APPLIED MATHEMATICS FOR ENVIRONMENTAL MODELLING

Attempt ANY FIVE questions

Time: three hours

Observe silence

Question 1. (20 Marks)

We consider here *Bessel's differential equation (of order n)*,

$$x^2 y'' + xy' + (x^2 - n^2)y = 0, \tag{1}$$

for integer $n \geq 0$. Recall, from the previous problem sheet, that a solution of this equation is given by the *Bessel function of first kind* J_n, which has the series expansion

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(-x^2/4\right)^k}{k!(k+n)!} \quad . \tag{2}$$

(a) Using (2), show that the following recursion relation is true for all integer $n \geq 0$:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

(b) For any integer $n \geq 0$, show that

$$\int_0^1 x \left[J_n(\alpha x)\right]^2 dx = \frac{1}{2} \left[J_n'(\alpha)\right]^2,$$

where α is a zero of J_n.
Hint: Substitute $z = \alpha x$ and use integration by parts, and that J_n satisfies Bessel's equation.

Question 2. (20 Marks)

(a) Use (1) in question 1 to determine the *bounded* eigenfunctions and eigenvalues of the singular Sturm-Liouville problem on $0 \leq x \leq 1$,

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) + \lambda xy = 0 \qquad y(1) = 0. \tag{3}$$

Hint: Use a change of variables of the form $r = \beta x$.

(b) Use (a) to obtain the eigenfunction expansion for the <u>*bounded*</u> solution of the inhomogeneous problem on $0 \leq x \leq 1$,

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x \qquad y(1) = 0. \tag{4}$$

Leave the c_k's in your final answer in terms of integrals containing Bessel functions.

Question 3. (20 Marks)

consider Legendre's equation

$$(1-x^2)y'' - 2xy' + \left(l(l+1) - \frac{m^2}{1-x^2}\right)y = 0, \tag{5}$$

and its self-adjoint form

$$((1-x^2)y')' + \left(l(l+1) - \frac{m^2}{1-x^2}\right)y = 0, \tag{6}$$

for integer values $0 \leq m \leq l$.

(a) You are given that $P_l(x)$ is a solution of (5) with $m = 0$. Show that the associated Legendre functions

$$P_l^m = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}\left(P_l(x)\right)$$

are solutions of (5) for $0 \leq m \leq l$.
Hint: Let $y(x) = (x^2-1)^{m/2}v(x)$, plug this into (5) to obtain a differential equation for v. Show that $v = d^m P_l/dx^m$ satisfies this equation, by differentiating

$$(1-x^2)P_l'' - 2xP_l' + l(l+1)P_l = 0$$

m times using Leibniz' theorem.

(b) Show that the following relation is true for $0 \leq m \leq k \leq l$:

$$\int_{-1}^{1} P_k^m(x)P_l^m(x)dx = \begin{cases} 0 & \text{if } l \neq k \\ \frac{2}{(2k+1)}\frac{(k+m)!}{(k-m)!} & \text{if } l = k. \end{cases}$$

Question 4. (20 Marks)

For each degenerate Fredholm integral operator:

(i) Determine all of the eigenfunctions and eigenvalues for the eigenvalues that are of finite multiplicity in $Ly = \lambda y$. What is the eigenvalue that has infinite multiplicity (λ_∞)?

(ii) Solve the inhomogeneous problem $Ly = f(x)$. Specify if no solution exists or if the solution is unique. If the solution is not unique, provide the simplest homogeneous solution[1] that may be added to your particular solution.

(a) $Ly \equiv \dfrac{1}{2}\displaystyle\int_{-1}^{1}(x+t)y(t)\,dt$ $\qquad f(x) = 6x+7$

(b) $Ly \equiv \displaystyle\int_0^1 \left(\tfrac{1}{5}+xt\right)y(t)\,dt$ $\qquad f(x) = \sin(\pi x)$

(c) $Ly \equiv y + \displaystyle\int_0^1 \left(\tfrac{1}{5}+xt\right)y(t)\,dt$ $\qquad f(x) = 31\sin(\pi x)$

(d) $Ly \equiv \dfrac{6}{5\pi}y + \displaystyle\int_0^1 \sin(2\pi x - 3\pi t)y(t)\,dt$ and

(α) $f(x) = 2\sin(2\pi x - \frac{\pi}{6})$,

(β) $f(x) = \sin(2\pi x)$,

(γ) For *general* f, formulate the solvability condition.

Question 5. (20 Marks)

(a) Find the eigenvalues and eigenfunctions for the following operator:

$$Lu(x) \equiv \int_0^\pi k(x,y)u(y)\,dy \tag{1}$$

where $k(x,y) = 2\sin(x/2)\cos(y/2)$ for $x<y$ and $k(x,y) = 2\cos(x/2)\sin(y/2)$ for $x>y$. *Hint: differentiate and turn into a Sturm Liouville problem.*

(b) Solve the integral equation

$$\int_0^\pi k(x,y)u(y)\,dy - 2u(x) = x$$

using an eigenfunction expansion, with $k(x,y)$ as above, i.e. give an explicit formula for the coefficients of an eigenfunction expansion.

Question 6. (20 Marks)

Infinite domains. Obtain the Green's function for each of the following boundary value problems and use it to express the solution for the given data.

(a) $Ly \equiv y''(x) - \mu^2 y'(x) = f(x), \; -\infty < x < 0,$

with boundary conditions:

$$y \to 0 \text{ as } x \to -\infty, \quad y(0) = 1. \tag{1}$$

Here μ is a constant.

(b) $Lu(x) \equiv u''(x) - (1 + x^2)u(x) = f(x), \quad -\infty < x < \infty,$

with the regularity boundary conditions that the solution vanishes at $\pm\infty$.

Hint: Show that $u_0 = e^{x^2/2}$ satisfies $Lu_0 = 0$. Then, seek a solution in the form $u = wu_0$. You may use without proof the fact that

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}.$$

Question 7. (20 Marks)

Green's function for Sturm-Liouville. Consider the Sturm-Liouville operator

$$Ly \equiv -(py')' + qy, \quad a < x < b$$

with boundary conditions

$$B_l(y)|_a \equiv y(a) = 0$$
$$B_r(y)|_b \equiv y(b) = 0.$$

In lectures we gave the following formula for the Green's function, derived via variation of parameters:

$$g(x,t) = \begin{cases} \frac{-y_l(x)y_r(t)}{W(t)p(t)} & a < x < t \\ \\ \frac{-y_l(t)y_r(x)}{W(t)p(t)} & t < x < b \end{cases} \tag{2}$$

where $Ly_l = 0 = Ly_r$, $B_l(y_l) = 0$, $B_r(y_r) = 0$, and $W = y_l y_r' - y_l' y_r$ is the Wronskian.

(a) Derive this formula in a different approach, by constructing the Green's function via the formulation $Lg(x,t) = \delta(x - t)$.

(b) Give an alternative expression for the Green's function, through an eigenfunction expansion, and show that the two formulas agree.

Hint: Expand the Green's function in Equation (2) in an eigenfunction expansion.

Question 8. (20 Marks)

. Determine the parameter values (A, B) that yield existence of a solution for each inhomogeneous boundary value problem:

(a) For $0 \le x \le 2\pi$:

$$\frac{d^2y}{dx^2} + y = A\sin x + B\cos x + 2\sin(x + \tfrac{\pi}{3}) + \sin^3 x \qquad y(0) = y(2\pi) \qquad y'(0) = y'(2\pi).$$

(b)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 1 \qquad y'(0) + y(0) = A \qquad y'(1) + y(1) = 3.$$

Hint: Recall Problem sheet 2 Q2: the homogeneous adjoint problem has solution $w_0(x) = e^x$

Question 9. (20 Marks)

Legendre's equation and the Fredholm Alternative – Consider *bounded* solutions of the eigenvalue problem

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - \lambda y = 0, \qquad -1 \le x \le 1. \tag{1}$$

(a) Writing (1) as $Ly = \lambda r y$, use the inner product relation to directly compute L^* and show that the boundary terms vanish. Why is no information given about the boundary terms?

(b) Convert to Sturm-Liouville form. What orthogonality relation do the eigenfunctions satisfy?

(c) When $\lambda = -n(n+1)$ for integer n, (1) is called Legendre's equation, whose solutions form a sequence of orthogonal polynomials. We will construct the first four, as follows:

  i. Let $y_0(x) = 1$. Show that this an eigenfunction with $\lambda_0 = 0$.

  ii. Let $u_k(x) = x^k$ for $k = 1, 2, 3, \ldots$. Use the Gram-Schmidt orthogonalization process[1] with the orthogonality relation from (b) to construct orthogonal functions y_1, y_2, y_3 from u_1, u_2, u_3.

  iii. Evaluate Ly_k for $k = 1, 2, 3$ to show that they are in fact eigenfunctions with $\lambda_k = -k(k+1)$.

(d) In order for the inhomogeneous problem $Ly = f(x)$ to have *a* solution in the form of an eigenfunction expansion, $y(x) = \sum_k c_k y_k(x)$, what condition must $f(x)$ satisfy?

(e) Consider the equation $Ly = -2y_1(x)$. Explain via Fredholm Alternative why this problem should have a solution, *but non-unique*. Verify by direct substitution that

$$y = y_1(x) + A\ln\left(\frac{1+x}{1-x}\right) + B$$

is a solution for any values of A and B.
What can you conclude about the constant A?

(f) Find the general solution of $Ly = 1$ by direct integration. Is this solution "*acceptable*"? Does this match your reasoning in (d)?

NOTE!

[1] Look up from Linear Algebra, or use this reminder: To construct an orthogonal set of vectors $\{\vec{\mathbf{v}}_k\}$ from an ordered set of linearly independent vectors $\{\vec{\mathbf{u}}_k\}$, subtract-off from $\vec{\mathbf{u}}_k$ all of the projections of $\vec{\mathbf{u}}_k$ onto the previously generated $\vec{\mathbf{v}}_j$ $(j = 0, \cdots, k-1)$ vectors:

$$
\begin{aligned}
\vec{\mathbf{v}}_0 &= \vec{\mathbf{u}}_0 \\
\vec{\mathbf{v}}_1 &= \vec{\mathbf{u}}_1 - \frac{\langle \vec{\mathbf{u}}_1, \vec{\mathbf{v}}_0 \rangle}{\langle \vec{\mathbf{v}}_0, \vec{\mathbf{v}}_0 \rangle} \vec{\mathbf{v}}_0 \\
\vec{\mathbf{v}}_2 &= \vec{\mathbf{u}}_2 - \frac{\langle \vec{\mathbf{u}}_2, \vec{\mathbf{v}}_0 \rangle}{\langle \vec{\mathbf{v}}_0, \vec{\mathbf{v}}_0 \rangle} \vec{\mathbf{v}}_0 - \frac{\langle \vec{\mathbf{u}}_2, \vec{\mathbf{v}}_1 \rangle}{\langle \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_1 \rangle} \vec{\mathbf{v}}_1 \\
\vec{\mathbf{v}}_3 &= \vec{\mathbf{u}}_3 - \frac{\langle \vec{\mathbf{u}}_3, \vec{\mathbf{v}}_0 \rangle}{\langle \vec{\mathbf{v}}_0, \vec{\mathbf{v}}_0 \rangle} \vec{\mathbf{v}}_0 - \frac{\langle \vec{\mathbf{u}}_3, \vec{\mathbf{v}}_1 \rangle}{\langle \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_1 \rangle} \vec{\mathbf{v}}_1 - \frac{\langle \vec{\mathbf{u}}_3, \vec{\mathbf{v}}_2 \rangle}{\langle \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_2 \rangle} \vec{\mathbf{v}}_2
\end{aligned}
$$

and so on. Observe that $\langle \vec{\mathbf{v}}_k, \vec{\mathbf{v}}_j \rangle = 0$ for any $k \neq j$.