2 Rivers

2.1 Simple models and the flood hydrograph

2.1.1 Mass conservation

Rivers are usually much longer than their width or depth, and are therefore modelled as quasi-one-dimensional features, described by the discharge (flux) Q ($\mathrm{m}^3\,\mathrm{s}^{-1}$), and cross-sectional area A (m^2). The essential ingredient is conservation of mass, expressed by

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = E, \tag{2.1}$$

where s is distance downstream, t is time, and E ($\mathrm{m}^2\,\mathrm{s}^{-1}$) is a source term, describing precipitation, runoff, and flow from tributaries.

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This equation can be derived in a number of different ways. Consider a small section of the river between x_1 and x_2. The rate of change of the volume of water in the region is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} A\ \mathrm{d}x = Q_1 - Q_2 + \int_{x_1}^{x_2} E\ \mathrm{d}x, \tag{2.2}$$

Using the fundamental theorem of calculus, we can write this as

$$\int_{x_1}^{x_2} \frac{\partial A}{\partial t}\ \mathrm{d}x = \int_{x_1}^{x_2} -\frac{\partial Q}{\partial x} + E\ \mathrm{d}x. \tag{2.3}$$

This holds for any x_1 and x_2 and therefore, assuming A and Q are continuously differentiable, we obtain (2.1).

2.1.2 Turbulent flow

To relate the discharge to the cross-sectional area we must consider the fluid mechanics. The flow in a river is typically turbulent, involving chaotic trajectories with velocity varying rapidly in time and space. This means it is not well described by either the inviscid or laminar viscous fluid theory that may be familiar. Turbulent flow occurs when the Reynolds number, which is a dimensionless measure of the flow speed, is larger than a critical value $Re_c \approx 10^3$ (the precise value depends on the situation). The Reynolds number is defined by

$$Re = \frac{[u][h]}{\nu}, \tag{2.4}$$

where $\nu \approx 10^{-6}\ \mathrm{m}^2\,\mathrm{s}^{-1}$ is the kinematic viscosity of water, and $[u]$ and $[h]$ are the scales for flow speed and depth.

We can describe the turbulent flow using the average downstream velocity $u = Q/A$, and a parameterisation of the drag force that the fluid exerts on the walls of the channel. A common parameterisation of this drag force is

$$\tau = f\rho u^2, \tag{2.5}$$

where $f \approx 0.01$ is a friction factor, ρ is the density, and u is the average flow speed.

2.1.3 Force balance

We assume the flow is slowly varying, which allows us to neglect accelerations. The force balance within a cross-section is then

$$\tau\ell = \rho g S A, \tag{2.6}$$

where τ is the wall shear stress acting over wetting perimeter ℓ, ρ is the density, g is the gravitational acceleration, $S = \sin\alpha$ is the slope, and A is the cross-sectional area. The left hand side is the force per unit length resisting the flow, and the right hand side is the component of weight per unit length in the downstream direction.

Using the parameterisation for shear stress (2.5), this gives

$$u = CR^{1/2}S^{1/2}, \tag{2.7}$$

where we define $C = (g/f)^{1/2}$, and the hydraulic radius

$$R = \frac{A}{\ell}. \tag{2.8}$$

This is referred to as the *Chèzy law*, and the constant C as the Chèzy coefficient (Chèzy, 1775).

An alternative empirical description of flow in an open channel that is commonly used is the *Manning law*,

$$u = \frac{R^{2/3}S^{1/2}}{n}, \tag{2.9}$$

where $n' \approx 0.01 - 0.1$ m$^{-1/3}$ s is the Manning coefficient (Manning, 1890). This is equivalent to a shear stress parameterisation

$$\tau = \frac{\rho g n^2 u^2}{R^{1/3}}, \tag{2.10}$$

and we can therefore identify f with $gn^2/R^{1/3}$, or $C = R^{1/6}/n$.

Either (2.7) or (2.9) can be combined with the definition of flux $Q = uA$, to provide a relationship between discharge and cross-sectional area.

The relationship includes the hydraulic-radius, which itself can be related to the cross-sectional area with an assumption about the geometry of the cross-section. For a canal-shaped cross-section of given width w and smaller height h, we have $A = wh$, and $\ell = w + 2h \approx w$, so that $R = A/\ell \approx A/w = h$. For a notch-shaped cross-section with lateral slope angle β, we have $A = \frac{1}{8}\ell^2 \sin 2\theta$, so $R = A/\ell = A^{1/2}\left(\frac{1}{8}\sin 2\theta\right)^{1/2}$.

In all cases, we can therefore write

$$Q = \frac{cA^{m+1}}{m+1}, \tag{2.11}$$

where, for the Chézy law, $m + 1 = 3/2$ for a canal-shape and $m + 1 = 5/4$ for a notch or circular shaped cross-section, and for the Manning law, $m + 1 = 5/3$ or $4/3$.

2.1.4 Characteristics and shocks

Putting the relationship (2.11) together with the mass conservation equation (2.1) we have

$$\frac{\partial A}{\partial t} + cA^m \frac{\partial A}{\partial x} = E. \tag{2.12}$$

This is a first order quasi-linear equation, and can be solved using the method of characteristics. Since it is non-linear, solutions will generically form shocks (discontinuities), discussed below.

As an example, consider solving the equation on an infinite domain, with no source, and with initial condition $A = A_0(x)$ (the water is assumed to have come from a previous storm, say, or from far upstream). The characteristic equations are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = cA^m, \qquad \frac{\mathrm{d}A}{\mathrm{d}\tau} = 0, \tag{2.13}$$

with initial conditions,

$$t = 0, \qquad x = \sigma, \qquad A = A_0(\sigma), \qquad \text{at} \quad \tau = 0. \tag{2.14}$$

Here τ is used as the variable parameterising each characteristic, and σ is used as the variable parameterising the initial data. We see immediately that $t = \tau$, as is common for this type of equation, and therefore we could just as well take t as the variable parameterising the characteristics.

We also see that $A = A_0(\sigma)$ is constant along each characteristic, which are therefore given by

$$x = \sigma + cA_0(\sigma)^m t, \tag{2.15}$$

and this implicitly defines the solution as

$$A(x,t) = A_0(x - cA^m t). \tag{2.16}$$

So the initial cross-sectional profile is advected downstream at a speed that depends upon the profile. Larger areas move faster, and this generically leads to the formation of shocks (discontinuities).

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Shocks must be described by interpreting the equation in weak form or (equivalently) by returning to the conservation arguments from which the equation was derived. In the frame of a shock moving with speed $\dot{x}_s$, the discharge into and out of the shock must balance, so

$$Q_- - A_- \dot{x}_s = Q_+ - A_+ \dot{x}_s, \tag{2.17}$$

where $-$ and $+$ denote upstream and downstream of the shock. Thus

$$\dot{x}_s = \frac{[Q]_-^+}{[A]_-^+} = \frac{Q_+ - Q_-}{A_+ - A_-}. \tag{2.18}$$

We can find when a shock will form either by looking for where neighbouring characteristics intersect, or by looking for where the gradient of the solution $|\partial A/\partial x|$ becomes infinite. Differentiating (2.16) implicitly gives

$$A_x = A_0'(\sigma)\left(1 - mctA^{m-1}A_x\right), \tag{2.19}$$

and therefore

$$A_x = \frac{A_0'}{1 + mctA_0^{m-1}A_0'}, \tag{2.20}$$

which blows up (tends towards $-\infty$) as $t \to -1/mcA^{m-1}A_0'$. This will happen at different times for different σ (provided $A_0'(\sigma)$ is negative), but we are interested in the first time at which it occurs, which is

$$t_s = \min_{A_0'(\sigma)<0}\left(-\frac{1}{mcA_0^{m-1}A_0'}\right). \tag{2.21}$$

After this moment, a shock must be inserted into the solution. The characteristic equations still hold on either side, but must not be continued through the shock, which moves according to (2.18).

2.1.5 Flood hydrograph

We want to understand how the water level in a river will vary during the course of a flood. We consider the case of a localised storm at $x = 0$, which can be modelled using the initial condition $A(x, 0) = V_0\delta(x)$, where $\delta(x)$ is the delta function.

This will immediately produce a shock that moves downstream. Indeed from the general solution (2.16) we see that $A = 0$ unless $x = cA^m t$, and the solution is therefore

$$A = 0 \qquad x > x_f, \tag{2.22}$$

$$A = (x/ct)^{1/m} \quad 0 < x < x_f, \tag{2.23}$$

where the front x_f moves according to

$$\dot{x}_f = \frac{Q_-}{A_-} = \frac{cA_-^m}{m+1} = \frac{1}{m+1}\frac{x_f}{t}. \tag{2.24}$$

Hence $x_f = Ct^{1/(m+1)}$, and the constant C is determined from the global mass constraint

$$\int_0^{x_f} A \, \mathrm{d}x = V_0, \tag{2.25}$$

which gives

$$C = \left(\frac{m+1}{m}\right)^{m/(m+1)} c^{1/(m+1)} V_0^{m/(m+1)}. \tag{2.26}$$

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2.2 St Venant equations

2.2.1 Force balance

We now consider cases where the acceleration of the water is non negligible. We will see that the magnitude of the acceleration terms is described by a dimensionless number called the *Froude number*, which is a measure of how 'rapid' a river is.

Consider a section of the river between x_1 and x_2. Conservation of momentum (ρAu) can be expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho Au \ \mathrm{d}x = -\left[\rho Au^2\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \rho gAS - \tau\ell \ \mathrm{d}x - \left[\overline{p}A\right]_{x_1}^{x_2}, \tag{2.27}$$

where the terms on the right are the net momentum flux into this section of the river, the gravitational and frictional forces (which were previously assumed to be the only important components of this balance), and the net pressure force. We ignore any runoff and tributaries, which would provide additional source terms on the right hand side. Using the fundamental theorem of calculus, and assuming A and u are continuously differentiable, we derive

$$\frac{\partial}{\partial t}(\rho Au) + \frac{\partial}{\partial x}(\rho Au^2) = \rho gAS - \tau\ell - \frac{\partial}{\partial x}(\overline{p}A). \tag{2.28}$$

Together with the mass conservation equation

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho Au) = 0, \tag{2.29}$$

these are referred to as the *St Venant equations.*

The equations are not yet closed, because we must prescribe a parameterisation for the shear stress τ, and establish how the wetted perimeter ℓ and average pressure $\overline{p}$ depend on the cross-sectional area.

For the average pressure, we assume that the pressure is hydrostatic, so $p = \rho g(\eta - z)$, where η is the elevation of the water surface. Then the average pressure is defined by

$$\overline{p}A = \int_{y_l}^{y_r}\int_{b}^{\eta} \rho g(\eta - z) \ \mathrm{d}z \ \mathrm{d}y, \tag{2.30}$$

where b is the bed elevation and the integral is taken over the whole cross section from the left bank (y_l) to the right bank y_r. We can write this as

$$\overline{p}A = \int_{y_l}^{y_r} \frac{1}{2}\rho gh^2 \ \mathrm{d}y, \tag{2.31}$$

where $h = \eta - b$ is the depth, which might vary with the transverse coordinate y, depending on the cross-sectional geometry. Regardless of the geometry,

$$\frac{\partial}{\partial x}(\overline{p}A) = \int_{y_l}^{y_r} \rho gh\frac{\partial h}{\partial x} \ \mathrm{d}y \approx \rho gA\frac{\partial \overline{h}}{\partial x}, \tag{2.32}$$

since it is reasonable to assume that the downstream slope of the depth will be approximately independent of the transverse coordinate. Here $\overline{h}$ is the mean depth across the cross-section.

Rearranging (2.28), making use of (2.29), we find

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - \frac{\tau\ell}{\rho A} - g\frac{\partial \overline{h}}{\partial x}. \tag{2.33}$$

We concentrate on the case of a canal of width w, so $\overline{h} = A/w$ and $\ell \approx w$. We also adopt the shear stress description $\tau = f\rho u^2$ from (2.5) and which leads to Chézy's law under steady conditions. In this case

$$\frac{\tau\ell}{\rho A} = \frac{fwu^2}{A}, \qquad g\frac{\partial \overline{h}}{\partial x} = \frac{g}{w}\frac{\partial A}{\partial x}, \tag{2.34}$$

so the equations become

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0. \tag{2.35}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - \frac{fwu^2}{A} - \frac{g}{w}\frac{\partial A}{\partial x}. \tag{2.36}$$

2.2.2 Non-dimensionalisation

We scale

$$x = [x]\hat{x}, \quad t = [t]\hat{t}, \quad A = [A]\hat{A}, \quad u = [u]\hat{u}, \tag{2.37}$$

and suppose the length scale $[x]$ is given, choosing other scales so that

$$[t] = \frac{[x]}{[u]}, \quad [A][u] = Q_0, \quad gS[A] = fw[u]^2. \tag{2.38}$$

Here Q_0 is a given discharge scale. It is common when studying rivers to suppose that this scale is known (for any given river we wish to study we could measure or estimate it); this informs the model of how large a river we are dealing with. The choice of timescale is the natural 'advective' timescale. The balance between gravity and friction force is motivated by the fact that this must be the balance under uniform and steady conditions. Rearranging these constraints on the scales, we have

$$[u] = \left(\frac{gSQ_0}{fw}\right)^{1/3}, \quad [A] = \left(\frac{fwQ_0^2}{gS}\right)^{1/3}. \tag{2.39}$$

For example, for the Thames at Oxford we might take $Q_0 = 20$ m^3 s^{-1}, $w = 10$ m, $f = 0.05$, $g = 10$ m s^{-2}, $S = 10^{-3}$, then $[u] = 0.7$ m s^{-1}, and $[A] = 27$ m^2.

The dimensionless equations become

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0, \tag{2.40}$$

$$\delta F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{A} - \delta\frac{\partial A}{\partial x}. \tag{2.41}$$

The dimensionless numbers are the Froude number,

$$F = \frac{[u]}{\sqrt{g[h]}}, \tag{2.42}$$

where $[h] = [A]/w$ is the depth scale, and the ratio of the depth gradient to bed slope,

$$\delta = \frac{[h]}{S[x]}. \tag{2.43}$$

The Froude number is a measure of how rapid the river is. Note that $\sqrt{gh}$ is the speed of surface waves on a layer of water of depth h, so the Froude number can be thought of as the ratio of the river speed to the speed of surface waves. This will have relevance below for the direction of propagation of surface waves on a river. If $F > 1$ the river flow is referred to as *supercritical*, whereas is $F < 1$ it is *subcritical*.

Note that it would be possible to choose the length scale $[x]$ so as to make $\delta = 1$. This would define a natural length scale on which the pressure gradients are comparable with the gravitational term and the shear stress. However, we usually imagine that an external length scale is imposed on the problem (for instance, we are interested in predicting flood conditions at certain locations).

2.2.3 Limits

The equations (2.40) and (2.41) are a pair of first order nonlinear hyperbolic equations. They can again be solved using the method of characteristics, although doing so analytically is not possible in general. Below we will consider steady states and linearised perturbations to the steady states. First, we note how the equations behave under certain limits.

In the limit $\delta \ll 0$ (long-wave theory), we recover the slowly varying model from earlier, since the force balance equation reduces to $u^2 \approx A$, and therefore $Q = Au = A^{3/2}$. If in addition $F \ll 1$ (the flow is sufficiently tranquil), then the approximation can be improved to give

$$u^2 \approx A\left(1 - \delta\frac{\partial A}{\partial x}\right). \tag{2.44}$$

Substituting this into the mass equation gives

$$\frac{\partial A}{\partial t} + \tfrac{3}{2}A^{1/2}\frac{\partial A}{\partial x} = \tfrac{1}{2}\delta\frac{\partial}{\partial x}\left(A^{3/2}\frac{\partial A}{\partial x}\right), \tag{2.45}$$

so the correction provides a non-linear diffusion term. This has the effect of smoothing out (regularising) the shocks that would form in the absence of the diffusion term. If $0 < \delta \ll 1$, the effect on the flood hydrograph is to provide a sharply rising curve, rather than a discontinuous jump up as in the case of $\delta = 0$.

If $\delta \gg 1$ (i.e. the length scale of consideration is sufficiently short), the momentum equation can be approximated as

$$F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial A}{\partial x} = 0, \tag{2.46}$$

which together with the mass equation (2.40) are the *shallow water equations*. These apply when we consider short length scales, or very shallow hill slopes, so that the downstream component of gravity and friction play a negligible role.

2.2.4 Stability

Consider the dimensionless model for the canal (since cross-section A and depth h are linearly related we can use either as the primary variable),

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{2.47}$$

$$F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{h} - \frac{\partial h}{\partial x}. \tag{2.48}$$

We have chosen the length scale so that $\delta = 1$.

There is a uniform steady state with $u^2 = h$ and $uh = 1$ (by the choice of the non-dimensionalisation we can set the dimensionless flux uh to 1 without loss of generality). Thus $u = h = 1$.

We consider the linear stability of this uniform state by writing $u = 1 + U$, $h = 1 + H$, where the capitalised variables will be supposed small. Substituting into the equations gives

$$H_t + H_x + U_x = 0, \tag{2.49}$$

$$F^2(U_t + U_x) = -2U + H - H_x, \tag{2.50}$$

which we can rearrange into

$$F^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 U = -2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) U - \frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}. \tag{2.51}$$

As usual with linear stability analyses, we look for exponential solutions,

$$U = \hat{U}\exp((\sigma t + ikx)/F^2) = \hat{U}\exp((\sigma_R t + i(kx + \sigma_I t))/F^2), \tag{2.52}$$

where we write $\sigma = \sigma_R + i\sigma_I$, and where σ_R/F^2 is the growth rate, k/F^2 is the wave number, and $-\sigma_I/k$ is the wave speed (the factors of F^2 are included in the exponential with hindsight, to make the algebra easier).

Substituting into the equation we obtain the dispersion relation

$$\sigma = -1 - ik + \left(1 - ik - \frac{k^2}{F^2}\right)^{1/2}, \tag{2.53}$$

which is most easily analysed by writing the square root as $p + iq$, say, so that

$$\sigma_R = -1 \pm p, \qquad -\frac{\sigma_I}{k} = 1 \mp \frac{q}{k}, \tag{2.54}$$

(we may take $p > 0$ without loss of generality). By the definition of $p + iq$, equating real and imaginary parts,

$$2pq = -k, \qquad p^2 - q^2 = 1 - \frac{k^2}{F^2}, \tag{2.55}$$

and so

$$L(p) := p^2 - \frac{k^2}{4p^2} = 1 - \frac{k^2}{F^2}. \tag{2.56}$$

The left hand side here is an increasing function of p, so $p > 1$ (which leads to $\sigma_R > 0$ and therefore instability) if and only if $L(p) > L(1)$, *i.e.* if and only if

$$1 - \frac{k^2}{F^2} > 1 - \frac{k^2}{4}, \tag{2.57}$$

which requires $F > 2$. Thus surface waves are unstable if the flow is sufficiently fast.

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The wave speeds are of opposite sign (*i.e.* one moves upstream and the other downstream) unless $-q < k$, which is the case if and only if $p > \frac{1}{2}$, or equivalently $L(p) > L(\frac{1}{2})$. This requires

$$1 - \frac{k^2}{F^2} > \frac{1}{4} - k^2, \tag{2.58}$$

or equivalently

$$\frac{3}{4} > k^2 \left(\frac{1}{F^2} - 1 \right). \tag{2.59}$$

Thus for $F > 1$, all waves travel downstream, and the flow is described as super-critical (although if $F < 2$ the waves are stable and will decay as they travel). For $F < 1$, waves can travel both up and downstream and the flow is described as subcritical (although waves with small enough wavenumber will all travel downstream even in this case).

In the case $F > 2$, the instability evolves to produce non-linear waves, referred to as *roll waves*. These are often visible in heavy rain on steep pavement. They form periodic trains of waves that can be analysed by looking for travelling wave type solutions of the equations $A = A(x - ct)$, $u = u(x - ct)$.

2.3 Sediment transport and Dunes

This section concerns the erosion and transport of sediment in rivers, and the resultant sculpting of the river bed to form *dunes* and *anti-dunes*. We focus on rivers, but many of the ideas are also relevant to subaerial erosion and transport (driven by the wind, rather than by water flow), and therefore to the formation of a larger class of landforms, including desert sand-dunes.

2.3.1 Patterns in rivers

Erosion and sediment transport are responsible for may different patterns in rivers. On a large scale, the path of the river channel is a result of the long-term interaction with the substrate (which may be rigid bedrock, or may be looser sediments that were deposited previously by floods, wind, or by deglaciation). *Meanders* are the most obvious manifestation of this patterning. These are associated with a secondary transverse flow that results in larger velocities and hence more erosion on the outer banks of bends. Although we will not discuss meandering further, there is a large and interesting literature on the mechanisms that lead to it.

In a river channel itself, there are several types of instability that can occur to the uniform flow state. The basic mechanism for instability is that erosion rates typically increases with water speed, which increases with water depth. Thus deeper areas undergo more erosion, and this provides a positive feedback. Instabilities can be broadly classified according to whether the bed profile variations are transverse or parallel to flow.

Transverse instabilities produce *lateral bars*. Such bars commonly form in gravel-bedded rivers, and often interact with a meandering instability to form alternating bars, on alternate sides of the river. In wide rivers, many bars may form across the channel and are referred to as multiple row bars. They often form when the river is high, and produce islands or beaches when the water level drops. For much of the time such rivers are split into many connecting braids, and are referred to as a *braided river*. Vegetation may form on the bars, and helps to stabilise them against further erosion.

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Longitudinal instabilities have undulations in the downstream direction, and are referred to as ripples, dunes or anti-dunes. Which of these occurs depends on the flow conditions, and particularly on the size of the Froude number F. Dunes form at low Froude number ($F < 1$); they move slowly downstream, are typically of small amplitude compared to the depth of the river, and have river surface perturbations that are out of phase with the bed. Anti-dunes form at high Froude numbers ($F > 1$); they move upstream and have large amplitude surface perturbations that are in phase with those in the bed. Anti-dunes are often seen on shallow streams flowing over a beach.

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We will primarily focus on understanding the mechanism for these longitudinal instabilities, and we will restrict attention to their linear (small amplitude) evolution. Much of the erosion of river channels occurs during floods, when the river channel is at its widest and the erosive power of the flow is greatest. Nevertheless, we will restrict our attention to steady flow conditions.

The mechanisms responsible for forming *aeolian* dunes are similar to those considered here, but the fact that the wind can change direction more easily than a river gives rise to a larger range of behaviour. Similar 'transverse' dunes do occur, at right angles to a strongly prevailing wind direction. They typically have a distinctive sharp crest, which is due to separation of the flow at this point to form a recirculating pocket on the lee side of the dune. This causes the upstream side to have a relatively shallow slope, and the downstream side to have a steep slope (typically at the limiting angle of friction).

Linear dunes can also form aligned with the mean wind direction, if two different prevailing directions alternately blow from each side of the dune; these are called seifs. Star-shaped dunes form when the wind blows from many different directions. Another common aeolian dune is the barchan dune, which has a crescent shape with the arms pointing in the direction of the wind. These typically form when there is a limited supply of sand. Anti-dunes do not form in the desert because the effective Froude number is not large enough.

2.3.2 Sediment transport mechanisms

Sediment transport occurs by two processes: *bedload* and *suspension*. For given flow conditions, larger particles will roll along the bed, dragged by the stress exerted by the fluid. This is the bedload transport. Smaller particles are entrained by turbulent eddies into the flow to be transported in suspension. Suspended particles also settle out of the water, due to their greater density, and a steady-state balance between entrainment and deposition determines the suspended sediment load.

Natural rivers have a large range of grain sizes: gravel is typically > 1 mm, sand in the range $60\ \mu\text{m} - 1$ mm, silt in the range $2 - 60\ \mu\text{m}$. Sufficiently large particles can be neither entrained nor dragged along the bed, and are essentially immobile. Empirical observations suggest that a dimensionless measure of the basal shear stress, referred to as the Shields stress, is a good indicator of whether given grains will be mobilised under a given shear stress. The Shields stress is defined by

$$\tau^* = \frac{\tau}{\Delta\rho g D_s}, \tag{2.60}$$

where τ is the shear stress, D_s is the grain size, and g the gravitational acceleration, and $\Delta\rho = \rho_s - \rho_w$ the density difference between the grain and water. Sediment transport occurs if τ^* is larger than a critical value τ_c^*, which itself depends on the particle size via the particle Reynolds number

$$Re_p = \frac{u_* D_s}{\nu_w}, \tag{2.61}$$

where $\nu_w \approx 10^{-6}$ m^2 s^{-1} is the kinematic viscosity of water, and u_* is the *friction velocity*, defined by

$$u_* = (\tau/\rho_w)^{1/2}. \tag{2.62}$$

The dependence of τ_c^* on Re_p is relatively weak ($\tau_c^* \approx 0.06$ except at low low rates). Above τ_c^*, sediment transport occurs by suspension (at low Re_p) or bedload (at high Re_p).

The shear stress can be related to the mean flow using the law (2.5),

$$\tau = f\rho_w u^2, \tag{2.63}$$

where f is a dimensionless friction factor in the range $0.01 - 0.1$ (in this case $u_* = f^{1/2}u$).

Descriptions for bedload transport and for erosion are ultimately empirical.

Bedload is described in terms of a bedload flux q_b, with units m^2 s^{-1} (the volume transported per unit stream width per unit time). This is typically a function of the basal shear stress, and therefore of the stream speed, through (2.63). One common description is the Meyer-Peter-Müller relation,

$$q_b = (\Delta\rho g D_s^3/\rho_w)^{1/2} q_*, \qquad q_* = K\left[\tau^* - \tau_c^*\right]_+^{3/2}. \tag{2.64}$$

Here q_* is a dimensionless bedload flux, with $\tau_c^* \approx 0.047$ the threshold Shields parameter defined above, $K \approx 8$ a dimensionless constant, and $[\cdot]_+ = \max(\cdot, 0)$.

Suspended sediment is described in terms of the concentration c of sediment in the water column (there is a choice of units for concentration; we use kg m^{-3}). Erosion from the bed (or entrainment of previously eroded sediments) is described using an erosion rate v_E (with units m s^{-1}), which is, like bedload, empirical function of the shear stress. Deposition occurs at a rate v_D that is proportional to the concentration of sediments, and to their settling velocity. The latter is a function of the particle size and density (it is referred to as Stokes settling velocity and can be derived using Stokes equations for low-Reynolds number flow),

$$v_s = \frac{\Delta\rho g D_s^2}{18\eta_w}. \tag{2.65}$$

We use this velocity scale to non-dimensionalise the erosion and deposition rates, writing

$$v_E = v_s E, \qquad v_D = v_s \frac{c}{\rho_s}, \tag{2.66}$$

where E is a dimensionless function of shear stress τ or velocity u.

2.3.3 Exner equation and suspended sediment concentration

We consider a two-dimensional river (e.g. a wide canal), with bed elevation $s(x,t)$, depth $h(x,t)$, and surface elevation $\eta = s + h$. The essential equation describing bedform evolution is the Exner equation,

$$(1-n)\frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = v_D - v_E. \tag{2.67}$$

This equation represents conservation of sediment in the bed; n is the bed porosity (assumed constant). The deposition and erosion terms on the right hand side represent a source and sink of sediment, which is passed into suspension. These appear as equal and opposite source terms in the equation for suspended sediment conservation

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(huc) = \rho_s(v_E - v_D), \tag{2.68}$$

Using mass conservation of the fluid this can also be written

$$h\frac{\partial c}{\partial t} + hu\frac{\partial c}{\partial x} = \rho_s(v_E - v_D). \tag{2.69}$$

2.3.4 Bedload transport

We consider first the evolution of a bed accounting only for bedload transport, so $v_E = v_D = 0$. Our aim is to establish under what conditions, if any, we can explain the formation of dunes or anti-dunes. We must combine the Exner equation (2.67) with an empirical law for the bedload flux $q_b(\tau)$, and a description of the fluid flow.

For the fluid flow, we adopt the St Venant equations in the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{2.70}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = g\left(S - \frac{\partial \eta}{\partial x}\right) - \frac{fu^2}{h}, \tag{2.71}$$

appropriate for a wide canal with depth h, and where we have used the parameterisation $\tau = f\rho u^2$. This is combined with the Exner equation, which we write in the form

$$(1-n)\frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = 0. \tag{2.72}$$

There are two timescales at play here, and the situation is simplified by the fact that these are typically quite different, the timescale for bedload transport being much longer than the advective timescale for the flow. This is made clear by non-dimensionalising the model.

We write

$$x = [x]\hat{x}, \quad t = [t]\hat{t}, \quad u = [u]\hat{u}, \quad s, h, \eta = [h]\left(\hat{s}, \hat{h}, \hat{\eta}\right), \quad q_b = [q_b]\hat{q}. \tag{2.73}$$

The balance of terms in the momentum equation suggests we choose scales so that

$$gS[h] = f[u]^2, \tag{2.74}$$

and we take the flux scale $Q = [h][u]$ as given, so choose scales

$$[h] = \left(\frac{fQ^2}{gS}\right)^{1/3}, \quad [u] = \left(\frac{gSQ}{f}\right)^{1/3}, \quad [t] = \frac{(1-n)[h][x]}{[q_b]}. \tag{2.75}$$

It is possible to choose the length scale $[x]$ to balance the gravity and pressure term in the momentum equation, as was done earlier, but this gives a length scale longer than typical bedform instabilities, and since our goal is to examine the latter we choose $[x]$ to be on the order of observed dune wavelengths. The scale for bedload flux $[q_b]$ is also assumed known once the scale for the velocity (and hence shear stress) has been chosen.

This results in the dimensionless equations

$$h = \eta - s, \tag{2.76}$$

$$\varepsilon \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{2.77}$$

$$F^2 \left(\varepsilon \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = \delta \left(1 - \frac{u^2}{h} \right) - \frac{\partial \eta}{\partial x}, \tag{2.78}$$

$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{2.79}$$

where the parameters are

$$\varepsilon = \frac{[x]}{[u][t]}, \quad F = \frac{[u]}{\sqrt{g[h]}}, \quad \delta = \frac{S[x]}{[h]}, \tag{2.80}$$

and where the dimensionless bedload function can be taken to be a function of velocity, $q = q(u)$ (this is derived from the dimensional $q_b(\tau)$, but τ is directly related to u, and since u is a primary variable in our model it is easier to write it this way).

Here ε is the ratio of the advective timescale to the sediment transport timescale, which we expect to be small (note $\varepsilon = [q_b]/(1-n)Q$ so represents the ratio of sediment flux to water flux). In addition, we expect δ to be small, since the dune wavelengths are typically small compared to the natural length scale of the river (e.g. $S = 10^{-2}$, $[h] = 1$ m and $[x] = 10$ cm gives $\delta = 10^{-3}$). We can therefore approximate the problem by taking $\varepsilon \to 0$ and $\delta \to 0$.

The model simplifies to

$$hu = (\eta - s)u = 1, \tag{2.81}$$

$$F^2 u \frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial x}, \tag{2.82}$$

$$\frac{\partial s}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{2.83}$$

The second equation can be integrated to give

$$\tfrac{1}{2} F^2 u^2 + \eta = \tfrac{1}{2} F^2 + 1, \tag{2.84}$$

where we use uniform upstream conditions $u = 1$, $\eta = 1$ to evaluate the constant of integration. Alternatively, since $\eta = s + 1/u$, we can write this as a relationship between u and s,

$$s = \tfrac{1}{2} F^2 (1 - u^2) + 1 - \frac{1}{u}. \tag{2.85}$$

There is a uniform steady state $s = 0$, $u = h = 1$, $q = q(1)$. We look for linear perturbations

$$u = 1 + U, \quad s = S, \tag{2.86}$$

and substituting into the equations (2.85) and (2.83) we have

$$S = (1 - F^2)U, \tag{2.87}$$

$$S_t + q'(1)U_x = 0. \tag{2.88}$$

Writing

$$S = e^{\sigma t + ikx}, \quad U = \hat{U}e^{\sigma t + ikx}, \tag{2.89}$$

we find

$$1 = (1 - F^2)\hat{U}, \tag{2.90}$$

and hence

$$\sigma = -\frac{ik}{1 - F^2}q'(1). \tag{2.91}$$

We see that there is no instability mechanism. Small perturbations neither grow not shrink, but are translated with speed $-\sigma_I/k = q'(1)/(1-F^2)$, which is positive if $F < 1$ and negative if $F > 1$.

Since $\eta = s + 1/u$, the perturbation in surface elevation is

$$S - U = \frac{F^2}{F^2 - 1}e^{\sigma t + ikx}, \tag{2.92}$$

which is in phase with the bed elevation if $F > 1$ and out of phase if $F < 1$.

This model therefore correctly predicts the propagation direction of dunes and anti-dunes, as well as the relative phase of the bed and the stream surface, but it does not explain their growth from a uniform state.

Note that the Exner equation can be written in the form of a nonlinear wave equation for s,

$$\frac{\partial s}{\partial t} + q'(u)u'(s)\frac{\partial s}{\partial x} = 0, \tag{2.93}$$

where, from the relationship (2.85), we have

$$u'(s) = \frac{1}{s'(u)} = \frac{u^2}{1 - F^2u^3}. \tag{2.94}$$

The wave speed can be both positive and negative depending on the value of F and on s, so the generic behaviour of many initial conditions is to form travelling waves that form shocks. In particular, dunes with $F < 1$ typically have downstream faces that steepen, whereas anti-dunes with $F > 1$ have upstream faces that steepen.

2.3.5 Suspended sediment

We now consider adding suspended sediment to the model. We make use of (2.66) and take dimensionless erosion rate $E = E(u)$, so the Exner equation and suspended sediment equation take the form

$$(1-n)\frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = -v_s\left(E - \frac{c}{\rho_s}\right), \tag{2.95}$$

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(huc) = \rho_s v_s\left(E - \frac{c}{\rho_s}\right), \tag{2.96}$$

together with the St Venant equations in the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{2.97}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = g\left(S - \frac{\partial \eta}{\partial x}\right) - \frac{fu^2}{h}, \tag{2.98}$$

We non-dimensionalise the model, using scales

$$x = [x]\hat{x}, \quad t = [t]\hat{t}, \quad u = [u]\hat{u}, \quad s, h, \eta = [h]\left(\hat{s}, \hat{h}, \hat{\eta}\right), \quad c = [c]\hat{c}, \quad E = [E]\hat{E}, \quad q_b = [q_b]\hat{q}. \tag{2.99}$$

As before, the balance of terms in the momentum equation suggests that we choose scales so that

$$gS[h] = f[u]^2, \tag{2.100}$$

and we take the flux scale $Q = [h][u]$ as given, so choose scales

$$[h] = \left(\frac{fQ^2}{gS}\right)^{1/3}, \quad [u] = \left(\frac{gSQ}{f}\right)^{1/3}, \quad [t] = \frac{(1-n)[h]}{v_s[E]}, \quad [c] = \rho_s[E], \quad [x] = \frac{Q}{v_s}. \tag{2.101}$$

Note that, unlike for the bedload-only model, there is a natural length scale for deposition in this model, and we choose the length scale $[x]$ according to this. We also now choose the timescale associated with erosion, rather than with bedload transport, and will therefore introduce a parameter below to measure the size of the dimensionless bedload flux.

The dimensionless equations are

$$h = \eta - s, \tag{2.102}$$

$$\varepsilon\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{2.103}$$

$$F^2\left(\varepsilon\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \delta\left(1 - \frac{u^2}{h}\right) - \frac{\partial \eta}{\partial x}, \tag{2.104}$$

$$\frac{\partial s}{\partial t} + \beta\frac{\partial q}{\partial x} = c - E, \tag{2.105}$$

$$\varepsilon\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(huc) = E - c, \tag{2.106}$$

where the parameters are

$$\varepsilon = \frac{[x]}{[u][t]}, \quad F = \frac{[u]}{\sqrt{g[h]}}, \quad \delta = \frac{S[x]}{[h]}, \quad \beta = \frac{[q_b][t]}{[h][x]}. \tag{2.107}$$

Both $E = E(u)$ and $q = q(u)$ are taken to be dimensionless functions of velocity.

As in the previous section, we can reasonably expect to have $\varepsilon \ll 1$ and $\delta \ll 1$ (note that now $\varepsilon = [E]/(1-n)$ is the ratio of the advective timescale to the erosion timescale, and since $[E]$ dictates the suspended volume fraction, we expect this to be 10^{-2} or smaller; $\delta = S[u]/v_s$ can now be interpreted as the ratio of vertical velocity to settling velocity).

The model simplifies to

$$hu = (\eta - s)u = 1, \tag{2.108}$$

$$F^2 u\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial x}, \tag{2.109}$$

$$\frac{\partial s}{\partial t} + \beta\frac{\partial q}{\partial x} = c - E = -\frac{\partial c}{\partial x}. \tag{2.110}$$

As earlier, the second equation can be integrated to give

$$\tfrac{1}{2}F^2u^2 + \eta = \tfrac{1}{2}F^2 + 1, \tag{2.111}$$

where we use uniform upstream conditions $u = 1$, $\eta = 1$ to evaluate the constant of integration.

There is a uniform steady state $s = 0$, $u = h = 1$, $c = E(1)$, $q = q(1)$. We look for linear perturbations

$$u = 1 + U, \quad c = E(1) + C, \quad s = S. \tag{2.112}$$

Substituting into the equations we have

$$S = (1 - F^2)U, \tag{2.113}$$

$$S_t + \beta q'(1)U_x = C - E'(1)U = -C_x. \tag{2.114}$$

Writing

$$S = e^{\sigma t + ikx}, \quad U = \hat{U}e^{\sigma t + ikx}, \quad C = \hat{C}e^{\sigma t + ikx}, \tag{2.115}$$

we find

$$1 = (1 - F^2)\hat{U}, \quad \hat{C} = \frac{E'(1)}{1 + ik}\hat{U}, \tag{2.116}$$

and hence

$$\sigma = \frac{E'(1)}{F^2 - 1}\frac{k^2}{1 + k^2} - \frac{ik}{1 - F^2}\left(\beta q'(1) + E'(1)\right). \tag{2.117}$$

On physical grounds we expect $E'(1) > 0$ and $q'(1) > 0$. We see that the growth rate σ_R is positive if $F > 1$ and negative if $F < 1$. The wave speed $-\sigma_I/k$ is positive if $F < 1$ and negative if $F > 1$.

This model therefore predicts the growth of upstream-propagating anti-dunes when $F > 1$. It correctly predict the direction of propagation of dunes for $F < 1$, but does not explain their growth. Note that the destabilisation for $F > 1$ comes entirely from the erosion term rather than the bed-load, consistent with the analysis in the previous section.

Note also that the growth rate is positive even for short wavelengths ($k \to \infty$). This is a sign of ill-posedness, since it suggests that very small scale perturbations in initial conditions would grow exponentially. Usually, this indicates that some other process that would regularise very small wavelength perturbations is missing from the model. In this case, we have neglected diffusion of the suspended sediment, and we have neglected the effect of gravity on the particles in the bed (which provides a modification to the bedload flux depending on the perturbed bed slope).

2.3.6 Eddy viscosity model

The above calculations have shown that it is not possible to predict the formation of dunes using the St Venant-type, depth-integrated, stream model. However, it *is* possible to do so if we use a description of the water flow that accounts for vertical variations in velocity. Models of this form include potential flow models (for inviscid flow), or eddy viscosity models for turbulent flow. We do not go into these here, but report that the results of a more detailed calculation (based on the solution of the Orr-Sommerfeld equation), lead to the following expression for the basal shear stress,

$$\tau = \frac{f\rho_w\overline{u}^2}{\overline{h}}\left[\overline{h} - s + \int_{-\infty}^{\infty} K(x - \xi)\frac{\partial s}{\partial \xi}(\xi)\,\mathrm{d}\xi\right], \tag{2.118}$$

with $K(x) = \mu x^{-1/3}\mathcal{H}(x)$, $\overline{u}$ the mean stream speed, and $\overline{h}$ the mean depth. This allows τ to depend on the bed slope $\partial s/\partial x$ upstream.

It can be shown (exercise) that a model including this modified relationship for the shear stress can predict the formation of dunes for $F < 1$, as observed.

2.3.7 Instability mechanism

The essential ingredient for the formation of dunes that appears to be lacking from the depth-integrated theories is that there should be a lag between local maxima in the bed and the maximum shear stress exerted on the bed.

In our models, the maximum shear stress occurs always at the same location as the maximum in the bed height, because the velocity there is largest (this is easiest to see from looking at the perturbations, where U and S are always proportional for $F < 1$). But for the system to be unstable to the formation of dunes, the maximum shear stress, and therefore the maximum bedload flux or erosion rate, must be slightly upstream of the maxima in the bed height. In that case the sediment load is largest upstream of each bump and sediment is therefore being deposited back onto the bed at the top of the bump, causing it to grow. (In the absence of erosion and deposition the rate of change of bed height is proportional to $-\partial q_b/\partial x$, and so the region ahead of the maxima of q_b will see grow taller and the region behind will fall; this also explains why the dunes move forwards over time).

A crude way to include this lag, that does exist in the mode detailed eddy-viscosity model, is to suppose that the shear stress is a function of the the velocity a short distance further downstream, so $\tau(x,t) = f\rho_w(u(x+\delta,t))^2$, where δ represents the lag. Combining with the Exner equation and the simplest assumption about fluid flow as in (??), we have the dimensionless model

$$\frac{\partial s}{\partial t} + q'(\tau)\frac{\partial \tau}{\partial x} = 0, \qquad \tau(x,t) = \frac{1}{(1 - s(x+\delta,t))^2}. \tag{2.119}$$

The steady state $s = 0$ can be perturbed to give perturbation $S(x,t)$ satisfying

$$\frac{\partial S}{\partial t}(x,t) + 2q'(1)\frac{\partial S}{\partial x}(x+\delta,t) = 0. \tag{2.120}$$

Solutions of the form $S = e^{\sigma t + ikx}$ require

$$\sigma = -2ikq'(1)e^{ik\delta}, \tag{2.121}$$

for which the growth rate is $\sigma_R = 2q'(1)k\sin k\delta > 0$ and the wave speed is $-\sigma_I/k = 2q'(1)\cos k\delta > 0$. This model therefore predicts the formation of dunes (note that we should restrict attention to values of k for which $k\delta \ll 1$, since the lag we artificially introduced should not be longer than the wavelength of the perturbations under consideration).