LINEAR PROGRAMMING PROBLEMS

Basic Concepts

Feasible region.
A region in which all the constraints are satisfied simultaneously is called a feasible region.

Feasible solution.
A solution to the LPP which satisfies the non-negativity restrictions of the LPP is called a feasible solution.

Optimal solution.
Any feasible solution which optimizes the objective function is called its optimal solution.

Basic solution and Basic feasible solution
Given a system of m linear equations with n variables (m<n), the solution obtained by setting n-m variables equal to zero and solving for the remaining m variables is called a basic solution. A basic solution in which all the basic variables are non-negative is called a basic feasible solution.

Unbounded solution.
If the values of the objective function Z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

Slack and Surplus variables
The non-negative variable which is added to LHS of the constraint to convert the inequality $\leq$ into an equation is called slack variable.

$\sum_{j=1}^{n} a_{ij}x_i + s_i = b_i (i = 1, 2, ..., m)$ where s_i are called slack variables.

The non-negative variable which is subtracted from the LHS of the constraint to convert the inequality $\geq$ into an equation is called surplus variable.

$\sum_{j=1}^{n} a_{ij}x_i - s_i = b_i (i = 1, 2, ..., m)$ where s_i are called surplus variables.

Optimality test in a LPP
By performing optimality test we can find whether the current feasible solution can be improved or not. This is possible by finding the $Z_j - C_j$ row. In the case of a maximization problem if all $Z_j - C_j$ are nonnegative, then the current solution is optimal.

Methods used to solve an LPP involving artificial variables

i) Big M method or penalty cost method
ii) Two-phase simplex method

Artificial variable
Any non negative variable which is introduced in the constraint in order to get the initial basic feasible solution is called artificial variable.

LPP posses a pseudo-optimal solution
An LPP possesses a pseudo-optimal solution if at least one artificial variable is in the basis at positive level even though the optimality conditions are satisfied.

Degeneracy
The concept of obtaining a degenerate basic feasible solution in a LPP is known as degeneracy. In the case of a BFS, all the non basic variables have zero value. If some basic variables also have zero value, then the BFS is said to be a degenerate BFS.

How to resolve degeneracy in a LPP
Divide each element of the rows (with tie) by the positive coefficients of the key column in that row.
Compare the resulting ratios, column by column, first in the identity and then in the body from left to right.
The row which first contains the smallest ratio contains the leaving variable.

Dual of LPP.
For every LPP there is a unique LPP associated with it involving the same data and closely related optimal solution. The original problem is then called the primal problem while the other is called its dual problem. If the primal problem is

$$\text{Maximize } Z = CX$$
$$\text{subject to } AX \le b$$
$$X \ge 0$$

Then the dual is

$$\text{Minimize } Z^* = b^T Y$$
$$\text{subject to } A^T Y \ge C^T$$
$$Y \ge 0$$

Characteristics of canonical form.
The characteristics of canonical form are

i) The objective function is of maximization type
ii) All constraints are "$\le$" type
iii) All variables X_i are non negative.

Characteristics of standard form.
The characteristics of standard form are

i) The objective function is of maximization type
ii) All constraints are expressed as equations
iii) RHS of each constraint is non- negative
iv) All variables X_i are non-negative.

General mathematical model of LPP in matrix form.

Max or Min Z = CX

Subject to AX ($\leq = \geq$) b

$X \geq 0$

Worked out Examples

1. Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix. Formulate this problem as a LPP.
Solution:

Let x_1 = number of grams of eggs to be consumed
x_2 = number of grams of milk to be consumed

The LPP is: Min $Z = 12x_1 + 20x_2$

Subject to

$6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

$x_1, x_2 \geq 0$

2. Write the procedure of the graphical method?
Solution:

Step 1: Consider each inequality constraint as equation.

Step 2: Plot each equation on the graph, as each will geometrically represent a straight line.

Step 3: Mark the region. If the inequality constraint corresponding to a line is $\leq$ type then the region below the line lying in the first quadrant is shaded. For the inequality constraint ≥ type, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region is the feasible region.

Step 4: Plot the objective function by assuming Z = 0. This will be a straight line passing through the origin. As the value of Z is increased from zero, the line starts moving, parallel to itself. Move the line till it is farthest away from the origin for maximization of the objective function. For a minimization problem the line will be nearest to the origin. A point of the feasible region through which this line passes will be the optimal point.

Step 5: Alternatively find the co-ordinates of the extreme points of the feasible region and find the value of the objective function at each of these extreme points. The point at which the value is maximum (or minimum) is the optimal point and its coordinates give the optimal solution.

3. Write the procedure of the big M method?
   Solution:

Step 1: Express the problem in the standard form.

Step 2: Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type ≥ or = . Assign a very large penalty cost (-M for Maximization and M for Minimization) with artificial variables in the objective function.

Step 3: Solved the modified LPP by simplex method, until any one of the three cases that may arise.

1. If no artificial variable appears in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.
2. If at least one artificial variable in the basis is at zero level and the optimality condition is satisfied then the current solution is an optimal basic feasible solution (though degenerate).
3. If at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied then the original problem has no feasible solution .The solution satisfies the constraints but does not optimize the objective function, since it contains very large penalty M and it is called pseudo optimal solution.

4. Solve the following LPP by graphical method.

Minimize $Z = 20x_1 + 10x_2$

Subject to

$x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$

$\& \; x_1, x_2 \geq 0$

Solution:

Replace all the inequalities of the constraints by equation

$x_1 + 2x_2 = 40 \quad \text{if} \begin{array}{l} x_1 = 0 \Rightarrow x_2 = 20 \\ x_2 = 0 \Rightarrow x_1 = 40 \end{array}$

$\therefore \; x_1 + 2x_2 = 40$ passes through (0,20)(40,0)

$3x_1 + x_2 = 30$ passes through (0,30)(10,0)

$4x_1 + 3x_2 = 60$ passes through (0,20)(15,0)

Plot each equation on the graph. The feasible region is ABCD.

C and D are points of intersection of lines

$\therefore \; x_1 + 2x_2 = 40$, $3x_1 + x_2 = 30$,

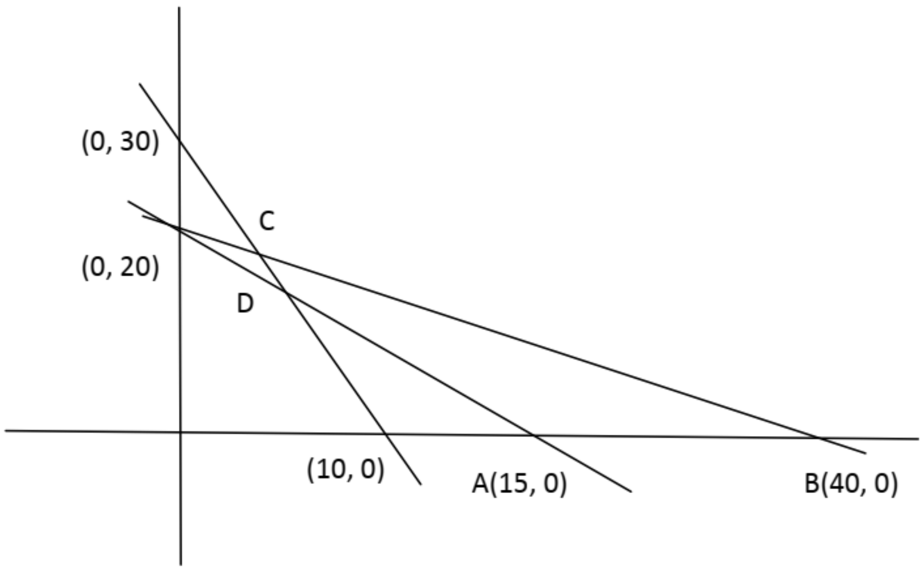
and $4x_1 + 3x_2 = 60$, $x_1 + x_2 = 30$

On solving we get C = (4, 18) D = (6, 12)

Corner points	Value of $Z = 20x_1 + 10x_2$
A (15,0)	300
B (40,0)	800
C (4,18)	260
D (6,12)	240 minimum value

$\therefore$ The minimum value of Z occurs at D (6, 12).

Hence the optimal solution is $x_1 = 6, x_2 = 12$.



5. Solve by the dual simplex method the following LPP

$$\text{Min } Z = 5x_1 + 6x_2 .$$

Subject to $x_1 + x_2 \geq 2$

$$4x_1 + x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

Solution:

By introducing slack variables S_1, S_2 we get the standard form of LPP as given below.

$$\text{Max } Z = -5x_1 - 6x_2 + 0s_1 + 0s_2$$

Subject to

$$-x_1 - x_2 + s_1 = -2$$
$$-4x_1 - x_2 + s_2 = -4$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Initial table

		C_J	-5	-6	0	0
C_B	B	X_B	X_1	X_2	S_1	S_2
0	S_1	-2	-1	-1	1	0

$\leftarrow 0$	S_2	-4	-4	-1	0	1
	Z_J	0	0	0	0	0
	$Z_J - C_J$		5↑	6	0	0

Since all $Z_J - C_J \geq 0$ optimality conditions are satisfied. Since all $X_{Bi} < 0$, the current solution is not a basic feasible solution.

Since $X_{B2} = -4$ is most negative, the basic variable S_2 leaves the basis.

Since Max {5/-4, 6/-1} = -5/4, X_1 enters the basis.

First iteration: Drop S_2 and introduce X_1

		C_J	-5	-6	0	0
C_B	B	X_B	X_1	X_2	S_1	S_2
$\leftarrow 0$	S_1	-1	0	$-3/4$	1	$-1/4$
-5	x_1	1	1	$1/4$	0	$-1/4$
	Z_J	-5	-5	$-5/4$	0	$5/4$
	$Z_J - C_J$		0	$19/4$	0	$5/4$ ↑

Since all $Z_J - C_J \geq 0$ optimality conditions are satisfied. Since some $X_{Bi} < 0$ the current solution is not a basic feasible solution.

S_1 leaves the current basis. Since Max {19/-3, 5/-1} =5/-1, S_2 enters the basis.

Second iteration:

		C_J	-5	-6	0	0
C_B	B	X_B	X_1	X_2	S_1	S_2
0	S_2	4	0	3	-4	1

-5	x_1	2	1	1	-1	0
	Z_J	-10	-5	-5	5	0
	$Z_J - C_J$		0	1	5	0

Since all $Z_J - C_J \geq 0$ and all $X_{Bi} \geq 0$, the current basic feasible solution is optimum. The optimal solution is $Z = 10$, $x_1 = 2$.

6. Write the algorithm of simplex method?
Solution:

Step 1: Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximization by Min Z = - Max (-Z)

Step 2: Check whether all b_i are positive. If any one of b_i is negative then multiply the inequation of the constraint by -1 so as to get all b_i to be positive.

Step 3: Express the problem in standard form by introducing slack/surplus variables, to convert the inequality constraints into equations.

Step 4: Obtain an initial basic feasible solution to the problem in the form $X_B = \mathbf{B}^{-1}b$ and put it in the first column of the simplex table.

Step 5: Compute the net evaluations $Z_J - C_J$ by using the relation

$$Z_J - C_J = C_B X_J - C_J \ .$$

Examine the sign of $Z_J - C_J$.

(i) If all $Z_J - C_J \geq 0$, then the current BFS is the optimal solution.

(ii) If at least one $Z_J - C_J < 0$, then proceed to the next step.

Step 6: If there are more than one negative $Z_J - C_J$ choose the most negative them. Let it be $Z_r - C_r$.

(i) If all $X_{ir} \leq 0 (i = 1, 2, ...m)$ then there is an unbounded solution to the given problem.

(ii) If at least one $X_{ir} > 0 (i = 1, 2, ...m)$ then the variable X_r (key column) enters the basis.

Step 7: Compute the ratio $\left\{ \frac{X_{Bi}}{X_{ir}} / X_{ir} > 0 \right\}$. Let the minimum of these ratios be $\frac{X_{Bk}}{X_{kr}}$. Then choose the variable X_k (key row) to leave the basic. The element at the intersection of the key column and the key row is called the key element.

Step 8: Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under C_B column. Convert the leading element to unity by dividing the key row by the key element and convert all other elements in the simplex table by using the formula

New element = Old element –

{(product of elements in key row and key column) / key element}

Go to Step 5 and repeat the procedure until either an optimal solution is obtained or there is an indication of unbounded solution.

7. Solve the following LPP using simplex method

$$\text{Max } Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$$

Subject to

$$2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$$
$$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$$
$$7x_1 + x_4 \leq 70$$
$$\& x_1, x_2, x_3, x_4 \geq 0$$

Solution:

Rewrite the inequality of the constraint into an equation by adding slack variables S_1, S_2 and S_3 the LPP becomes,

$$\text{Max } Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0.S_1 + 0.S_2 + 0.S_3$$

Subject to

$$2x_1 + x_2 + 5x_3 + 6x_4 + S_1 = 20$$
$$3x_1 + x_2 + 3x_3 + 25x_4 + S_2 = 24$$
$$7x_1 + x_4 + S_3 = 70$$
$$\& \ x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$$

Initial basic feasible solution is

$S_1=20$
$S_2=24$
$S_3=70$
Initial simplex table

		C_j	15	6	9	2	0	0	0	
c_B	B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	Min $\frac{X_B}{X_1}$
0	S_1	20	2	1	5	6	1	0	0	10
0	S_2	24	3	1	3	25	0	1	0	8
0	S_3	70	7	0	0	1	0	0	1	10
	Z_j	0	0	0	0	0	0	0	0	
	Z_J-C_J		-15	-6	-9	-2	0	0	0	

Since all $Z_J - C_J \leq 0$ and the current basic feasible solution is not optimum.

First iteration: X_1 enters the basis and S_2 leaves the basis.

	B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	Min $\frac{X_B}{X_2}$
$\leftarrow 0$	S_1	4	0	$1/3$	3	$-32/3$	1	$2/3$	0	12
15	X_1	8	1	$1/3$	1	$25/3$	0	$1/3$	0	24
0	S_3	14	0	$-7/3$	-7	$-172/3$	0	$-7/3$	1	-
	Z_J	120	15	5	15	125	0	5	0	

	Z_J-C_J		0	-1 ↑	6	123	0	5			

Since some $Z_J - C_J \le 0$ the current basic feasible solution is not optimum

Second iteration:

X_2 enters the basis and S_1 leaves the basis.

The new table is

C_B	B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3
6	X_2	12	0	1	9	-32	3	-2	0
15	X_1	4	1	0	-2	57/3	-1	1	0
0	S_3	42	0	0	14	-132	7	-7	1
	Z_j	132	15	6	24	93	3	3	0
	Z_J-C_J		0	0	15	91	3	3	0

Since all $Z_J - C_J \ge 0$ and the current basic feasible solution is optimum

and is given by Max Z = 132, $X_1 = 4$, $X_2 = 12$, $X_3 = 0$, $X_4 = 0$.

8. Solve by the big M method

   Minimize $Z = 4x_1 + 3x_2$

   Subject to

   $$2x_1 + x_2 \ge 10$$
   $$-3x_1 + 2x_2 \le 6$$
   $$x_1 + x_2 \ge 6$$
   $$\& \; x_1, x_2 \ge 0$$

Solution:

Given Minimize $Z = 4x_1 + 3x_2$

Subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$\& \; x_1, x_2 \geq 0$$

That is $\quad$ $\text{Max } Z = -4x_1 - 3x_2$

Subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$\& \; x_1, x_2 \geq 0$$

By introducing the non negative slack, surplus and artificial variables, the standard form of the LPP is

$$\text{Max } Z = -4x_1 - 3x_2 + 0.s_1 + 0.s_2 - MR_1 - MR_2$$

Subject to

$$2x_1 + x_2 - s_1 + R_1 = 10$$
$$-3x_1 + 2x_2 + s_2 = 6$$
$$x_1 + x_2 - s_3 + R_2 = 6$$
$$\& \; x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0$$

Initial basic feasible solution is

R_1=10
R_2=6
S_2=6

Initial iteration:

C_j	-4	-3	0	0	0	-M	-M

C_B	B	X_B	X_1	X_2	S_1	S_2	S_3	R_1	R_2	Min
-M	R_1	10	2	1	-1	0	0	1	0	5
0	S_2	6	-3	2	0	1	0	0	0	-
-M	R_2	6	1	1	0	0	-1	0	1	6
	$Z_j - C_j$	-16M	-3M+4	-2M+3	M	0	M	0	0	

Since some $Z_J - C_J \leq 0$ and the current basic feasible solution is not optimum.

First iteration:

C_j	-4	-3	0	0	0	-M	-M

C_B	B	X_B	X_1	X_2	S_1	S_2	S_3	R_2	Min
-4	x_1	5	1	1/2	-1/2	0	0	0	10
0	S_2	21	0	7/2	-3/2	1	0	0	42/7
-M	R_2	1	0	½	1/2	0	-1	1	2
	$Z_j - C_j$	-M-20	0	$\frac{-M+2}{2}$	$\frac{-M+4}{2}$	0	M	0	

Since some $Z_J - C_J \leq 0$ and the current basic feasible solution is not optimum.

Second iteration:

C_j -4 -3 0 0 0

C_B	B	X_B	X_1	X_2	S_1	S_2	S_3
-4	X_1	4	1	0	-1	0	1
0	S_2	14	0	0	-5	1	7
-3	x_2	2	0	1	1	0	-2
	$Z_j - C_j$	-22	0	0	1	0	2

Since all $Z_J - C_J \geq 0$ the current basic feasible solution is optimum.

and is given by Min Z = 22, $X_1 = 4$, $X_2 = 2$.

9. Use two phase simplex method to

$$\text{Max } Z = 5x_1 + 3x_2$$

Subject to

$$2x_1 + x_2 \leq 1$$
$$x_1 + 4x_2 \geq 6$$
$$\& \; x_1, x_2 \geq 0$$

Solution:

By introducing the non negative slack, surplus and artificial variables, the standard form of the LPP is

$$\text{Max } Z = 5x_1 + 3x_2 + 0.s_1 + 0.s_2 - R_1$$

Subject to

$$2x_1 + x_2 + s_1 = 1$$
$$x_1 + x_2 - s_2 + R_1 = 6$$
$$\& x_1, x_2, s_1, s_2, R_1 \geq 0$$

Initial basic feasible solution is

$S_1=1$

R1=6

Initial iteration:

C_j 0 0 0 0 -1

C_B	B	X_B	X_1	X_2	S_1	S_2	R_1	Min
0 -1	S_1 R_1	1 6	2 1	1 4	1 0	0 -1	0 1	1 6/4
	$Z_j - C_j$	-6	-1	-4	0	1	0	

Since some $Z_J - C_J \leq 0$ and the current basic feasible solution is not optimum.

First iteration:

C_j 0 0 0 -1 -1

C_B	B	X_B	X_1	X_2	S_1	S_2	R_1
0 -1	x_2 R_1	1 2	2 -7	1 0	1 -4	0 -1	0 1
	$Z_j - C_j$	-2	7	0	4	1	0

Since all $Z_J - C_J \geq 0$ and the current basic feasible solution is optimal to the auxiliary LPP. Since an artificial variable is in the current basis at positive level, the given LPP has no feasible solution.

10. Use dual simplex method to solve the LPP
    Max $Z = -3x_1 - 2x_2$

    Subject to

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3$$
$$\& \; x_1, x_2 \geq 0$$

Solution:

The given LPP is

Max $Z = -3x_1 - 2x_2$

Subject to

$$-x_1 - x_2 \leq 1$$
$$x_1 + x_2 \leq 7$$
$$-x_1 - 2x_2 \leq 10$$
$$x_2 \leq 3$$
$$\& \; x_1, x_2 \geq 0$$

By introducing the non negative slack variables s_1, s_2, s_3 and s_4 the LPP becomes

Max $Z = -3x_1 - 2x_2 + 0.s_1 + 0.s_2 + 0.s_3 + 0.s_4$

Subject to

$$-x_1 - x_2 + s_1 = -1$$
$$x_1 + x_2 + s_2 = 7$$
$$-x_1 - 2x_2 + s_3 = -10$$
$$x_2 + s_4 = 3$$
$$\& \; x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Initial iteration BFS is

$S_1 = -1$, $S_2=7$, $S_3=-10$, $S_4=3$

Initial Iteration:

C_B	B	X_B	X_1	X_2	S_1	S_2	S_3	S_4
0	S_1	-1	-1	-1	1	0	0	0
0	S_2	7	1	1	0	1	0	0
0	S_3	-10	-1	-2	0	0	1	0
0	S_4	3	0	1	0	0	0	1
	$Z_j - C_j$	0	3	2	0	0	0	0

First iteration

C_B	B	X_B	X_1	X_2	S_1	S_2	S_3	S_4
0	S_1	4	-1/2	0	1	0	-1/2	0
0	S_2	2	1/2	0	0	1	1/2	0
-2	x_2	5	1/2	1	0	0	-1/2	0
0	S_4	-2	-1/2	0	0	0	1/2	1
	$Z_j - C_j$	-10	2	0	0	0	1	0

Second iteration:

C_B	B	X_B	X_1	X_2	S_1	S_2	S_3	S_4
0	S_1	2	0	0	1	0	-1	-1
0	S_2	0	0	0	0	1	1	1
-2	x_2	3	0	1	0	0	0	1
-3	x_1	4	1	0	0	0	-1	-2
	$Z_j - C_j$	-18	0	0	0	0	3	4

Since all $Z_J - C_J \geq 0$ the current basic feasible solution is optimum and is given by Min Z = -18, $X_1 = 4$, $X_2 = 3$.