the glacier occurs in a network of such tunnels analogous to a normal river network.

The theory for these subglacial tunnels - often called channels - was developed by Röthlisberger, after whom they are sometimes named. The channels are thought to open and close each year, since viscous deformation of the ice causes them to creep closed during the winter. In summer, heat generated by dissipation in the turbulent water flow causes them to enlarge again. This melting of the ice around the channel walls is analogous to the erosion that forms river channels; the interesting difference for the subglacial channels is that they creep closed whenever the discharge decreases and the melting is therefore reduced. River channels to not generally fill in so quickly when the discharge is low.

If the subglacial channels are equivalent to rivers, forming an artery-like drainage network, there must also be some equivalent of the distributed overland or groundwater flow that enables water to get into the channels. Sub-glacially this distributed flow is believed to occur by seepage of water between the cavities that form behind bed bumps (as described in the previous section). There is evidence, both from beneath current glacier beds and from deglaciated bedrock, of the water flowing between these cavities, which act in effect as 'pores' of a macroscopic porous medium. The flow through the cavities can be modelled using a turbulent-flow generalisation of Darcy's law that describes flow in a porous medium.

We discuss the dynamics of channels and cavities in turn. An important concept in doing so is the hydraulic potential ϕ, which is given by

$$\phi = p_w + \rho_w g b. \tag{3.85}$$

Here p_w is the water pressure, ρ_w is the water density, g is the gravitational acceleration, and $z = b(x)$ is the bed elevation. The gradient of the potential provides the driving force for water movement (if the potential is constant, the pressure is hydrostatic so there is no movement).

Notice that in a river, where p_w is approximately equal to the constant atmospheric pressure, the gradient of ϕ is simply proportional to gravity times bed slope, which is the driving force used for modelling rivers. If the bed is flat, the potential gradient is the same as the pressure gradient and this corresponds to pressure-driven flow, along a pipe for example.

It is convenient to write the potential in terms of the effective pressure $N = p - p_w$, where $p = \rho g H$ is the hydrostatic ice pressure at the bed. Then,

$$-\frac{\partial \phi}{\partial x} = -\rho_w g \frac{\partial b}{\partial x} - \rho g \frac{\partial H}{\partial x} + \frac{\partial N}{\partial x}. \tag{3.86}$$

In some cases, particularly for mountain glaciers, the gradient of ϕ is dominated by the bed elevation and we can approximate it by

$$-\frac{\partial \phi}{\partial x} \approx \rho_w g \sin\theta, \tag{3.87}$$

where θ is the bed slope.

3.4.1 Subglacial channels

A model for the subglacial channels can be derived in a similar way to that for a river, based on the ingredients of mass conservation and force balance. In addition, since the size of the channel changes quite quickly, we need an equation for the evolution its cross-section (equivalent to the Exner equation for a river).

We assume that the tunnel has a semi-circular cross-section, with area $S(x,t)$ and discharge $Q(x,t)$. Mass conservation is expressed as

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = \frac{M}{\rho_w}, \tag{3.88}$$

where the source term on the right is due to the melting rate M of the ice from the tunnel walls It is also possible to include a source term due to seepage from the surrounding bed and due to water input from the surface through a moulin or crevasse, but we include the latter here as an upstream boundary condition instead.

Force balance is expressed as

$$\tau \ell = -S \frac{\partial \phi}{\partial x}, \tag{3.89}$$

where $\tau = f \rho_w U^2$ is the wall stress (different from the viscous shear stress in the ice, though the notation is the same), $\ell = S^{1/2}(2+\pi)/2^{1/2}\pi^{1/2}$ is the wetted perimeter, and $U = Q/S$ is the average water speed. This rearranges to give

$$FQ^2 = S^{5/2}\left(-\frac{\partial \phi}{\partial x}\right), \tag{3.90}$$

where $F = (2+\pi) f \rho_w / 2^{1/2}\pi^{1/2}$.

The evolution equation for the cross-sectional area is written

$$\frac{\partial S}{\partial t} = \frac{M}{\rho} - \frac{2A}{n^n} S N^n, \tag{3.91}$$

where M is the wall melting rate, ρ is the ice density, N is the effective pressure and A and n are the coefficients in Glen's flow law, which describes the creep of the ice walls inwards.

Finally, the melting rate is determined by an energy balance between then heat generated by turbulent dissipation in the water, given by $Q(-\partial\phi/\partial x)$, and the latent heat required to melt the walls, LM,

$$M = \frac{Q}{\rho L}\left(-\frac{\partial\phi}{\partial x}\right). \tag{3.92}$$

Combining (3.88), (3.90), (3.91) and (3.92) provides a model for the evolution of the tunnel cross section and the effective pressure. It is forced by the input to the channel, which we take to be given as a boundary condition at the upstream end.

The equations can be simplified if we approximate the potential gradient as in (3.87). In this case, taking the steady-state version of (3.91), the equations can be re-arranged to find

$$N = \frac{n(\rho_w g \sin\theta)^{7/5n}}{(2A\rho L F^{2/5})^{1/n}} Q^{1/5n}. \tag{3.93}$$

This implies an increasing relationship between discharge and effective pressure, so suggests that the water pressure will be lower for larger discharge ($p_w = p - N$). This is the opposite of our usual experience, when forcing more water through a pipe, for example, requires a larger water pressure. The reason is that this channel selects its own size; larger discharge results in a larger channel and less resistance as a consequence.

The increasing relationship between Q and N means that if two neighbouring channels are hydraulically connected (by seepage through sediments or cavities for example), water will tend to move from the smaller one (which has a higher water pressure) to the larger one. This 'capturing' property of the channels means that at least locally the water focusses into one main channel; on a larger scale it leads to an arterial-like channel network, similar to a normal river network.

Note that the dependence of effective pressure on Q is rather weak, since $n \approx 3$, and a $1/15$th power-law is almost indistinguishable from a constant unless Q changes over many orders of magnitude.

The relationship in (3.93) also allows us to estimate the size of the pressure gradient in (3.86) and check whether it was consistent to ignore that term. We find that this is a reasonable approximation except near the glacier terminus, where a boundary layer is necessary in order to satisfy the condition that the water pressure must equal atmospheric pressure there.

3.4.2 Linked cavities

Flow between cavities can be described using a law of the form

$$Q \propto -K(h)\frac{\partial\phi}{\partial x}, \tag{3.94}$$

where Q is the discharge, and $K(h)$ is the hydraulic conductivity, which depends on the average depth of the water-filled cavities across the bed, h. This is very similar to Darcy's law, with the conductivity taking the place of the permeability.

The depth of the water-filled cavities depends on the roughness of the bed and, as discussed in the previous section, on the effective pressure and also the sliding speed. Thus we may take

$$h = h(u_b/N^n), \tag{3.95}$$

an increasing function of the ratio u_b/N^n. Since $K(h)$ is also increasing, this suggests that, for given potential gradient $-\partial\phi/\partial x \approx \rho_w g \sin\theta$, the discharge through the linked cavities decreases with N, *i.e.*

$$\frac{\partial Q}{\partial N} < 0. \tag{3.96}$$

This is different from the behaviour in the Röthlisberger channels, and helps explain why the water flow through cavities does not focus into a channel network.

This behaviour also implies that if all drainage occurs through the cavities and the water discharge increases (which it must do during summer to evacuate all the meltwater coming from the ice surface), the effective pressure must decrease. Returning to the sliding law of the previous section (3.82), this suggests that the sliding speed would increase when there is more water flowing (since the shear stress τ_b is determined by the ice depth so can't change quickly).

However, with larger discharge through the cavities it is possible for some of the larger ones to grow into Röthlisberger channels, and thus change the drainage system into a channel network through which the majority of the water flows. The discharge then becomes an increasing function of effective pressure (3.93), and so the sliding law indicates that the sliding speed decreases with more water.

This discussion helps to explain the confusing observation that some glaciers are seen to speed up during the summer whereas some are found to slow down. It makes predictions about how glacier speed will change in response to future climate particularly challenging.

3.5 Glacier surging

Glacier *surges* are sudden dramatic increases in the speed of a glacier that occur with an approximately regular period. The speed-up can be 100 fold (to speeds of around 10 km y^{-1}), and lasts for several months. During this period the glacier advances and thins. It is followed by a quiescent period, in which the glacier moves at a more usual glacial pace, and which lasts for several years or more. We attempt to understand this phenomena as a form of relaxation oscillation.

The most well-documented example of a surging glacier is Variegated glacier in Alaska, where a surge during 1982 and 1983 was studied with a large field campaign. There are, however, many hundreds of surging glaciers around the world. It is currently not known why some glaciers undergo this surging behaviour while others do not.

A leading explanation for the surge behaviour is that it corresponds to a sudden change in the structure of the subglacial drainage system. The idea is that the surge corresponds to a state in which channel drainage is unstable and the water is forced to flow through an inefficient system of linked cavities. The cavity system results in smaller values of effective pressure which, according to the sliding law (3.82) result in faster sliding speeds. When a channel system dominates the drainage, the effective pressure is larger and the sliding speeds more subdued.

The reason why a channel-dominated drainage system might become unstable is a little mysterious, but is related to the size of the cavities. If the cavities become sufficiently large, the channels will shut down and all the water flows through the cavities. The critical parameter is therefore u_b/N^n, which controls cavity size. Below a certain value of this parameter, Λ_c say, channels are stable, and above it they are not.

We denote the effective pressure in a channel system and a cavity system by N_R and N_K respectively, and suppose that these are constants (they depend on the water discharge, but only weakly as noted earlier). The generalised Weertman sliding law (3.83) then gives the corresponding sliding velocities as

$$u_b = C^{-m}\tau_b^m N_R^{-m/p}, \qquad u_b < \Lambda_c N_R^n, \tag{3.97}$$

$$u_b = C^{-m}\tau_b^m N_K^{-m/p}, \qquad u_b > \Lambda_c N_K^n, \tag{3.98}$$

where the basal shear stress is $\tau_b \approx -\rho g H \sin\theta$. Note that since $N_R > N_K$, there is a range of sliding velocities over which both of these options are possible, and since u_b is controlled in each case by the basal shear stress this corresponds to a range of ice depths. The relationship between sliding speed and ice depth is therefore multivalued. Converting into a relationship between ice flux q and depth H (we ignore internal shearing of the ice, since the surge is associated with sliding), this gives

$$q = (\rho g \sin\theta/C)^m N_R^{-m/p} H^{m+1}, \qquad H < H_+, \tag{3.99}$$

$$q = (\rho g \sin\theta/C)^m N_K^{-m/p} H^{m+1}, \qquad H > H_-, \tag{3.100}$$

where $H_\pm = (C/\rho g \sin\theta)\Lambda_c^{1/m} N_{R/K}^{n/m+1/p}$.

The lower branch corresponds to the channel drainage system with slower sliding speed, and the upper branch corresponds to the cavity drainage system with faster sliding speed. Such a multivalued relationship can then be combined with the mass conservation equation,

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a, \tag{3.101}$$

to explain the occurrence of surge cycles as follows.

The growing glacier occupies the lower branch of the ice flux/depth relationship, and gets thicker over time, growing towards a steady state that would require the ice to be thicker than H_+. Before it reaches the steady state, it therefore reaches the end of the lower branch; the channel drainage system shuts down and the glacier is forced onto the upper fast-moving branch. Only a small section of the glacier will have exceeded H_+ initially, but the theory goes that once some part of the ice has accelerated it forces the rest of the glacier with $H > H_-$ to move onto the upper branch too (this is due to the longitudinal stress that have been neglected in our lubrication theory, but which prevent there from being large downstream gradients in velocity). This corresponds to the initiation of the surge, which then causes the glacier to lengthen and thin (the steady state to which the glacier is now evolving has thickness $H < H_-$ since the same steady-state ice flux now requires thinner ice). Once the ice has thinned sufficiently it falls off the upper surging branch and returns to the lower branch. The quiescent period of slow thickening then begins again. The surge period is dominated by the slow growth from H_- to H_+, since the thinning during the surge occurs more rapidly.

A similar explanation of surges in terms of a multivalued ice flux-depth relationship is related to the thermal state of the glacier. The basic mechanisms of having two overlapping 'branches' of the relationship is the same, but in this case the slow branch corresponds to the glacier having a frozen bed and therefore little or no sliding, whereas the fast branch corresponds to the glacier having a temperate bed with rapid sliding. The cold branch operates at ice depths below a critical thickness H_+, beyond which the insulating effect of the ice allows its bed to be warmed to the melting point by geothermal heating. The increased frictional heating due to sliding when the bed is at the melting point means that the warm sliding state can operate down to lower thicknesses $H > H_-$, so that there is again a range of depths for which both states are possible.

As a simple model to illustrate this we assume a conduction-dominated temperature profile (this is not usually a good model, since advection is also important, but it illustrates the point). In this case the steady-state temperature profile is governed by

$$0 = k\frac{\partial^2 T}{\partial z^2}, \qquad b < z < s, \tag{3.102}$$

with boundary conditions

$$T \le 0, \quad -k\frac{\partial T}{\partial z} = G \qquad \text{or} \qquad T = 0, \quad -k\frac{\partial T}{\partial z} \ge G + \tau_b u_b \qquad \text{at} \quad z = b, \tag{3.103}$$

and

$$T = T_s \quad \text{at} \quad z = s, \tag{3.104}$$

where G is the geothermal heat flux and $T_s < 0$ is the surface temperature.

In the cold-bedded case, $T = T_s + (G/k)(s - z)$, $u_b = 0$, and this applies while $H \le H_+ = kT_s/G$. In the warm-bedded case, $T = T_s(1 - (s - z)/H)$, $u_b \ge 0$ (related to τ_b by the sliding law), and this applies while $H \ge H_-$ where H_- satisfies $kT_s/H = G + \tau_b u_b$ (for example $kT_s/H_+ = G + C(\rho g \sin\theta)^m H_+^m$, using the Weertman sliding law (3.70) and $\tau_b = \rho g H \sin\theta$).

This thermal mechanism has been used on a larger scale to explain huge surges of the Laurentide ice sheet originating in Hudson bay during the last glacial period. These surges supposedly discharged vast quantities of ice bergs to the North Atlantic, which deposited frozen-on debris as they melted. The debris is recorded in layers of sediments taken from the ocean floor, and the deposition events are referred to as Heinrich events.

3.6 Ice sheets

The canonical model for an ice sheet is very similar to that for a glacier, but without a particular direction of bed slope. We consider an isothermal two-dimensional ice sheet, with net accumulation $a(x, t)$, bed elevation $z = b(x)$ and surface elevation $z = s(x, t)$. The expressions for mass conservation (3.6) and flux (3.15) therefore hold, together with the results for ice pressure, shear stress, and velocity profile,

$$p = \rho g(s - z), \qquad \tau = -\rho g(s - z)\frac{\partial s}{\partial x}, \tag{3.105}$$

$$u = u_b - \frac{2A(\rho g)^n}{n+1}\left|\frac{\partial s}{\partial x}\right|^{n-1}\frac{\partial s}{\partial x}\left(H^{n+1} - (s - z)^{n+1}\right), \tag{3.106}$$

$$q = u_b H - \frac{2A(\rho g)^n}{n+2}\left|\frac{\partial s}{\partial x}\right|^{n-1}\frac{\partial s}{\partial x}H^{n+2}, \tag{3.107}$$

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = a. \tag{3.108}$$

After non-dimensionalising, assuming no basal sliding and a flat bed $b = 0$ for simplicity, we have

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[H^{n+2}\left|\frac{\partial H}{\partial x}\right|^{n-1}\frac{\partial H}{\partial x}\right] + a. \tag{3.109}$$

The generalisation to two horizontal dimensions, when $H = H(x, y, t)$, is

$$\frac{\partial H}{\partial t} = \nabla \cdot \left[H^{n+2}\left|\nabla H\right|^{n-1}\nabla H\right] + a, \tag{3.110}$$

where $\nabla = (\partial x, \partial y)$ is the two-dimensional gradient. Generalisations that allow sliding at the ice-sheet bed are straightforward to derive in the same way as for glaciers.

3.6.1 Steady states

Returning to one horizontal dimension, as for a glacier, a steady-state ice sheet terminating on land has boundaries x_{m-} and x_{m+} such that

$$\int_{x_{m-}}^{x_{m+}} a(x')\,\mathrm{d}x' = 0. \tag{3.111}$$

If we assume symmetry at $x = 0$, so $q = 0$ there, then the right-hand margin x_m is such that

$$\int_0^{x_m} a(x')\,\mathrm{d}x' = 0, \tag{3.112}$$

and the ice thickness is given by

$$H^{n+2}\left(-\frac{\partial H}{\partial x}\right)^n = \int_0^x a(x')\,\mathrm{d}x'. \tag{3.113}$$

Assuming $H = 0$ at $x = x_m$, this can be integrated to give

$$H_0(x) = \left[\int_x^{x_m} \frac{2(n+1)}{n}\left(\int_0^{x''} a(x')\,\mathrm{d}x'\right)^{1/n}\mathrm{d}x''\right]^{n/2(n+1)}. \tag{3.114}$$

The situation is different if the ice sheet terminates in the ocean, because in this case neither the ice thickness nor the flux need be zero there, so the condition that the integrated accumulation should be zero no longer holds. Instead, if q_m is the ice flux at the margin x_m, we have (in a steady state),

$$\int_0^{x_m} a(x')\,\mathrm{d}x' = q_m. \tag{3.115}$$

This ice flux might break off immediately as icebergs, or it might flow into a floating ice shelf that is attached to the end of the ice sheet. In this case, x_m is referred to as the *grounding line*, since it is where the ice shelf grounds on the sea floor.

A detailed theory of the flow across the grounding line suggests that it may be reasonable to impose $q_m = Q_m(x_m)$ as a known function of space, and to impose $H(x_m) = H_m = 0$. (The reason for this is that the stress balance where the ice becomes afloat forces the ice thickness and flux to take certain values that depend on the depth of the water there, but the ice thickness is relatively small and can be approximated as zero). In this case the net mass balance statement (3.115) still determines the position of the margin, as for the land-terminating ice-sheet. If $H_m = 0$, then (3.114) also determines the ice thickness.

3.6.2 Accumulation-elevation feedback

An interesting aspect of ice sheets is that their surface elevation is essentially self-determined, unlike a glacier where the surface elevation is largely set by the bed elevation. The ice in an ice sheet is usually much thicker, and the surface topography does not necessarily bear much resemblance to the bed topography. This means that the net accumulation a, which depends strongly on the air temperature at the surface, is not externally prescribed; it depends on the solution H.

In fact it does so in a way that generically suggests 'blow-up' behaviour, since we would expect a to be an increasing function of H, and the mass conservation equation becomes

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[H^{n+2}\left|\frac{\partial H}{\partial x}\right|^{n-1}\frac{\partial H}{\partial x}\right] + a(H). \tag{3.116}$$

The equivalent linear problem would be the reaction diffusion equation,

$$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + H. \tag{3.117}$$

Our nonlinear generalisation of this equation has the same general property that the 'reaction' term can cause 'blow-up' ($H \to \infty$), perhaps limited by the diffusion term depending on the size of the domain. The ice sheet also has a free boundary, and the reaction term can therefore lead to an ice sheet that grows ever wider and higher. In reality, it is limited either by flowing into the ocean or by spreading to low enough latitudes that the net accumulation becomes sufficiently negative to balance the increased accumulation at the surface.

A simple toy model that illustrates dependence of accumulation on surface elevation and latitude is to take $n = 1$ (Newtonian ice) and

$$a = \frac{1}{4}(H^4 - x). \tag{3.118}$$

Here x can be considered as distance from the pole, and we assume the ice grows symmetrically from $x = 0$ (though in one horizontal dimension still, for simplicity). The steady state then satisfies

$$0 = \frac{\partial}{\partial x}\left(H^3\frac{\partial H}{\partial x}\right) + \frac{1}{4}(H^4 - x), \tag{3.119}$$

with boundary conditions

$$H^3\frac{\partial H}{\partial x} = 0 \quad \text{at} \quad x = 0, \qquad H^3\frac{\partial H}{\partial x} = H = 0 \quad \text{at} \quad x = x_m. \tag{3.120}$$

The conditions serve to determine the position of the margin x_m.

DIAGRAM

Letting $y = H^4$, the problem simplifies to

$$x = \frac{\partial^2 y}{\partial x^2} + y. \tag{3.121}$$

with

$$\frac{\partial y}{\partial x} = 0, \quad \text{at} \quad x = 0, \qquad \frac{\partial y}{\partial x} = y = 0 \quad \text{at} \quad x = x_m. \tag{3.122}$$

The solution is found to be

$$y = x - \sin x + \frac{1 - \cos x_m}{\sin x_m} \cos x, \tag{3.123}$$

where x_m satisfies

$$x_m \sin x_m = 1 - \cos x_m. \tag{3.124}$$

In fact there are two viable solutions here; ones has $x_m \approx 2.33$, and the other has $x_m = 0$. The latter solution corresponds to there being no ice at all. One of the features of the accumulation-elevation feedback is that there can be multiple steady states. It is believed that the current environment of the Greenland ice sheet probably gives rise to such multiple states. The ice sheet is relatively close to an equilibrium (though by no means in equilibrium; it is losing mass quite quickly), but if all the ice were somehow removed instantaneously it is unlikely it would grow back under present conditions. The climate of Antarctica on the other hand is sufficiently cold that were that ice sheet to be suddenly removed, at least a large part of it would grow back again.