Singular points of differential equations

In this section we will seek solutions of the nth order linear differential equation

$$Ly = y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y^{(n)}(x) + p_0(x)y(x) = 0, \quad (90)$$

in the form a *series expansion* in the neighbourhood of $x = x_0$. How we proceed, and the nature of the solution, depends on how well-behaved the functions $p_j(x)$ are around x_0.

Ordinary points

The point x_0 is an ordinary point of the ODE if all $p_j(x)$ are analytic there, i.e. they can be expanded as a convergent power series. The procedure in this case is pretty straightforward: (i) write $y(x) = \sum_0^\infty a_k x^k$ as a power series, (ii) plug into (90), using the power series expansions of each of the p_j, then (iii) obtain a sequence of equations for the coefficients a_k that can be solved recursively.
This is the simplest (and least interesting) case, so we won't really spend any time on it, but it is worth noting a few things about ordinary points:

- All n linearly independent solutions of (90) are analytic at x_0.
- The radius of convergence of the series solution $\geq$ distance (in $\mathbb{C}$) to next singular point.

Example:

$$(x^2 + 1)y' + 2xy = 0$$

Here $x_0 = 0$ is an ordinary point. Nearest singular points are $x = \pm i$, distance 1 from 0 $\Rightarrow$ radius of convergence ≥ 1.
Here we can obtain the solution $\frac{1}{(1+x^2)}$ via easier routes, but we note that

$$\frac{1}{(1 + x^2)} = 1 - x^2 + x^4 - \dots$$

with a radius of convergence $= 1$, is the series solution one would obtain.

Singular points

The point x_0 is called **a singular point of the ODE** if one of the $p_j(x)$ is not analytic there.
In this case, the general solution y may have a singularity at $x = x_0$ (but not necessarily). This means that the general solution y may not be analytic at x_0: y or its derivatives can "blow-up" as $x \to x_0$.

Where do ODEs with singular points arise?

- As we will see later, equations with singular points at $x = 0$ commonly arise from linear PDEs in polar/spherical co-ordinates (where $x = 0$ corresponds to radius $r = 0$).
- SL-BVPs with a singular point at the boundary - such problems often require the solution to be bounded at the boundary point (rather than prescribing a specific value).

Note: $x_0 = \infty$ can also be classified as an ordinary or singular point by changing the independent variable via the substitution $t = 1/x$, $u(t) = y(x)$, (i.e. $\mathrm{d}y/\mathrm{d}x = -t^2 \mathrm{d}u/\mathrm{d}t$, etc.) and classifying the point $t = 0$ for the resulting ODE for $u(t)$.

Some Examples

(a) $y'' = \mathrm{e}^x y$: every x_0 is an ordinary point

(b) $x^5 y''' = y$: $x_0 = 0$ is a singular point, every $x_0 \neq 0$ is an ordinary point.

Let's consider more closely a simple example to see the types of solutions near singular points. Consider the equation:

$$y' + x^{-m} y = 0, \quad m \geq 0 \text{ an integer}$$

The general solution can be found via separation of variables, and depends on the value of m:

1. For $m = 0$, the point $x = 0$ is ordinary and $y(x) = C\exp(-x)$. This function can be expanded into a power series at $x = 0$ which converges for all $x \in \mathbb{C}$.
2. For $m = 1$, $y(x) = C/x$. This solution clearly has a singularity at $x = 0$, but a rather benign one (a simple pole).

3. For $m = 2$ (and similarly for larger m), $y(x) = C\exp(1/x)$; this solution has a very strong singularity: $x = 0$ is an essential singularity of $\exp(1/x)$ in the complex plane.

This example suggests that the solution at a singular point of an ODE tends to have a stronger singularity the higher the order of the poles in the coefficients in front of the lower order terms of the ODE. In fact, this is the key idea behind the classification of singular points:

Regular singular points: x_0 is a regular singular point, if all $\tilde{p}_j(x) \equiv p_j(x)(x - x_0)^{n-j}$ are analytic at $x = 0$ (for $j = 0, \ldots, n-1$).

Irregular singular points: all other singular points.

For regular singular points, you might think of it as the singularities are not "too bad", we are essentially able to remove the trouble in p_j by multiplying by a power of $(x - x_0)$ decreasing with the degree of derivative. In this case, a modification of the power series approach can be used.
For irregular singular points, though, there is no general theory!

Cauchy-Euler. One of the simplest and most instructive examples is the Cauchy-Euler equation

$$x^2y'' + axy + by = 0,$$

which clearly has a regular singular point at $x = 0$. As you've seen before, the general solution can be found via the ansatz $y = x^m$. The characteristic equation for m is $m(m-1) + am + b = 0$. There are two cases:

(i) The characteristic equation has two distinct roots m_1 and m_2. Then, the general solution is:

$$y(x) = C_1x^{m_1} + C_2x^{m_2}.$$

(ii) The characteristic equation has a double root m. Then, the general solution is:

$$y(x) = C_1x^m + C_2x^m \ln(x).$$

Note that if the roots are two distinct non-negative integers, then the general solution in (i) is analytic (even though the ODE has a singular point). In general, however, the behaviour as $x \to 0$ is a fractional or even complex power of x.

This behaviour carries over to the general situation for regular singular points, except that the functions x^m are multiplied by an analytic function (i.e. a power series in x). Next, we'll look at the general theory for regular singular points.

Frobenius method for 2nd order ODEs

From now on, we'll restrict to 2nd order equations. If the ODE

$$Ly \equiv y'' + P(x)y' + Q(x)y = 0 \tag{91}$$

has a regular singular point at $x = x_0$, then

$$p(x) := P(x)(x - x_0)$$

and

$$q(x) := Q(x)(x - x_0)^2$$

are analytic, i.e.

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n \tag{92}$$

$$q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n \tag{93}$$

The idea is to seek a solution in the form of a *Frobenius series*

$$y(x) = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{94}$$

(In terms of the Cauchy-Euler example, α is playing the role of m and $\sum a_n(x-x_0)^n$ is the analytic function with coefficients a_n to be determined.) We may assume that $a_0 = 1$ (by choosing α appropriately and normalizing). Now plug in and equate coefficients. At the lowest power ($(x - x_0)^{\alpha-2}$, we find

$$\alpha(\alpha - 1) + p_0\alpha + q_0 = 0$$

This polynomial plays an important role, let's give it a name:

$$F(\alpha) = \alpha(\alpha - 1) + p_0\alpha + q_0.$$

The equation $F(\alpha) = 0$ is called the **indicial equation**, it determines the possible indicial exponents α_1, α_2. Note that these exponents can be complex! We'll order them such that $\Re(\alpha_1) \geq \Re(\alpha_2)$.

Let's carry on with equating coefficients of powers of $(x - x_0)$. We find after some algebra that the coefficients of $(x - x_0)^{n+\alpha-2}$ satisfy

$$F(\alpha + n)a_n = -\sum_{k=0}^{n-1}[(\alpha + k)p_{n-k} + q_{n-k}]a_k \tag{95}$$

Setting $\alpha = \alpha_1$, we know that $F(\alpha_1 + n) \neq 0$ for any integer $n \geq 1$ (do you see why?), thus we can use (95) to solve for all the coefficients a_n, and we obtain one solution

$$y_1(x) = (x - x_0)^{\alpha_1} \underbrace{\sum_{n=0}^{\infty} a_n(x - x_0)^n}_{A(x)} . \tag{96}$$

Thus the **first solution** can always be expressed as a Frobenius series with indicial exponent $\alpha = \alpha_1$.

For the **second solution**, we have to distinguish between several cases and sub-cases[11].

Case I: $\alpha_1 - \alpha_2$ is not an integer (in particular $\neq 0$).
In this case, $F(\alpha_2 + n) \neq 0$ for all $n \geq 1$. Thus we can solve (95) for all coefficients – let's call them b_n to distinguish from previous coefficients. Thus, we obtain with no problems a second solution also as a Frobenius series, with indicial exponent α_2,

$$y_2(x) = (x - x_0)^{\alpha_2} \underbrace{\sum_{n=0}^{\infty} b_n(x - x_0)^n}_{B(x)}$$

Case IIa: $\alpha_1 = \alpha_2$
In the case of a double root we clearly only get one solution with the Frobenius method, and we have to multiply by logs to get a second solution

[11] and sub-sub-cases. It's a bit of a headache...

(similar to the case of a double root in Cauchy-Euler). In particular, the second solution is of the form

$$y_2(x) = y_1(x)\ln(x - x_0) + (x - x_0)^{\alpha_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where y_1 is the first solution. We can determine the c_n in the usual manner. A derivation of this form can be done using the so-called derivative method, which is outlined in Section 7.4.

Case IIb: $\alpha_1 - \alpha_2 = N$, where $N > 0$ is an integer. In this case, we will potentially run into trouble in (95) at $n = N$. There are two possibilities:

(i) For $n = N$, RHS $\neq 0$ in (95). Then we have a contradiction, and the solution method doesn't work. To get a second solution, we use the same form as Case IIa:

$$y_2(x) = y_1(x)\ln(x - x_0) + (x - x_0)^{\alpha_2} \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

and determine the c_n by substituting into the ODE. Note that the indicial exponent for the second term is α_2 (whereas y_1 is given by the Frobenius series using the exponent α_1).

(ii) For $n = N$, RHS $= 0$ in (95).
There is no contradiction, but any choice for a_n (or b_n) will satisfy (95) $\Rightarrow$ 2nd solution has Frobenius form

$$y_2(x) = (x - x_0)^{\alpha_2} \underbrace{\sum_{n=0}^{\infty} b_n (x - x_0)^n}_{B(x)}. \tag{97}$$

where b_0 can be chosen to be $b_0 = 1$ and b_N is also arbitrary. Notice that changing b_N changes (97) by multiples of y_1.

Example

Find a series solution about the regular singular point $x_0 = 0$ for the differential equation

$$4x^2 y'' + 4xy' + (4x^2 - 1)y = 0. \tag{98}$$

Step 1 Assume a solution of form

$$y = x^\alpha \sum_{k=0}^{\infty} a_k x^k \tag{99}$$

with the assumption $a_0 \neq 0$. Compute the corresponding series for y', y'' by differentiating "term by term".

Step 2 Plug the series into the ODE and multiply everything out.

$$0 = \underbrace{\sum_{k=0}^{\infty} 4(k+\alpha)(k+\alpha-1)a_k x^{k+\alpha}}_{4x^2y''} + \underbrace{\sum_{k=0}^{\infty} 4(k+\alpha)a_k x^{k+\alpha}}_{4xy'} - \underbrace{\sum_{k=0}^{\infty} a_k x^{k+\alpha}}_{y} + \underbrace{\sum_{k=0}^{\infty} 4a_k x^{\alpha+k+2}}_{4x^2y} \tag{100}$$

The indicial equation comes from the balance at lowest order, in this case x^α:

$$F(\alpha) = 4\alpha^2 - 1. \tag{101}$$

Step 3 The indicial exponents are the roots of F

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2} \tag{102}$$

Step 4 Shift the terms in the series above in oder to combine all terms into one series, i.e. the goal is to obtain a form

$$\sum [\text{stuff not involving x}] x^{\text{something}} + \text{possibly extra terms from start of series}$$

For this example, we need only shift the index in the last sum, so all series have sum with $x^{\alpha+k}$. Thus, writing

$$\sum_{k=0}^{\infty} 4a_k x^{\alpha+k=2} \overset{n=k+2}{\widehat{=}} \sum_{n=2}^{\infty} 4a_{n-2} x^{n+\alpha}$$

we obtain

$$0 = a_0 F(\alpha) x^\alpha + a_1(4\alpha^2 + 8\alpha + 3)x^{\alpha+1} + \sum_{n=2}^{\infty} [(4(n+\alpha)^2 - 1)a_n + 4a_{n-2}] x^{n+\alpha} \tag{103}$$

We have chosen the α so that the equation balances at x^α, and hence a_0 is free. Balancing at all other orders will determine the coefficients a_n

Step 5 Treat $\alpha = \alpha_1$ first. Setting $\alpha = \alpha_1 = 1/2$ in (103), we obtain

$$a_1 = 0, \quad a_n = \frac{-1}{n(n-1)} a_{n-2}, \; n = 2, 3, \ldots$$

Step 6 Use the recursion formula to determine a formula for the a_k in terms of a_0. A good idea is to write out a few terms, and look for a pattern.

$$\begin{aligned} a_2 &= \frac{-1}{2 \cdot 3} a_0 \\ a_3 &= 0 \\ a_4 &= \frac{-1}{4 \cdot 5} a_2 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} a_0 \\ &\ldots \\ a_{2k} &= \frac{(-1)^k a_0}{(2k+1)!}, \quad a_{2k+1} = 0, \quad k = 1, 2, 3 \ldots \end{aligned} \tag{104}$$

Step 7 Input the formula for the coefficients to obtain the first solution

$$y_1(x) = a_0 x^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} \tag{105}$$

Step 8 Repeat the process for the second root α_2, being careful to treat the right case depending on whether $\alpha_1 - \alpha_2$ is an integer. In this case, $\alpha_1 - \alpha_2 = 1$ is an integer, so we are in Case IIb. At $n = N$, we obtain $0 * b_1 = 0$. There is no contradiction, and b_1 is arbitrary and can be set to zero (CaseIIb(ii)). Following the recursion forward with $b_0 \neq 0$, similar computations as above yield

$$y_2(x) = b_0 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \tag{106}$$

Step 9 The general solution is a combination of the two solutions. Thus the general solution is

$$y(x) = C_1 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + C_2 x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

But you might recognise the series for sine and cosine here, with a root x out front! In fact, the general solution to (98) (which is Bessel's equation of order 1/2) is

$$y(x) = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}$$

Derivative method

Suppose α_1 is a double root of $F(\alpha)$. Let $a_0 = 1$ and solve (95) for $a_1, a_2 \ldots$ with arbitrary α (i.e. $F(\alpha)$ not generally $= 0$. Thus, the $a_n = a_n(\alpha)$, and we can think of α as a parameter in the series

$$y(x;\alpha) \equiv (x-x_0)^\alpha + \sum_{n=1}^{\infty} a_n(\alpha)(x-x_0)^{n+\alpha}$$
$$\Rightarrow Ly = L(x-x_0)^\alpha = (x-x_0)^{\alpha-2}F(\alpha) \tag{107}$$

We know that $y(x;\alpha_1)$ is a solution. But α_1 is double root, which implies

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F|_{\alpha=\alpha_1} = 0$$

The idea is to differentiate (107), then set $\alpha = \alpha_1$. Since L has no dependence on α,

$$\frac{\partial}{\partial\alpha}(Ly)|_{\alpha=\alpha_1} = 0 = L\left(\frac{\partial y}{\partial\alpha}\right)|_{\alpha=\alpha_1}$$
$$\Rightarrow y_2 = \left(\frac{\partial y}{\partial\alpha}\right)|_{\alpha=\alpha_1}$$

is also a solution. Specifically, to get a more concise form:

$$\begin{aligned}\left(\frac{\partial y}{\partial\alpha}\right)|_{\alpha=\alpha_1} = &\sum_{n=0}^{\infty} a_n(\alpha_1)(x-x_0)^{n+\alpha_1}\ln(x-x_0) + \\ &\sum_{n=0}^{\infty}\frac{\mathrm{d}a_n}{\mathrm{d}\alpha}|_{\alpha=\alpha_1}(x-x_0)^{n+\alpha_1} \\ &= y_1(x)\ln(x-x_0) + \underbrace{\sum_{n=0}^{\infty} b_n(x-x_0)^{n+\alpha_1}}_{=(x-x_0)^{\alpha_1}C(x)}\end{aligned} \tag{108}$$

- Derivative method can be used to determine b_n, however, a closed form for $a_n(\alpha)$ for general α is required for this! It is usually easier to just use the appropriate form of the series.
  Plug it into the equation, and compare coefficients (as you would do for a power series expansion). After plugging in, the terms containing the log terms should cancel.