CALCULUS OF VARIATIONS

Functional:

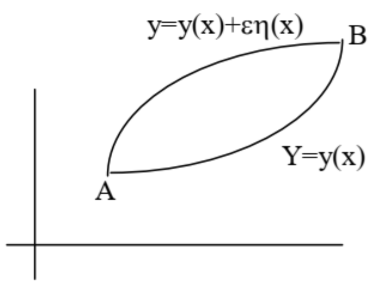
A functional is a type of function where the independent variable itself is a function.

Euler's Equation:

Necessary condition for $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ to be an extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) = 0$$

Proof:



y = y(x) be the curve joining points A(x_1, y_1), B(x_2, y_2) which makes I extremum

Let

$$\boxed{Y = y(x) + \xi\eta(x)} \qquad \text{-------------- (1)}$$

Be neighbouring curve joining A & B such that $\eta(x) = 0$ at A and B.

The value of I along (1)

$$I = \int_{x_1}^{x_2} f\left(x, y + \varepsilon, y' + \varepsilon\eta'(x)\right) dx$$

As $\varepsilon \to 0$ $\frac{dI}{d\varepsilon} = 0$

By Leibniz's Rule

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x}, \frac{\partial r}{\partial \varepsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \varepsilon} \right) dx$$

ε is independent of x $\Rightarrow \dfrac{\partial x}{\partial \varepsilon} = \eta'(r)$

$$\frac{\partial y}{\partial \varepsilon} = \eta(x), \frac{\partial y'}{\partial \varepsilon} = \eta'(r)$$

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(r) + \frac{\partial f}{\partial y'}\eta'(x) \right) dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x) + \int_{x_1}^{r_2} \frac{\partial f}{\partial y'}\eta'(x)\, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x) + \left[\frac{\partial f}{\partial y'}\eta'(x) \right]_{x1}^{x2} \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'(x)\, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x) + \left[\frac{\partial f}{\partial y'}\eta(x) \right]_{x1}^{x2} \int_{x_1}^{x_2} \eta(x) \frac{\partial}{\partial x}\, dx = 0$$

as η(x) equal to zero both at x and x_2

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'} \right) \right) \eta(x)\, dx = 0$$

$$\boxed{\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0}$$

Other Forms of Euler's Equation

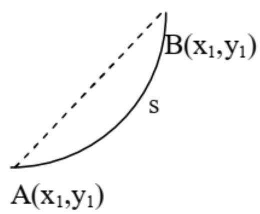
1. $$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial x} = 0$$

2. Extended form of Euler's Equation

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0$$

Problems:

1. Prove by variational principles that the straight line is the shortest path between two given points.



$$S = \int_{x_1}^{x_2} ds$$

$$= \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

$$f = \sqrt{1 + y'^2}$$

Euler's Eqn. is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad (1)$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{1 + y'^2}} - 2y'$$

Sub in (1)

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = a \text{ constant}$$

$$\Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = a \text{ constant}$$

$$\Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = c^2$$

$$\Rightarrow y'^2 = C^2 + c^2 y'^2$$

$$\Rightarrow y'^2 - c^2 + c^2 y'^2 = c^2$$

$$\Rightarrow y'^2 \frac{c^2}{1 - c^2} = m^2 (say)$$

$$\Rightarrow y'^2 \frac{c^2}{1 - c^2} = m^2 (say) \Rightarrow y' = m$$

$$\frac{dy}{dx} = m \Rightarrow y = mx + c \text{ (on integrating c is constant of in)}$$

$\Rightarrow$ y is a straight line. Hence the result

2. Find the curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about x-axis gives a minimum surface area.

We know that

$$s = 2\pi \int_{x_1}^{x_2} y ds$$

$$= 2\pi \int_{x1}^{x2} y\sqrt{1+y'^2}\, dx$$

$$f = y\sqrt{1+y'^2}$$

Euler's Eqn. is $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}.\frac{dy}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}.\frac{dy}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{d}{dy}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow f - y'\frac{\partial f}{\partial y'} = a \text{ constant}$$

$$\Rightarrow y\sqrt{1+y'^2} - y'.\frac{y.y'}{\sqrt{1+y'^2}} = a \text{ constant}$$

$$\Rightarrow \frac{y + yy'^2 - yy'^2}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow y = c\left(\sqrt{1+y'^2}\right)$$

$$y^2 = c^2 + c^2 y'^2$$

$$y'^2 = \frac{y^2 - c^2}{c^2}$$

$$y' = \sqrt{\frac{y^2 - c^2}{c^2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - c^2}}{c}$$

$$c = \frac{d}{\sqrt{y^2 - c^2}} = dx$$

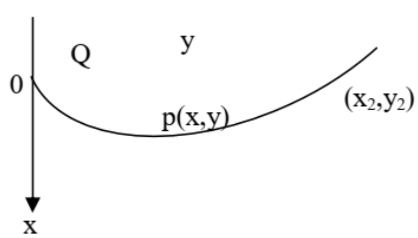
on integrating

$$c - Cos\,h^{-1}\left(\frac{y}{c}\right) = x + k$$

$$Cos\,h^{-1}\left(\frac{y}{c}\right) = \frac{x + k}{c}$$

Which is Equation of catenary

3. Find the path on which a particle in the absence of friction, will slide from one point to another in the shortest time under the action of gravity



Starting point
$(x_1, y_1) = (0, 0)$
Terminal Point
(x_2, y_2)

v = Valocity at any point P(x,y)

m = mass

g = gravity

By Principle of conservation of energy

$$\frac{1}{2}mv^2 = mgx$$

$$v^2 = 2gx$$

$$v = \frac{ds}{dt} = \sqrt{2gx}$$

$$dt = \frac{ds}{\sqrt{2gx}} = \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2g}\sqrt{x}}$$

$$\sqrt{2g}\,dt = \sqrt{\frac{1+y'^2}{x}}dx$$

$$\sqrt{2g}\,t = \int_0^{x_2} \sqrt{\frac{1+y'^2}{x}}dx$$

$$f = \sqrt{\frac{1+y'^2}{x}}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{x}\sqrt{1+y'^2}}.y'$$

Fuler's Eqn. is $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{x(1+y'^2)}}\right) = 0$$

$$\Rightarrow \left(\frac{y'}{\sqrt{x(1+y'^2)}}\right) = a\ cons\tan t = c(say)$$

$$\Rightarrow y'^2 = x\left(1 + y'^2\right) = c^2$$

$$\Rightarrow y'^2 = c^2 x + c^2 x y'^2$$

$$\Rightarrow y'^2 = \left(1 - c^2 x\right) = c^2 x$$

$$y'^2 = \frac{c^2 x}{1 - c^2 x}$$

$$y' = \sqrt{\frac{c^2 x}{1 - c^2 x}}$$

$$y = \int_0^x \sqrt{\frac{c^2 x}{1 - c^2 x}} dx$$

Put $c^2 x = \text{Sin}^2\theta$

$$c^2 dx = 2\text{Sin}\theta \text{ Cos } \theta \text{ d}\theta$$

$$y = \int \frac{\sin\theta}{\cos\theta} . \frac{2\sin\theta\cos\theta}{c^2} d\theta + c_1$$

$$= \frac{1}{c^2} \int 2\sin^2\theta d\theta + c_1$$

$$= \frac{1}{c^2} \int \left(\theta - \frac{\sin^2\theta}{2} \right) + c_1$$

$$= \frac{1}{c^2} \left(\theta - \frac{\sin^2\theta}{2} \right) + c_1$$

$$\text{y} = \frac{1}{2c^2}\left(2\theta - \sin 2\theta\right) + c_1$$

when $\quad x = 0 \quad y=0 \quad 2\theta=0$

$$0 = \frac{1}{2c^2}\left(0 - 0\right) + c_1 \Rightarrow c_1 = 0$$

$$y = \frac{1}{2c^2}\left(2\theta - \sin 2\theta\right)$$

Taking $\phi = 2\theta \;\; 2 \;\; \frac{1}{2c^2} = a$

y = a(ϕ - sinϕ)

x = a. 2 $\sin^2\theta$ = a(1-Cos 2θ)

= a(1-cosϕ)

which is equation of a cycloid

4. Find the function: y(x) for which $\int_{x1}^{x2}\left(1 + x^2 y'\right) y' dx$ is stationary

f = y' + x^2 y'^2

Euler's Equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$0 - \frac{d}{dx}\left(1 + 2x^2 y'\right) = 0$$

4xy' + $2x^2$y'' = 0

x^2y'' + 2xy' = 0

$$x^2 \frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} = 0 \Rightarrow \left(x^2 D^2 + 2xD\right) y = 0 \;\; D = \frac{d}{dx}\text{x}$$

x = ez & z=$\log^x$

x^2D^2 = D'(D'-1), xD = D' where D' = $\frac{d}{dz}$

(D'(D'-1) + 2D') y=0

(D'^2 + D') y = 0

$m^2 + m = 0$

$m = 0, -1$

$y = A + B\, e^{-z}$

$= A + B/x$

$$\boxed{y = A + B.\frac{1}{x}}$$

10. Find the extremal of the functional $v[y(x)] = \int_0^1 \left(y^2 + x^2 y'\right)dx;\ y(0) = 0,\ y(1) = a$

$$2y - \frac{d}{dx}\left(x^2\right) = 0$$

$2y - 2x = 0$

$x = y$

For extremum "a" should be equal to 1

Isoperimetric Problems:

Extremum of $I = \int_{x_1}^{x_2} f\left(x, y, y'\right)dx$ (1)

Subject to (or keeping the Int.) $J = \int_{x_1}^{x_2} g(x, y, y')dx$ as constant (2)

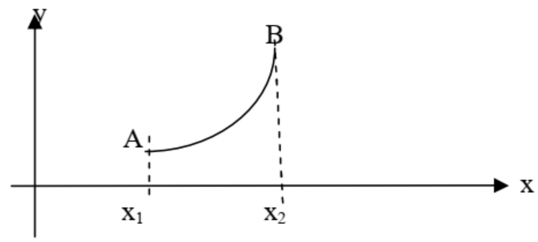
Let $H = f + \lambda g$ (Lagrangian Multiplies)

Necessary condition for $\int_{x_1}^{x_2} H\ dx$ is to be an Extremum is $\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$

λ - being determined from (1), (2) in the boundary conditions.

Problem 1:

Find the plane curve of fixed perimeter & maximum area



Area: $A = \int_{x_1}^{x_2} y\,dx$

Perimeter: $I = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$ (a fixed constant)

$$f = y$$

$$g = \sqrt{1+y'^2}$$

Let $H = y + \lambda\sqrt{1+y'^2}$

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

Integrating w.r. to x

$$x - \frac{\lambda y'}{\sqrt{1+y'^2}} = a \text{ (a constant)}$$

$$x - a\frac{\lambda y'}{\sqrt{1+y'^2}} \Rightarrow y' = \frac{x-a}{\sqrt{\lambda^2 - (x-a)^2}}$$

Integrating $y = -\sqrt{\lambda^2 - (x-a)^2} + 6 \Rightarrow (x-a)^2 + (y-b)^2 = \lambda^2$

Which is a circle.

Problem 2:-

Prove that the sphere is the solid figure of revolution which for a given Area, has maximum Volume

Given : Surface Area

Maximize : Volume

Volume, $V=\int_{0}^{x} \prod y^{2}\, dx$

Surface Area, $S=\int_{o}^{a} 2\pi y\, ds$

$$S=\int_{o}^{a} 2\pi y\sqrt{1+y^{1^{2}}}\, dx$$

$$S=\int_{o}^{a} 2\pi y\sqrt{1+y^{1^{2}}}\, dx$$

$$f=\pi y^{2}$$

$$g=2\pi y\sqrt{1+y^{1^{2}}}$$

$$H=f+\lambda g$$

$$=\pi\left(y^{2}+2y\lambda\sqrt{1+y^{12}}\right)$$

$$\frac{\partial H}{\partial y}-\frac{d}{dx}\left(\frac{\partial H}{\partial y^{1}}\right)=0$$

$$\frac{\partial H}{\partial y}-\frac{d}{dy}\frac{dy}{dx}\left(\frac{\partial H}{\partial y^{1}}\right)=0$$

$$\frac{\partial H}{\partial y}-\frac{d}{dy}\left(y^{1}\frac{\partial H}{\partial y^{1}}\right)=0$$

$$H-y^{1}\frac{\partial H}{\partial y^{1}}=a\,\text{constant}$$

$$y^2 + 2y\lambda\sqrt{1+y^{12}} - y^1\left(\frac{\lambda 2y y^1}{\sqrt{1+y^{12}}}\right) = c$$

$$y^2\sqrt{1+y^{1^2}} + 2y\lambda + 2\lambda y y^{1^2} - 2\lambda y y^{1^2} = c\sqrt{1+y^{1^2}}$$

$$y^2\sqrt{1+y^{1^2}} + 2\lambda y = c\sqrt{1+y^{1^2}}$$

The curve passes through A for which $y = 0$

$$\Rightarrow 0 + 0 = C\sqrt{1+y^{1^2}} \Rightarrow c = 0.$$

$$\therefore \qquad y^2\sqrt{1+y^{1^2}} + 2\lambda y = 0.$$

$$y^4(1+y^{1^2}) = 4\lambda^2 y^2$$

$$y^1 = \sqrt{\frac{4\lambda^2 - y^2}{y}}$$

Integrating on both sides

$$x = k - \sqrt{4\lambda^2 - y^2}$$

$$x = 0,\ y = 0 \Rightarrow k = 2\lambda$$

$$\therefore\ (x - 2\lambda)^2 + y^2 = 4\lambda^2$$

which is a circle. Hence the figure formed by revolution is a sphere.

Problem 3:

Find a function y(x) for which $\int_0^1 (x^2 + y^2)\,dx$ is stationary given that $\int_0^1 y^2\,dx = 2, y(0) = 0, y(1) = 0.$

$$f = x^2 + y^{1^2}$$

$$g^1 = y^2$$

$$H = x^2 + y^{1^2} + \lambda y^2$$

$$\frac{\partial H}{\partial y}-\frac{d}{dx}\left(\frac{\partial H}{\partial y^1}\right)=0$$

$$2\lambda y-\frac{d}{dx}(2y^1)=0$$

$$y^{11}-\lambda y=0$$

$$(D^2-\lambda)\,y=0$$

Case 1:- λ is + ve.

$$m^2=\lambda$$

$$m=\pm\sqrt{\lambda}$$

$$y=c_1\,e\sqrt{\lambda x}+c_2\,e^{-\sqrt{\lambda}x}$$

$$y(0)=0\Rightarrow \quad c_1+c_2=0\Rightarrow c_1=-c_2$$

$$y(1)=0\Rightarrow \quad c_1\,e^{\sqrt{\lambda}}+c_2e^{-\sqrt{\lambda}}=0$$

$$c_1\left(e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}\right)=0 \quad \Rightarrow \quad c_1=0$$

$$c_2=0 \quad \Rightarrow y=0$$

$$\int_0^1 o\,dx=2 \quad \Rightarrow 0=2 \quad \text{Impossible}$$

$\therefore \lambda$ is a $-$ ve Qty Let $\lambda=-k^2$

$$\Rightarrow \quad (D^2\ k^2)\,y=0$$

$$\Rightarrow \quad m^2+k^2=0$$

$$m=\pm k i$$

$$y=A\cos kx+B\sin kr.$$

$$y(0)=0\Rightarrow \quad A=0$$

$$y(1)=0\Rightarrow \quad B\sin k=0$$

$$\sin k=0 \qquad B\neq 0.$$

$$k=n\pi$$

$$y=B\sin n\pi x$$

$$\int_0^1 y^2\,dx=2$$

$$\int_0^1 B^2\sin^2 n\pi x\quad dx=2$$

$$\frac{B^2}{2}\left(x-\frac{\cos 2n\pi x}{2}\right)_0^1=2.$$

$$B^2(1)=4\Rightarrow B=\pm 2$$

$$y=\pm 2\sin n\pi x$$

Functions Involving Higher Order Derivatives

A necessary Condition for

$$I=\int_{x_1}^{x_2} f(x,y,y',y'')\,dx$$

to be an extremum is

$$\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0\right]$$

1) Find the extremal of the functional $\gamma[y(x)]=\int_0^1 (1+y^{11^2})\ dx$ given that $y(0)=0, y(1)=1, y^1(0)=1, y^1(1)=1$

$$f=1+y''^2$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial y^1}=0,\ \frac{\partial f}{\partial y^{11}}=2y''$$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0$$

$$\Rightarrow\frac{d^2}{dx^2}(2y'')=0$$

$$y^{1v}=0$$

$$\Rightarrow y = ax^3 + bx^2 + cx + d$$

$$y^1 = 3ax^2 + 2bx + c$$

Applying the Boundary Conditions

d = 0

a + b + c = 1 ⇒ a + b = 0 ⇒ a = -b

c = 1

3a + 2b + c = 1 ⇒ 3a + 2b = 0

⇒ -3b + 2b = 0 ⇒ b = 0 ⇒ a = 0.

y = x

2. Find the extremal of the functional $\gamma[y(x)] = \int_{-a}^{a}\left(\frac{\mu}{2}y''^2 + \rho y\right)dx.$ given $y(-a)=0, y(a)=0, y^1(-a)=0, y1(a)=0.$

$$f = \frac{\mu}{2}y''^2 + \rho y$$

$$\frac{\partial f}{\partial y} = \rho, \frac{\partial f}{\partial y'} = 0, \frac{\partial f}{\partial y''} = \mu y''.$$

By Euler's Eqn.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$\rho - \mu y^{1v} = 0$$

$$y^{1v} = \frac{\rho}{\mu}$$

Integrating

$$y = \frac{p}{\mu}\left[\frac{x^4}{2^4} + c_2\frac{x^3}{6} + c_3\frac{x^2}{2} + c_4 x + c_5\right]$$

sub. the Boundary & Initial conditions $y = \frac{-\rho}{24\mu}(x^2 - a^2)^2$

3) Find the external of the functional $v[y(x)]=\int_0^{\pi/2}(y^{11}-y^2+x^2)\,dx$ that satisfies

$y(0)=1, y'(0)=0, y\left(\frac{\pi}{2}\right)=0, y'\left(\pi/2\right)=11$

$f=y''-y^2+x^2$

$\frac{\partial f}{\partial y}=-2y, \frac{\partial f}{\partial y'}=0, \frac{\partial f}{\partial y''}=2y''$

By Euler's Eqn.

$$-2y+\frac{d^2}{dx^2}(2y'')=0$$

$$D^4y - y=0$$

$$(D^4-1)y=0$$

$$m^4-1=0$$

$m^2-1=0 \qquad m^2+1=0$

$m=\pm1 \qquad m=\pm i$

$y=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x.$

sub. the conditions

we get c_1, c_2, c_3, c_4

SEVERAL DEPENDENT VARIABLES.

A necessary conditions for $I=\int_{x_1}^{x_2}f(x,y_1,\ldots..y_n,y'_1,y'_2,\ldots...y'_n)\,dx$ to be an extremum is that $\frac{\partial f}{\partial y_i}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'_1}\right)=0,\quad i=1,2,\ldots\ldots.n.$ where y_1, y_2 …… y_n are functions of x

Problem

Find the external of the functional

$$\int_O^1 \left(2yz - 2y^2 + y'^2 - Z'^2\right)dx$$

$$y(0) = 0, y(1) = 1, z(0) = 0, z(1) = 1$$

$$f = 2yz - 2y^2 \mid y'^2 = z'^2$$

$$\frac{\partial f}{\partial y} = 2z - 4y, \frac{\partial f}{\partial y'} = 2y'$$

$$\frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z'} = 2y'$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \Rightarrow \quad 2z - 4y - 2y'' = 0 \qquad (1)$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = 0 \quad \Rightarrow \quad 2y + 2z'' = 0 \;\; z'' = -y \qquad (2)$$

diff (1) twice w, r, to x

$$2z'' - 4y'' - 2y^{iv} = 0$$

$$-y - 2y'' - y^{iv} = 0$$

$$\left(D^4 + 2D^2 + 1\right) y = 0$$

$$\eta^2 = -1, -1$$

$$y = (Ax + B)\cos x + (cx + n)\sin r$$

$$z = 2y + y''$$

$$sub \;\&\; solving \quad y = z = \frac{\sin x}{\sin'}$$

MOVING BOUNDARY PROBLEMS

f = f (x, y, y') be three times diff. fn

$y = \phi(x), \quad y = \Psi(x)$ be two curves in xy - plans (1)

$\phi(x) \in C_1 \{a,b\}$ *and* $\Psi(x) \in C_2 \{a,b\}$

We consider the functional

$$J(y) = \int_{r} F(x, y, y) dx \qquad (2)$$

$y_0 = \phi(x_0), y_1 = \psi(x_1)$

To find extremals of (2) we have the transversality conditions

$$\left(F + (\phi' - y') \frac{\partial f}{\partial y'} \right)_{x=x_o} = 0$$

$$\left(F + (\psi' - y') \frac{\partial f}{\partial y'} \right)_{x=x_1} = 0$$

To solve the problem of Moving Boundaries

1. y = f (x, c_1, c_2) be the extremal
2. Transversality condition

   $y_0 = f(x_0, c_1, c_2) = \phi(x_0)$

   $y_1 = f(x_0, c_1, c_2) = \psi(x_1)$

We have to compute c_1, c_2, x_0, x_1 to find the extremal

Problem

1. Find the shortest distance between A (-1, 5) and the parabola $y^2 = x$

$$J = \int_{-1}^{x_1} \sqrt{1 + y'^2} \, dx$$

Initial point $(-1, 5)$

End point $\left(x_1, \sqrt{x_1}\right)$

The shortest distance is always thro' the straight line.

$$Y = C_1x + C_2 \tag{1}$$

with the conditions

x = -1, y = 5

$x = x_1$, $y = \sqrt{x_1}$

$5 = - C_1 + C_2-$ (2) ie $C_2 = C_1 + 5$

$= \sqrt{x_1}\ C_1 x_1 + C_2$ (3)

The transversality condition for $f = \sqrt{1 + y'^2}$ is

$$\{\psi'\}_{x=x_1} = \left\{\frac{-1}{y'}\right\}_{x=x_1}$$

$$\{\varphi'\}_{x=x_0} = \left\{\frac{-1}{y'}\right\}_{x=x_0}$$

$$\frac{1}{2\sqrt{x_1}} = \frac{-1}{C_1} \tag{4}$$

Sub in (3)

$$\frac{-C}{2} = C_1\left(\frac{C_1^{\ 2}}{4}\right) + C_1 + 5$$

$$-2C_1 = C_1^{\ 3} + 4C_1 + 20$$

$$C_1^{\ 3} + 6C_1 + 20 = 0$$

Solving $C_1 = -2 \Rightarrow C_2 = 3 \Rightarrow \sqrt{x_1} = \frac{-C_1}{2} = 1 \Rightarrow x_1 = 1$

$$\therefore y = -2x + 3$$

$$\therefore \quad J = \int_{x_0}^{x_4} \sqrt{1 + y'^2} \quad dx$$

$$Y_0 = \phi(x0) = x_0^{\ 2}$$

$$Y_1 = \psi(x_1) = x_1 - 5$$

$$Y = C_1 x + C_2$$

$$C_1 x_0 + C_2 = X_0{}^2$$

$$C_1 x_1 + C_2 = X_1 - 5$$

Transversality condition for *for* $f = \sqrt{1 + y'^2}$ *is*

$$\{\phi'\}_{x=x_0} = \left\{\frac{-1}{y'}\right\}_{x=x_0}$$

$$\{\psi'\}_{x=x} = \left\{\frac{-1}{y'}\right\}_{x=x_1}$$

5. $$2x_0 = \frac{-1}{C_1}$$

$$1 = \frac{-1}{C_1} \Rightarrow C_1 = -1$$

$$x_0 = 1/2$$

$$-1.\frac{1}{2} + C_2 = \left(1/2\right)^2$$

$$C_2 = 1/4 + 1/2 = 3/4$$

$$-1.\, x_1 + 3/4 = x_1 - 5$$

$$2\, x_1 = \frac{23}{4}$$

$$x_1 = \frac{23}{8}$$

$$y = -x + 3/4$$

$$y' = \frac{dy}{dx} = -1$$

$$J = \int_{1/2}^{23/8} \sqrt{1 + (-1)^2}\, dx = \sqrt{2} - \left(\frac{23}{8} - \frac{1}{2}\right) = \frac{19\sqrt{2}}{8}$$