Special functions

We have seen in the previous section a method for constructing solutions to ODEs with non-constant coefficients and singular points. For any given problem, the success of the method and the utility of the solution depends on whether one can obtain a direct formula for the series coefficients.

In this section, we explore several special functions, which occur commonly enough to have a name, and for which the "hard work", the series solution method, has already been done.

Bessel Functions

These particularly common functions can be motivated by considering the vibrating membrane of a circular drum. Let $U(x, y, t$ be the position of the membrane at time t and position (x, y), compared to the flat horizontal rest state.
The governing equations for the membrane are (wave equation):

$U_{tt} = c^2 \Delta U \qquad \text{for } x^2 + y^2 < 1 \qquad$ (Newton's second law & elastic stresses)
$U = 0 \qquad \text{at } x^2 + y^2 = 1 \qquad$ (Membrane pinned at boundary.)

Separation of variables $U(x, y, t) = v(t)u(x, y)$ yields

$$\frac{v_{tt}}{v} = \frac{\Delta u}{u} = \text{const} \equiv -\lambda,$$

$$\text{i.e.} \qquad \Delta u = -\lambda u.$$

Switch to polar coordinates: u now depends on r, θ:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0 \qquad 0 < r < 1, \quad 0 \leq \theta \leq 2\pi,$$

$$u = 0 \qquad r = 1, \quad 0 \leq \theta \leq 2\pi$$

$$u \text{ periodic in } \theta.$$

This is a PDE eigenvalue problem, we need to find λ for which there are non-trivial solutions $u(r, \theta)$.
What about $r = 0$? Note that in

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

the first derivative of u has a singular coefficient. As for ODEs with singular points, we may therefore expect that this will give rise to singularities in the solutions of the PDE and thus we impose the condition

$$u \text{ bounded as } r \to 0,$$

since we expect the solution of the governing equations for the drum membrane to be bounded.
Since u is periodic in θ we can expand u into a Fouries series in θ:

$$u(r,\theta) = U_0(r) + \sum_{n=1}^{\infty} U_n(r)\cos n\theta + V_n(r)\sin n\theta;$$

substituting this into the previous set of equations gives ($' = d/dr$)

$$\frac{1}{r}\left(rU_n'\right)' + \left(\lambda - \frac{n^2}{r^2}\right)U_n = 0, \quad \text{for } 0 \le r < 1, \tag{109a}$$

$$U_n = 0 \quad \text{at } r = 1, \tag{109b}$$

$$U_n \text{ bounded as } r \to 0. \tag{109c}$$

The same equations hold for the V_n. Now eliminate λ by rescaling: $U_n(r) = y(x)$, $x = \sqrt{\lambda}r$,

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

and we arrive at Bessel's equation (for integer $n \ge 0$.).

Bessel functions of first and second kind

Bessel's equation (BE) has a regular singular point at $x = 0$, with indicial equation $\alpha(\alpha-1) + \alpha - n^2 = 0$, the solutions of which are $\alpha_1 = n$, $\alpha_2 = -n$ (double root for $n = 0$).

The general solution is given as a linear combination of two linearly independent solutions. A detailed discussion along the lines of Section 7 reveals that one is locally given at $x = 0$ by a Frobenius series with the exponent $\alpha_1 = n$ and the other by a Frobenius series with exponent $\alpha_2 = -n$ plus $\ln(x)$ times the first solution (case II(b)(i) in our general discussion of ODEs with singular points).

The first Frobenius series with a specific normalization of the leading coefficient of the expansion defines the *Bessel functions of first kind*

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k}$$

for integer $n \geq 0$.
Similarly, a specifically normalized choice for the second expansion defines the *Bessel functions of second kind*

$$Y_n(x) = \frac{2}{\pi}\ln(x/2)J_n(x) - \frac{(x/2)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(x^2/4\right)^k$$
$$-\frac{(x/2)^n}{\pi}\sum_{k=0}^{\infty}\left[\psi(k+1)+\psi(n+k+1)\right]\frac{\left(-x^2/4\right)^k}{k!(n+k)!},$$

where $\psi(m) = -\gamma + \sum_{k=1}^{m-1} k^{-1}$, $(m \geq 1)$ and $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. (Details regarding these expansions will be studied in the problem sets on problem sheet 7.)

Some Bessel function fun facts:

- Since Bessel's equation has only one singular point for finite x, the series for J_n and in Y_n have an infinite radius of convergence.

- Also, J_n and Y_n are oscillating functions that decay as $x \to \infty$. They have an infinitude of discrete zeros for $x \geq 0$, which are quite important and have therefore been tabulated (for example in Abramowitz and Stegun).

- At $x = 0$, the behaviour of the two kinds of Bessel functions is quite different. For J_n, we have $J_n(0) = 0$ if $n > 0$, and $J_0(0) = 1$, while $Y_n \to \infty$ as $x \to 0$.

- Two recursion relations, which can be derived from the local expansions:

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x),$$
$$J_{n+1}(x) = -2J_n'(x) + J_{n-1}(x).$$

The same relations also hold for the Y_n's. In principle, these relations can be used to compute the Bessel functions, however, the straightforward way – calculating the values for larger n from those for smaller n – is usually numerically unstable and is therefore not recommended.

Many more relations as well as theory exist. A vast collection of results for the Bessel functions can be found in particular in Abramowitz and Stegun; some derivations in Riley et al. But for now, back to the vibrating drum. We can now express the general solution for (109a) in terms of Bessel functions as

$$U_n(r) = aJ_n\left(\sqrt{\lambda}r\right) + bY_n\left(\sqrt{\lambda}r\right).$$

The boundedness condition (109c) requires $b = 0$. A non-trivial solution therefore requires that we set $a \neq 0$, without loss of generality: $a = 1$. Thus, the boundary condition (109b) at $r = 1$ leads to

$$J_n\left(\sqrt{\lambda}\right) = 0,$$

i.e. $\sqrt{\lambda}$ has to be one of the zeros of J_n. If we label the zeros of J_n by α_m $m = 1, 2, \ldots$, (sorted, for example, in ascending order), then the eigenvalues for (109) are

$$\lambda = \alpha_m^2 \qquad m = 1, 2, \ldots$$

with corresponding eigenfunctions $J_n(\alpha_m r)$, $m = 1, 2, \ldots$. (Note that n is kept fixed, and that the zeros depend on n!)
The differential equation (109a) can be written in Sturm-Liouville form by multiplying through with r; i.e. (109) is a singular SL problem with weighting function r, and we therefore have the following orthogonality relations between the eigenfunctions

$$\int_0^1 rJ_n(\alpha_l r)\, J_n(\alpha_m r)\, dr = 0 \qquad \text{for } l \neq m;$$

a separate calculation is required for $l = m$:

$$\int_0^1 rJ_n^2(\alpha_m r)\, dr = \frac{1}{2}\left(J_n'(\alpha)\right)^2.$$

Legendre equation and Legendre functions

Legendre equations/functions arise from studying eigenvalue problems for the 3D Laplace operator in spherical coordinates.
The *associated Legendre equation* is given by

$$(1-x^2)y'' - 2xy' + \left(l(l+1) - \frac{m^2}{1-x^2}\right)y = 0,$$

or in self-adjoint form

$$((1-x^2)y')' + \left(l(l+1) - \frac{m^2}{1-x^2}\right) y = 0.$$

The numbers m and l can in general be complex; here, we will focus on the case where m and l are non-negative integers. The solutions of the associate Legendre equation are the *associated Legendre functions* and are denoted by P_l^m; for $m = 0$, we drop the 'associated' and speak of the Legendre equation and functions, usually denoted by P_l.

Properties

1. The points $x = \pm 1$ and $x = \infty$ are regular singular points of the associated Legendre equation. The indicial exponents for $x = 1$ are $-m/2$ and $m/2$. Thus, the local expansion yields one bounded and one unbounded solution at $x = 1$. The same is true for $x = -1$.

2. If we replace $l(l+1)$ in the self-adjoint form of the associated Legendre equation by λ,

$$((1-x^2)y')' + \left(-\frac{m^2}{1-x^2}\right) y + \lambda y = 0, \tag{110}$$

   and consider bounded solutions on $-1 < x < 1$, we see that boundedness imposes two conditions, one at each end of the interval. This suggests that (110) is a singular Sturm-Liouville problem (with coefficient functions $p = 1 - x^2$, $q = -m^2/(1-x^2)$, $r = 1$) with discrete eigenvalues. Indeed, the eigenvalues are exactly of the form $\lambda = l(l+1)$ with integer $l \geq m$. The eigenfunctions are the corresponding associated Legendre functions, i.e. P_l^m. From Sturm-Liouville theory, we infer the orthogonality relation

$$\int_{-1}^{1} P_l^m(x) P_n^m(x) dx = 0 \qquad \text{for } l \neq n.$$

   The case $l = n$ requires explicit calculation, see problem sheet.

3. For $\underline{m = 0}$ the Legendre functions (without 'associated') are polynomials, and are given explictly by a so-called *Rodrigues' formula*:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left[(x^2-1)^l\right].$$

Of course, to find the general solution of the Legendre equation we need a second, linearly independent solution, and this is given by the Legendre function of 2^{nd} kind, denoted by Q_n. These solutions are unbounded at $x = \pm 1$. For the case $n = 0$, the solution Q_0 was stated on problem sheet 4:

$$Q_0(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).$$

4. For the general case, $0 \leq m \leq l$, the associated Legendre functions of first and second kind are given by

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_l}{dx^m}$$

and

$$Q_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m Q_l}{dx^m}$$

respectively. The associated Legendre functions P_l^m are polynomials, if and only if m is even.

5. There are several recurrence relations for (associated) LF, e.g.: These relations, and further properties can be found in Abramowitz and Stegun, and some derivations in Riley at al.

Generalisation: Orthogonal Polynomials

There are other second order linear ODEs with families of orthogonal polynomials as solutions. Often, the orthogonality relations

$$\int_a^b p_m(x) p_n(x) \omega(x) dx = 0 \qquad m \neq n$$

with a fixed weighting function $\omega(x)$ (in general non-trivial, i.e. $\not\equiv 1$) can by inferred by formulating appropriate Sturm-Liouville (eigen)problems. Orthogonal polynomials play an important role in approximation theory, for the construction of numerical methods to discretized differential equations, and in many applications, e.g. from physics.

One can in fact give a complete classification of all infinite families of orthogonal polynomials that can arise from second order linear differential equations (we omit here some specific conditions that are needed to make this a precise statement). The most important ones include (n is a non-negative integer):

1. The "Jacobi-like" polynomials, to which the Legendre, the Chebychev, and the Gegenbauer polynomials belong. These arise from DEs of the type
$$x(1-x)y'' + (a+bx)y' + \lambda y = 0,$$
with constants a and b and an appropriate discrete set of λ.

2. The Laguerre and associated Laguerre polynomials, which are solutions of
$$xy'' + (k+1-x)y' + \lambda y = 0,$$
for $k \neq -1, -2, \ldots$ and an appropriate discrete set of λ. (For the Laguerre polynomials, i.e. without 'associated', we have $k = 0$, and $\lambda = n$.)

3. Hermite polynomials, which are solutions of the Hermite equation
$$y'' - 2xy' + \lambda y = 0 \qquad \text{with } \lambda = 2n.$$

These families share many similar structural properties, e.g. they are given explicitly by Rodrigues' formulae and have similar recurrence relations, see Abramowitz and Stegun.