LAPLACE TRANSFORM SOLUTION TO WAVE EQUATION

Laplace Transform of u(x,t):

Suppose that x and t are two independent variables: consider t as the principal variable and x as the secondary variable. When the Laplace transform is applied with t as a variable, the PDE is reduced to an ordinary differential equations of t - transform U(x,s), where x is the independent variable. The general solution U(x,s) of the ODE is then fitted to the boundary conditions of the original problem. Finally, the solution u(x,t) is obtained by the inversion formula.

(i) $L\left[\frac{\partial u}{\partial t}; s\right] = sU(x, s) - u(x, 0)$

(ii) $L\left[\frac{\partial^2 u}{\partial t^2}; s\right] = s^2U(x, s) - su(x, 0) - u_t(x, 0)$

(iii) $L\left[\frac{\partial u}{\partial x}; s\right] = \frac{dU(x, s)}{dx}$

(iv) $L\left[\frac{\partial^2 u}{\partial x^2}; s\right] = \frac{d^2}{dx^2}U(x, s)$

(v) $L\left[\frac{\partial^2 u}{\partial x \partial t}; s\right] = s\frac{d}{dx}U(x, s) - \frac{d}{dx}u(x, 0)$.

where $U(x, s) = L[u(x, t); s]$.

(i) $$L\left[\frac{\partial u}{\partial t}; s\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial t}\,dt = \operatorname*{Lt}_{p\to\infty}\int_0^p e^{-st}\frac{\partial u}{\partial t}\,dt$$

$$= \operatorname*{Lt}_{p\to\infty}\left[\{e^{-st}u(x, t)\}_0^p + s\int_0^p e^{-st}u(x, t)\,dt\right]$$

$$= -u(x, 0) + s\int_0^\infty e^{-st}u(x, t)\,dt$$

Therefore,

$$L\left[\frac{\partial u}{\partial t}; s\right] = sU(x, s) - u(x, 0)$$

(ii) $L\left[\frac{\partial^2 u}{\partial t^2}; s\right] = L\left[\frac{\partial V}{\partial t}; s\right], \quad V = \frac{\partial u}{\partial t}$

$$= sL[V; s] - V(x, 0)$$

$$= s\{sU(x, s) - u(x, 0)\} - u_t(x, 0)$$

$$L\left[\frac{\partial^2 u}{\partial t^2}; s\right] = s^2 U(x, s) - su(x, 0) - u_t(x, 0)$$

(iii) $L\left[\frac{\partial u}{\partial x}; s\right] = \int_0^\infty e^{-st} \frac{\partial u}{\partial x} dt = \frac{d}{dx}\int_0^\infty e^{-st} u(x, t) dt = \frac{dU}{dx}(x, s)$

(iv) $L\left[\frac{\partial^2 u}{\partial x^2}; s\right] = L\left[\frac{\partial \bar{u}}{\partial x}; s\right] = \frac{d}{dx}\left(\frac{dU}{dx}\right) = \frac{d^2 U}{dx^2}(x, s), \quad \bar{u} = \frac{\partial u}{\partial x}$

(v) $L\left[\frac{\partial^2 u}{\partial x\, \partial t}; s\right] = \frac{d}{dx}[sU(x, s) - u(x, 0)] = s\frac{dU}{dx}(x, s) - \frac{du}{dx}(x, 0)$

Problems

1. Solve the initial boundary value problem given, using Laplace Transform Method: $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$; $u(0, t) = u(1,t) = 0$, $t > 0$,

   $u(x,0) = \sin\pi x$, $u_t(x,0) = -\sin\pi x$, $0 < x < 1$.

   Solution:
   Given $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$
   Taking Laplace transform on both sides, and applying the initial conditions,
   $\frac{d^2U}{dx^2} = s^2U(x, s) - s\,u(x, 0) - u_t(x, 0)$.

   The general solution is $U(x, s) = Ae^{sx} + Be^{-sx} + \frac{(s-1)\sin \pi x}{(\pi^2 + s^2)}$.

   Using the boundary conditons,

   $U(x, s) = \frac{(s-1)\sin \pi x}{(\pi^2 + s^2)}$

   Taking inverse Laplace transform, the required solution is
   $u(x, t) = \sin \pi x\left(\cos \pi t - \frac{\sin \pi t}{\pi}\right)$

2. Using Laplace transform method, solve the IBVP $u_{xx} = \frac{1}{c^2} u_{tt} + k$, $0 \le x \le \ell$, $t > 0$. $u(0, t) = 0$, $u_x(\ell, t) = 0$, $t > 0$, $u(x, 0) = u_t(x,0) = 0$, $0 < x < \ell$

Solution:

Given

$$u_{xx} = \frac{1}{c^2} u_{tt} + k,\ 0 \le x \le \ell,\ t > 0.$$

Taking Laplace transform on both sides, and applying the initial conditions,

$$\frac{d^2U}{dx^2} = \frac{1}{c^2}\left[s^2 U(x,s) - s\,u(x,0) - u_t(x,0)\right] + \frac{k}{s}$$

The general solution is $U(x, s) = Ae^{(s/c)x} + Be^{-(s/c)x} - \frac{kc^2}{s^3}$.

Using the boundary conditons in the general solution,

$$U(x, s) = \frac{kc^2}{s^3} \frac{\cosh\left[\frac{s}{c}(l-x)\right]}{\cos\left(\frac{s}{c}\ell\right)} - \frac{kc^2}{s^3}$$

Applying the complex inversion formula and taking inverse Laplace transform, we get the required solution,

$$u(x, t) = \frac{kx}{\ell}(x - 2\ell) + \frac{8k^2 l^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\left[\frac{2n+1}{2\ell}\pi ct\right]\cos\left[\frac{2n+1}{2\ell}\pi(l-x)\right]$$

3. Using Laplace transform method solve

PDE: $u_{xx} = \frac{1}{c^2}u_{tt} - \cos\omega t, \qquad 0 \le x < \infty, \qquad 0 \le t < \infty$

BCs: $u(0, t) = 0, \qquad u$ is bounded as x tends to ∞

ICs: $u_t(x, 0) = u(x, 0) = 0$

Solution Taking the Laplace transform of PDE we obtain

$$\frac{d^2U}{dx^2} = \frac{1}{c^2}[s^2U(x,s) - su(x,0) - u_t(x,0)] - \frac{s}{s^2+\omega^2}$$

Using the ICs, we get

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U(x,s) = -\frac{s}{s^2+\omega^2}$$

Its general solution is found to be

$$U(x,s) = Ae^{(s/c)x} + Be^{-(s/c)x} + \frac{c^2}{s(s^2+\omega^2)}$$

As $x \to \infty$, the transform should also be bounded which is possible if $A = 0$; thus,

$$U(x, s) = Be^{-(s/c)x} + \frac{c^2}{s(s^2 + \omega^2)}$$

Taking the Laplace transform of the BC, we get

$$U(0, s) = 0$$

Using this result in Eq. (6.73), we have

$$B = -\frac{c^2}{s(s^2 + \omega^2)}$$

Hence,

$$U(x, s) = \frac{c^2}{s(s^2 + \omega^2)}[1 - e^{-(s/c)x}]$$

Now, taking its inverse Laplace transform, we get

$$u(x, t) = c^2 L^{-1}\left[\frac{1}{s(s^2 + \omega^2)}; t\right] - c^2 L^{-1}\left[\frac{e^{-(s/c)x}}{s(s^2 + \omega^2)}; t\right]$$

But,

$$L^{-1}\left[\frac{1}{s(s^2 + \omega^2)}; t\right] = \frac{1}{\omega^2}\left\{L^{-1}\left[\frac{1}{s}; t\right] - L^{-1}\left[\frac{s}{s^2 + \omega^2}; t\right]\right\} = \frac{1}{\omega^2}(1 - \cos \omega t),$$

$$L^{-1}\left[\frac{e^{-(x/c)s}}{s(s^2 + \omega^2)}; t\right] = \frac{1}{\omega^2}\left\{1 - \cos \omega\left(t - \frac{x}{c}\right)\right\} H\left(t - \frac{x}{c}\right),$$

4. A string of length l *is* stretched and fixed between the points (0, 0) and (l, 0). Motion is initiated by displacing the string in the form $u = \lambda \sin\left(\frac{\pi x}{l}\right)$ and released from rest at time t = 0. Solve for the boundary conditions of the BVP.

<u>Solution:</u>

The displacement u(x, t) of the string is governed by

$u_{tt} = c^2 u_{xx}$, $0 \le x \le \ell$, $t > 0$.

BCs: $u(0, t) = 0$, $u(\ell, t) = 0$; ICs: $u(x, 0) = \lambda \sin\left(\frac{\pi x}{l}\right)$, $u_t(x, 0) = 0$.

Taking the Laplace transform of the given PDE, we have

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) = c^2 \frac{d^2 U}{dx^2}$$

Using the ICs, we get

$$\frac{d^2 U}{dx^2} - \frac{s^2 U}{c^2} = -\frac{\lambda s}{c^2} \sin \frac{\pi x}{l}$$

Its general solution is found to be

$$U(x, s) = Ae^{(s/c)x} + Be^{-(s/c)x} + \frac{\lambda s \sin(\pi/l)x}{s^2 + \pi^2 c^2/l^2}$$

The Laplace transform of the BCs is given by

$$U(0, s) = 0, \quad U(l, s) = 0$$

Applying these conditions in Eq. (6.87), we obtain

$$A + B = 0$$

$$Ae^{sl/c} + Be^{-sl/c} = 0$$

On solving the above set of equations, we get only the trivial solution, viz.

$$A = B = 0$$

Thus,

$$U(x, s) = \frac{\lambda s \sin(\pi/l)x}{s^2 + \pi^2 c^2/l^2}$$

$$u(x, t) = \lambda L^{-1}\left[\frac{s}{s^2 + \pi^2 c^2/l^2}; t\right] \sin\frac{\pi x}{l} = \lambda \cos\left(\frac{\pi c}{l} t\right) \sin\frac{\pi x}{l}$$

FOURIER TRANSFORM METHOD

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}; n\right] = \int_0^L \frac{\partial^2 f}{\partial x^2} \sin\frac{n\pi x}{L} dx = \left(\frac{\partial f}{\partial x}\sin\frac{n\pi x}{L}\right)_0^L - \frac{n\pi}{L}\int_0^L \frac{\partial f}{\partial x}\cos\frac{n\pi x}{L} dx$$

$$= -\frac{n\pi}{L}\left\{\left(f \cos\frac{n\pi x}{L}\right)_0^L + \frac{n\pi}{L}\int_0^L f \sin\frac{n\pi x}{L} dx\right\}$$

$$= -\frac{n\pi}{L}\left[\frac{n\pi}{L}\mathcal{F}_s[f; n] - [f(0, t) - f(L, t)\cos n\pi]\right]$$

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_s[f; n] + \frac{n\pi}{L}\{f(0, t) - f(L, t)\cos n\pi\}$$

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_s[f; n]$$

Similarly, it can be shown that

$$\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_c[f; n] - \{f_x(0,t) - f_x(L,t)\cos n\pi\}$$

In case $\partial f/\partial x$ vanishes at the ends $x = 0$ and $x = L$, it simplifies to

$$\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_c[f; n]$$

By repeatedly applying these results, we can deduce that

$$\mathcal{F}_s\left[\frac{\partial^4 f}{\partial x^4}; n\right] = +\frac{n^4\pi^4}{L^4}\mathcal{F}_s[f; n],$$

if f_x and f_{xx} vanish at both the ends $x = 0, x = L,$ and

$$\mathcal{F}_c\left[\frac{\partial^4 f}{\partial x^4}; n\right] = +\frac{n^4\pi^4}{L^4}\mathcal{F}_c[f; n]$$

$$\mathcal{F}\left(\frac{\partial}{\partial t} u(x,t)\right)(\omega) = \frac{d}{dt}\mathrm{U}(s, t);$$

$$\mathcal{F}\left(\frac{\partial^n}{\partial t^n} u(x,t)\right)(\omega) = \frac{d^n}{dt^n}\mathrm{U}(s, t), \quad n = 1, 2, \ldots;$$

$$\mathcal{F}\left(\frac{\partial}{\partial x} u(x,t)\right)(\omega) = i\omega\mathrm{U}(s, t);$$

$$\mathcal{F}\left(\frac{\partial^n}{\partial x^n} u(x,t)\right)(\omega) = (i\omega)^n\mathrm{U}(s, t), \quad n = 1, 2, \ldots.$$

Problems

5. Use the complex Fourier transform to solve the wave equation,
$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, -\infty < x < \infty, t > 0$ *given that* $y(x,0) = f(x)$ *and* $\frac{\partial y}{\partial t}(x,0) = 0$.

<u>Solution</u>:

Given $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, x > 0, t > 0$

Taking complex Fourier transform on both sides,

$$F\left(\frac{\partial^2 y}{\partial t^2}\right) = a^2 F\left(\frac{\partial^2 y}{\partial x^2}\right), x > 0, t > 0$$

$$i.e., \frac{d^2 F(y)}{dt^2} = a^2(-1)s^2 F(y)$$

$$i.e., \frac{d^2\bar{y}}{dt^2} + a^2 s^2 \bar{y} = 0 \quad \text{where } \bar{y} = F(y)$$

The general solution is

$\bar{y}(s,t) = A(s)\cos ast + B(s)\sin ast$

Taking the Fourier Transform on the given boundary conditions,

$$\bar{y}(s,0) = \bar{f}(s) = A(s) \quad \text{and} \quad \frac{d}{dt}\bar{y}(s,0) = 0 = B(s)$$

Therefore $\bar{y}(s,t) = \bar{f}(s)\cos ast$

Taking inverse Fourier transform,

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s)\cos ast\, e^{-isx}\, ds$$

Now, $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx}\, \bar{f}(s) ds$$

Therefore, $$f(x - at) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is(x-at)}\, \bar{f}(s) ds$$

And $$f(x + at) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is(x+at)}\, \bar{f}(s) ds$$

Hence, $$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s)\cos ast\, e^{-isx}\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s) \frac{e^{iast} + e^{-iast}}{2} e^{-isx}\, ds$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is(x-at)}\, \bar{f}(s) ds + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is(x+at)}\, \bar{f}(s) ds\right]$$

$$= \frac{1}{2}[f(x - at) + f(x + at)]$$

6. Solve the equation $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0$ under the given condition $u = u_0$ at $x = 0$, $t > 0$ with the initial condition $u(x, 0) = 0$, $x > 0$.

<u>Solution:</u>

$$\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0$$

Taking Fourier Sine Transform on both sides,

$$F_s\left(\frac{\partial u}{\partial t}\right) = h^2 F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin sx dx = h^2 \left[s\sqrt{\frac{2}{\pi}} u(o) - s^2 F_c(u)\right]$$

Using the given boundary conditions,

$$\frac{\partial \bar{u}}{\partial t} + h^2 s^2 \bar{u} = h^2 s u_0 \sqrt{\frac{2}{\pi}} \quad where \quad \bar{u} = F_s(u)$$

$$\Rightarrow \bar{u}(s,t) = As^{-h^2 s^2 t} + \frac{u_0}{s}\sqrt{\frac{2}{\pi}}$$

Taking the Fourier Sine Transform on the given boundary conditions,

$$\bar{u}(s,o) = A \Rightarrow A = -\frac{u_0}{s}\frac{\sqrt{2}}{\pi}$$

$$\bar{u}(s,t) = \frac{u_0}{s}\sqrt{\frac{2}{\pi}}\left[1 - e^{-h^2 s^2 t}\right]$$

Taking inverse Fourier sine transform,

$$\text{u (x, t)} = \frac{2u_0}{\pi}\int_0^\infty \left[\frac{\left[1 - e^{-h^2 s^2 t}\right]}{s}\right]\sin sx ds$$