Applied Mathematics for environmental modelling Assignment 1
attempt all the questions

Distributions

1. Scalar property of delta function. Show from its interpretation as an integral that $\delta(ax) = \frac{1}{|a|}\delta(x)$ for any constant a.

2. Convergence of distributions. Show that the following function sequences converge to the δ-distribution as $n \to \infty$.

(a) $$f_n(x) = \begin{cases} n/2 & \text{for } -1/n < x < 1/n, \\ 0 & \text{else} \end{cases}$$

(b) $$f_n(x) = \frac{e^{-nx^2/4}}{\sqrt{4\pi/n}} \quad \left(\text{using } \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \text{ without proof}\right)$$

You have to show that $\lim_{n\to\infty} < f_n, \phi > = < \delta, \phi >$ *holds for all* $\phi \in C_0^\infty(\mathbb{R})$*, i.e. that* $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \phi(0)$

Hints: In (a), split the integral as

$$\int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \int_{-\infty}^{\infty} f_n(x)\,[\phi(x) - \phi(0)]\,dx \,+\, \phi(0)\int_{-\infty}^{\infty} f_n(x)dx.$$

The second integral can be evaluated explicitly. What can you say about the first? A similar approach for (b) though a bit more work. In my model solution I split the integral into 4 parts, including a term

$$\int_{-\rho}^{\rho} f_n(x)\,[\phi(x) - \phi(0)]\; dx$$

with ρ chosen appropriately.

3. Derivatives of distributions.

(a) For $n \geq 0$, $a \in \mathbb{R}$, let $D_n : C_0^\infty \to \mathbb{R}$ be defined by

$$< D_n, \phi > = \left.\frac{d^n\phi}{dx^n}\right|_{x=a} = \phi^{(n)}(a) \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}).$$

Show that D_n is a distribution, and then, by induction, that

$$D_n = (-1)^n \delta^{(n)}(x - a),$$

where $\delta^{(n)}$ is the n-th distributional derivative of the δ-distribution.

(b) Let $T : C_0^\infty \to \mathbb{R}$ be a distribution (not necessarily one that is induced by a continuous function). Using the definition of distributional derivative, translation of distributions and convergence of distributions, show that

$$\lim_{\alpha\to 0} \frac{T(x+\alpha) - T(x)}{\alpha} = T'(x)$$

You may use (without proof) that for any $\phi \in C_0^\infty(\mathbb{R})$*,*

$$\rho_\alpha(x) \equiv \frac{\phi(x) - \phi(x-\alpha)}{\alpha}$$

converges to $\phi'(x)$ *uniformly in* x *as* $\alpha \to 0$*.*

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Eigenfunction expansions

1. Adjoint problems. Use the adjoint relation, $\langle w, Ly\rangle = \langle L^*w, y\rangle$, to determine the differential operator and boundary conditions for the adjoint problem. In each case case state if the operator and/or the full system is self-adjoint.

   (a) $Ly \equiv \dfrac{d^2y}{dx^2}$, $\quad 2y(0) + y'(0) = 0, \quad y(1) + y'(1) = 0.$

   (b) $Ly \equiv \dfrac{d^4y}{dx^4} - \dfrac{dy}{dx}$, $\quad y'(0) - y''(0) = 0, \quad y'''(0) = 0, \quad y(1) = 0, \quad y'(1) - y'''(1) = 0.$

2. Adjoint condition. Let

$$Ly \equiv a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x), \quad a < x < b$$

   such that $L^* = L$. Show that the system is fully self adjoint for any general unmixed boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned} \tag{1}$$

3. An inhomogeneous problem.

   (a) Find the general homogeneous solution of the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (1+\alpha)y = 0, \tag{2}$$

   where α is a given positive constant.

   (b) Use (a) to determine the eigenvalues and eigenfunctions of the self-adjoint problem

$$\frac{d}{dx}\left(x^3\frac{dy}{dx}\right) + \lambda xy = 0 \qquad y(1) = 0 \qquad y(\mathrm{e}) = 0. \tag{3}$$

   (c) Obtain the eigenfunction expansion for the solution of the inhomogeneous problem

$$\frac{d}{dx}\left(x^3\frac{dy}{dx}\right) = x \qquad y(1) = 0 \qquad y(\mathrm{e}) = 0. \tag{4}$$

   (Give the coefficients explicitly, i.e. compute the integrals.)

4. Eigenfunction expansion gone wrong. What is wrong with the below argument?

   Let L be a second-order differential operator as usual, with boundary conditions BC. Its eigenfunctions satisfy $Ly_k = \lambda_k y_k$ (or $L^*w_k = \lambda_k w_k$) with homogeneous boundary conditions $BC = 0$ (or $BC^* = 0$) and are complete. The solution of any inhomogeneous problem $Ly = f$, $BC \neq 0$, can then be written $y(x) = \sum_k c_k y_k(x)$, and in lectures we gave a method for finding c_k in terms of f and the boundary conditions.

Now suppose I say

$$
\begin{aligned}
Ly &= f \\
\Rightarrow L\sum_k c_k y_k &= f \\
\Rightarrow \sum_k c_k L y_k &= f \\
\Rightarrow \sum_k c_k \lambda_k y_k &= f \\
\Rightarrow w_j \sum_k c_k \lambda_k y_k &= w_j f \\
\Rightarrow \sum_k c_k \lambda_k \langle w_j, y_k \rangle &= \langle w_j, f \rangle \\
\Rightarrow c_j &= \frac{\langle w_j, f \rangle}{\lambda_j \langle w_j, y_j \rangle}
\end{aligned}
$$

where I have (legitimately) used orthogonality to get the last line. The answer must be wrong because it contains no reference to the boundary conditions, but what is wrong? (Carefully consider the justification for going from one line to the next).

[You might try going through the steps with the system $Ly \equiv y'' = 0$, $y(0) = 0$, $y(1) = 1$. Here the problem is self-adjoint with eigenfunctions $\sin n\pi x$, and the recipe above gives $c_k = 0$, which are obviously not the Fourier sine coefficients of the solution $y = x$. The recipe in lectures *does* give the right answer, of course.]